$$N - 4p - (m - 2) + 2r - \rho = \bar{I} + 2r - \bar{\rho} + 2,$$

where $\bar{\rho}$, $\bar{I}$, $\bar{\rho}$ are the invariants ρ and those of Zeuthen-Segre for $\bar{F}$. If I is the same invariant for F, since $\bar{I} - \rho = I - \rho$, we have

$$I = N - 4p - m = \Sigma s - 4p - m.$$

Hence Σs is equal to the "equivalence" in nodes of the point (a, b, c) in the evaluation of the invariant I for F by means of the pencil C_y. This property can be shown directly for the following cases: 1°. F has only ordinary nodes. 2°. In the vicinity of any of these nodes there lie other nodes, or ordinary infinitesimal multiple curves. In these cases it is easy to show that in the vicinity of the nodes all the numbers such as h are equal to 2. It would be interesting to know if such is always the case, but the preceding investigation shows that for the applications this does not matter.

UNIVERSITY OF KANSAS,
October 17, 1914.

THE FOURTH DIMENSION.

Geometry of Four Dimensions. By HENRY PARKER MANNING, Ph.D. New York, The Macmillan Company, 1914. 8vo. 348 pp.

EVERY professional mathematician must hold himself at all times in readiness to answer certain standard questions of perennial interest to the layman. "What is, or what are quaternions?" "What are least squares?" but especially, "Well, have you discovered the fourth dimension yet?"

This last query is the most difficult of the three, suggesting forcibly the sophists' riddle "Have you ceased beating your mother?" The fact is that there is no common locus standi for questioner and questioned. To the professional mathematician the fourth dimension usually suggests a manifold of objects depending upon four independent parameters, which it is convenient to describe in geometrical language. Occasionally he does not make any use of analysis, but builds up what the Italians call a "Sistema ipotetico deduttivo" of abstract assumptions and conclusions. The whole thing is professional, and unromantic. Such ideas, are, naturally, unintelligible to the "man in the street." Moreover, could he understand them, they would not answer the questions in which he is interested. What he desires is to know whether the fourth dimension, whatever it be, throws any light on the phenomena of spiritism, or the immortality of the soul. Here the professional mathematician is helpless. He is quite as ignorant as is his questioner in regard to the physical existence of a four-dimensional universe enclosing our space of experience. The only sure thing is that neither of them has been there. If the mathematician be wise he will refer his questioner to such books as our present author's "Fourth Dimension Simply Explained"* or E. A. Abbott's classic "Flatland, by A Square" and leave the matter there.

Between the non-mathematicians and the professionals there is a third class composed of amateurs whose interest in the fourth dimension is somewhat different. These are persons of moderate mathematical learning who feel a commendable curiosity as to what geometrical relations obtain in spaces of more dimensions than we know, and are glad to apply such ability as they possess to the study of the question. It is for this class of people, heretofore neglected by English-speaking writers, that the present work was written. Let us say loudly that we believe that the author has succeeded well in meeting their needs. The problems which he takes up are the problems in which such readers are interested, the methods of attack are those with which they are familiar. It must be remembered that the only allowable methods are those of elementary algebra, geometry, and trigonometry, and in the present work we have elementary geometrical methods exclusively. Moreover, these readers know little and care less about systems of logically independent axioms; they will swallow any reasonable number of assumptions with no fear of indigestion; and in the present work very little attention is given to axiom grinding. The book is admirably fitted for its end.

Unfortunately we can not leave the matter there. Whoever writes a mathematical book in these days, especially a book which introduces the reader to new realms of thought, must state clearly somewhere what assumptions he makes, and stand steadfastly by them. He need not, indeed, devote page after page to proving thoroughly familiar theorems by

* New York, Munn and Company, 1910.

means of some new system of independent axioms, but there should never be any doubt about what facts he has actually assumed. This sort of definite statement is just what our present author never makes; there is always a doubtful zone surrounding his hypotheses. Let us exhibit some examples.

Chapter II is introduced (page 74) with these words: "The subjects now to be taken up belong more particularly to metrical geometry; and we shall assume the axioms of metrical geometry, and employ its terms without special definition. In fact, we shall assume all of the theorems of the ordinary geometry, except, for the present, those which depend on the axiom of parallels." (In a note the author refers to Moore and to Veblen for systems of axioms for geometry.) Let us see what conclusions a malevolent reader might draw from this statement. Our author says, on page 23, that two figures intersect if they have a point or points in common. Let us then remember that the leading theorem in solid geometry says that if two planes intersect, their intersection is a straight line. If we hold to this we need read no further, we have excluded a fourth dimension. We must consult the footnote to see how the author would avoid this. As for parallels, how is a reader whose mathematical equipment is such as this work presupposes, to decide whether a theorem really depends upon the parallel axiom or not? The usual proof that all plane angles of a diedral angle are equal is based upon the properties of parallel lines, but an independent proof is easily devised. On the other hand the theorem that all angles inscribed in the same circular arc are equal, does not appear to depend necessarily upon the parallel axiom, yet it does so in fact, and is untrue in the non-euclidean spaces. No writer should leave to his reader the responsibility of settling questions of this sort.

We used the words "non-euclidean geometry," just now; they suggest a second point wherein we take sharp issue with the author. He is very particular to insist again and again that he has made no use of the parallel axiom (before Chapter VI) and that the theorems developed hold equally well in euclidean, and non-euclidean space. Now when an author is careful to exhibit explicitly all of his assumptions, and to refer constantly to them, a statement of this sort inspires confidence; but when he pursues the optimistic course followed in this book, we feel the need of caution in accepting the dictum. Let us be more explicit. On page 27 under the heading *order* we read: "If A and B are two distinct points, then A comes *before* B and B lies *beyond* A in one *direction along the line* AB while B comes before A and A lies beyond B in the *opposite direction.*" One naturally concludes that this is an explicit assumption which the author makes about a straight line; but he does not like to put things in such a bald fashion, for he says in the preceding sentence that the relation of order "may be explained somewhat in detail as follows." What he really does is to use *order* to explain *betweenness* and then say, presumably by way of an axiom, page 28: "Given any three points on a line, one of them lies between the other two." Now these assumptions are characteristic of a geometry where a straight line is an open locus, and the reader familiar with non-euclidean geometry would naturally assume that the writer in making such assumptions and building upon them intended consciously to exclude elliptic geometry. Such an inference is the polar opposite of the truth. The author acknowledges in a footnote on page 29 that in elliptic space we have cyclic order on a line, and suggests that the reader either look ahead to page 213 where a new set of axioms of cyclic order are set up, or confine himself to a restricted portion of a line. Let us consider these two suggestions in turn. It is true that on pages 213 ff. the author gives a set of axioms for order in what he calls "double elliptic geometry," i. e., spherical geometry where coplanar lines intersect twice, and if these be carefully followed we can reach the theorems of Chapter I for the spherical case. But he completely overlooks the more usual case of single elliptic geometry, where a straight line is a closed circuit, yet coplanar lines meet but once. Here we encounter a really serious difficulty. We naturally say that a point is *between* two others if it lie upon the smaller segment of their common line, assuming the two segments not equal. Let us then take two coplanar lines and mark on one of them three points ABC in such fashion as to divide the whole length into three equal parts. The other line will then contain a point of just one of the three segments BC, CA, AB. Now move one of the three points a very slight distance from the line, without removing it from the plane. We have a highly attenuated triangle ABC, and a line coplanar therewith which intersects one side, and two sides produced. But this is in flat contradiction to the axiom of Pasch, "A line intersecting one side of a triangle, and another side produced, intersects the third side," upon which the whole book rests.

The author would probably reply that the axiom of Pasch was not necessarily true in the whole of the elliptic plane, but only in a *restricted region.* Let us see what is meant by this phrase which is cropping up continually; as on page 153, where a certain theorem is announced for the plane but a footnote says: "Or at least *in a restricted portion of a plane.*" Just what does our author mean by this oft recurring phrase? We turn to the index which refers us to page 19 and here we are referred to page 6 of the author's "Non-Euclidean Geometry."* There we read: "The following propositions are true, at least for figures whose lines do not exceed a certain length. That is, if there is an exception, it is in a case where we can not apply the theorem, or some step in the proof, on account of the lengths of some of the lines. For convenience, we shall use the word *restricted* in this sense, and say that the theorem is true for restricted figures, or in a restricted portion of the plane." Surely, these are dark sayings. The meaning seems to be that certain theorems are usually true, and if they are not true, why that is because some of the lines involved are too long. But how are we going to tell in any particular case whether this difficulty is going to arise or no? We ask in vain; the author vouchsafes no reply. What he has in mind is, probably, something of this sort. Let us start with a complete elliptic space, and then restrict ourselves to such a region as the interior of a sphere whose radius is not greater than one quarter of the total length of a line. In this region two points determine a single definite segment, every segment may be extended beyond either end, points on a straight line have an open order, and Pasch's axiom is true. We seem to have removed all of our difficulties at a stroke. Alas no! In this vale of tears nothing is obtained without cost, and the price which we have paid has been to sacrifice the right to extend a given segment by a preassigned amount. Equally disagreeable is the fact that we can no longer surely drop a perpendicular on a line from an outside point. We are, in fact, in a very parlous state.

No, we take issue entirely with the author on the whole non-euclidean question. We feel that the book would have been stronger and better if he had taken his stand frankly at

* Boston, Ginn and Company, 1901.

the outset on some recognized system of euclidean axioms. Few readers of the type for whom the book was written would have missed the non-euclidean cases, fewer we believe than the number of those who are in danger of being confused by the present arrangement. Such a course would have entailed some modification in the highly ingenious treatment of the regular hypersolids, but this handicap would have been far outweighed by the gain in clearness and precision.

The work begins with an introductory chapter giving a historical account of the subject. The bibliographical references are many, and seem to have been compiled with care. The apology for the exclusive use of elementary synthetic methods is less convincing. The real reason for such a course is that the class of readers especially interested in the subject matter understand no other methods. The author is just right in saying (pages 13 ff.) that the study of the higher spaces throws a flood of light upon the lower ones, but such a statement is far more convincing if illustrated by a few concrete analytical examples. For instance, the statement that the system of spheres in our space gives a sensuous representation of the points in four dimensions does not drive the matter home half so convincingly as when we point out that if we refer the point (X, Y, Z, T) in euclidean S_4 to the oriented sphere in euclidean S_3 whose center is (X, Y, Z) and radius $-\sqrt{-1}T$, the distance of two points in S_4 will have the same analytic expression as the length of a common tangent of the corresponding spheres.

In Chapter I we have the systematic foundation for four-dimensional geometry. The axiom of Pasch, already mentioned, is fundamental in all of this work. The plane, hyperplane (S_3), and four-dimensional space are actually built up by a series of triangle transversal constructions. This is what has come to be recognized as the standard method, in recent years, and the present author employs it with skill and success. A good deal of attention is given to convex figures, which are carefully defined, though it is not clear whether the author is familiar with the recent work of Whittemore and Lennes dealing with such. The chapter ends with a discussion of various graphical properties of solids and hypersolids. Some of the latter are more easy to understand than others. We pass naturally from the pyramid in S_3 to the hyperpyramid in S_4. Much more elusive are the double pyramids and double cones (pages 66 and 71) obtained by drawing lines from the points of the interior of a closed plane curve, to those between two points not in the same S_3 as the curve.

The author's avoidance of existence postulates gives a startling sound to some of his statements, as this on page 59: "*A space of four dimensions* consists in the points we get if we take five points, not points of one hyperplane, etc." The reader almost instinctively says: "Dear me, this is so sudden!" Some definitions are also open to objection as, page 32: "A point is said to be *collinear with a triangle* when it is collinear with any two points of the triangle." Leaving aside the secondary consideration that it is a little inelegant to use the word *collinear* in connection with a two-dimensional figure, it may be contended with some force that according to this definition a point is collinear with a triangle when it is collinear with *each* two points thereof. We reach the author's true meaning if we omit the word *any*.

In spite of these criticisms we feel that the reader who has read the present chapter understandingly, knows a good deal more about geometry than he did before.

The second chapter is devoted almost exclusively to perpendicularity and is decidedly interesting. The author makes very clear the distinction between *simply* perpendicular planes, which lie in an S_3 and are the usual perpendicular planes of commerce, and *absolutely* perpendicular planes where each contains a pencil of concurrent lines perpendicular to the other. It is perhaps a pity that he refers only in a note (page 85) to *half-perpendicular* planes, each of which contains just one perpendicular to the other. It is regrettable that one serious misstatement recurs several times in the chapter. For instance, we read on page 77, theorem 3: "*Through any point outside a hyperplane passes one, and only one, perpendicular to the hyperplane.*" How can a writer so desirous to "bless and preserve to our use" elliptic space appear to forget that there all lines through a point might be perpendicular to a hyperplane? An equivalent mistake will be found on pages 81 and 82. Another slip occurs in the theorem on page 94. If two planes are not in an S_3 the lines in one coplanar with lines in the other have been defined as *linear elements.* We read: "*Given two planes, not in a hyperplane, if any two of their linear elements have a common perpendicular line, they* *all have a common perpendicular hyperplane.*" Leaving out the question of the ambiguity of the word *any*, the author clearly means a common perpendicular line in their plane, for two intersecting lines have plenty of common perpendiculars not in their plane.

Chapter III is devoted to various sorts of angles. It starts with a proof that two skew lines have always one common perpendicular. This is highly ingenious, being based on considerations of continuity and irrational numbers; of course the author has cut himself off from the usual simple proof based upon parallels. We next come to a study of the angles of two planes which have but one common point, and a discussion of the curious figure of isocline planes which, though only intersecting once, have yet an infinite number of equal minimal angles. They are dual figures to the Clifford parallel lines of elliptic space. The author pays some attention to what he rather infelicitously calls *point geometry*, i. e., the geometry of a bundle of concurrent lines, and *edge geometry*, generated by coaxal planes. The two geometries are dual to one another, and have elliptic measurement.

The fourth chapter takes up the concepts of symmetry, order, and motion. This, to the reviewer, is by far the least satisfactory chapter of the book. To begin with, the language is frequently so obscure that the reader vainly tries to find out what is assumed, what is defined, and what is proved. For instance we read on page 160: "We shall say then that the order of a triangle can not be changed by any motion of the triangle in its plane, regarding this statement as, in part, a definition of the phrase *motion in a plane*." The reviewer's guess is that this means, motion in a plane is a transformation of the plane which leaves invariant the distance of each pair of points, and the order of each triangle. What, then, is the order of a triangle? We read (page 154): "We have two principles on which we can base the theory of order in a plane:

"*I. A and B being any two points of a plane, a point which is on one side of the line AB is on the opposite side of the line BA.*

"*II. O, A, and B, being any three non-collinear points of a plane, B is on one side of the line OA and A is on the other side of the line OB.*"

Exactly what rôle in the drama is played by a *principle* the reviewer does not know. The natural inference is that we have here a definition of the phrases *same side, and opposite* *sides.* Yet this can hardly be the case for we can not say of a definition that it is true or not true, yet we continually meet statements like that in the corollary (page 155): *If I holds true of BC and II holds true of OA and OB, then II will hold true of OA and OC.* The reviewer has had lucid intervals when he believed that he understood what these words meant, but they were infrequent and of uncertain duration. Similar statements occur frequently in the following pages. A very definite error occurs in the theorem of page 170: "*If after a motion of a hyperplane on itself there is no point which occupies the position that it occupied before, then every point occupies a position that could have been reached by the motion on itself of some plane of the hyperplane, or by a screw motion.*"

The proof begins as follows: "Let A be the first position, and B the second position of some point, let C be the second position of the point whose first position was B, and let D be the second position of the point whose first positions was C. We will assume that A, B, C, and D are not collinear." "But," asks the careful reader, "what right have you to assume that these points are not collinear?" The answer is "None whatever." As a matter of fact, in the most striking form of motion of that elliptic space which is so dear to our author, the points are collinear, for each (real) point moves along that line of a congruence of Clifford parallels which passes through it, and each plane rotates about one such line. This error invalidates the proofs of the two important theorems on page 174.

Chapter V is devoted to hyperpyramids, hypercones, and hyperspheres, and calls for no special comment. In Chapter VI we have, for the first time, the explicit euclidean assumption about parallels, and an adequate discussion of parallel planes, half-parallel planes, and parallel hyperplanes. The treatment is good, but would have been better if the author had omitted the section on the "hyperplane at infinity." He begins (page 230): "We express certain facts of parallelism as if they were matters of intersection, from which, indeed, they are derived by limiting processes. Thus we say that two lines intersect at infinity only as another way of saying that they are parallel." This is just exactly right. Now we know what we mean by "intersect at infinity," we have as yet, however, no meaning for the phrase "point at infinity." If two lines intersect in the usual sense, they have a common point, if they intersect at infinity they have no common point, so that "point at infinity" must mean some other undefined thing which they share. Yet our author goes serenely on: "Points at infinity are sometimes called *ideal points*. A line has a single *point at infinity*, its intersection with any parallel line." The strange thing is that it is perfectly easy to put the whole subject upon a satisfactory basis. We begin by defining a point at infinity as a bundle of parallel lines, and say that an infinite point lies on a (finite) line, if the latter be a member of the bundle defining the former. Starting thus, the geometry of the infinite domain can be built up in a simple and rigorous way. The latter part of the chapter goes to hyperprisms and double prisms, hypercylinders, and double cylinders. The treatment is careful, but the subjects are sometimes difficult to grasp, especially the double prisms and double cylinders.

Chapter VII is the important metrical chapter, and is entirely devoted to volumes and hypervolumes. The author determines carefully, and with not a little ingenuity the limiting volumes, and the hypervolumes of all of the more important hypersolids. It is regrettable, though perhaps not surprising, that, as usual, he shies at definitions. For instance, we read (page 270): "A hypersolid is supposed to have a *hypervolume* which can be computed from the measurements of certain segments and angles, and which can be expressed in terms of the hypervolume of a given hypercube, taken as a unit." Here again, a few simple assumptions would put the matter in a much better light. Hypervolume is a numerical coefficient attached to a hypersolid. Congruent hypersolids have the same hypervolume, and if a hypersolid be the sum of two others, its hypervolume is the sum of theirs. It would probably be well also to assume the DeZolt theorem, a hypersolid can not have the same hypervolume as a part of itself, and perhaps also the proposition that an infinitely short hyperprism has an infinitesimal hypervolume.

Chapter VIII is the last and deals with the regular hypersolids. Four of these, the pentahedroid, hypercube, 16-hedroid, and 24-hedroid are immediately reached, and easily discussed. Then follows a half-hearted discussion of the Eulerian formula for hypersolids

$$\text{No. of vertices} + \text{No. of faces} = \text{No. of edges} + \text{No. of cells.}$$

We say that the discussion is half-hearted because the author says in a note on page 302 that the formula will not be used again, and may be omitted. The formula is said only to hold for *simple* polyhedrons, though we are never informed exactly which these are. We are, indeed, told on page 63 that the term "polyhedroid" will be applied only to certain simple figures defined individually. This promise is never carried out, and we are led to the idea that the formula is true for simple polyhedroids, since simple polyhedroids are those for which it is true. It would have been better to give more precise definitions or else leave it out entirely. For instance, the following hypersolid is simple, in the sense that it is easily described. We start with a rectangular hyperparallelepiped (heaven save the mark!) whose dimensions are $3 \times 3 \times 3 \times 1$. This can be constructed from 27 abutting unit hypercubes, and fulfills the formula above. We now construct a "hyperdoughnut" by removing the middle hypercube. By this process we add to our original hypersolid all of the vertices, edges, and faces of the hypercube, and all but two of its cells. The formula is no longer applicable.

The author next passes to nets of regular polyhedrons, meaning thereby systems which entirely fill our space. His wording is here sometimes unfortunate, so that the reader is in danger of being confused. Let us cite two instances. The first is page 305: "The only sets of regular polyhedrons that can be used to form nets are 4 tetrahedrons, cubes or dodekahedrons at a point," etc.

The reader naturally concludes that space could be filled with cubes abutting in fours, and wonders how these would appear. But what the author means is that if space *can* be filled with regular polyhedrons meeting four at a point, these solids must be tetrahedrons, cubes, or dodekahedrons. An even more cryptic utterance occurs on page 306: "Any combination which more than fills the part of euclidean space about a point belongs to hyperbolic geometry, and any combination which does not fill the part of euclidean space about a point, belongs to elliptic geometry." These mystic words cover a highly ingenious bit of geometrical reasoning, which is worth explaining.

We start with the geometry of a bundle of concurrent lines in S_3, defining the angle of two such lines, after Laguerre, as a constant multiple of the natural logarithm of a cross ratio which they determine with two generators of a certain cone. This definition is the same in euclidean and the classical non-euclidean geometries, hence the geometry of lines of a bundle is the same in all. We next take a regular tetrahedron, and drop half-lines from its center perpendicular to its faces. These will determine four trihedral angles which together fill up the space about this center. We then wonder whether these trihedral angles are congruent to those of a regular tetrahedron, so that the space could have been equally well filled by four abutting tetrahedrons, but we find that in euclidean space the face angles of these central trihedral angles are greater than those which appear in a regular tetrahedron. Not so in elliptic space. Here the area of a triangle is measured by the excess of the sum of its angles over π. If we take our regular tetrahedron large enough, the face angles will be just those of the four central trihedral angles. We thus complete the first step in the proof that elliptic space can be completely filled by regular tetrahedrons, meeting in fours. The author closes with an account of two pleasant little figures, the 600-hedroid, and the 120-hedroid. The treatment is admirable considering the complexity of the subject.

What shall we offer as our final opinion? We have praised the book in general and damned it enthusiastically in detail. Let the praise be remembered and the blame forgotten. Our author has written a book well fitted to interest and stimulate the audience he had in mind. He has done far more for his day and generation than has Dryasdust, who never makes a mistake because he never has anything to say.

J. L. COOLIDGE.