$$\frac{P_n + A_n R^{1/\lambda} + B_n R^{2/\lambda} + C_n R^{3/\lambda} + \cdots + K_n R^{(\lambda-2)/\lambda} + L_n R^{(\lambda-1)/\lambda}}{Q_n}$$

and the corresponding convergent by α_n/β_n, Professor Lehmer has obtained certain interesting inequalities connecting the P's and Q's, which show that there can be only a finite number of P's which have the same value. In a former paper it was shown that the Q's satisfy the indeterminate equation

$$(-1)^{n-1}Q_n = \alpha_n^\lambda - R\gamma_n^\lambda,$$

and by a general theorem due to Axel Thue (Christiania, *Videnskabs-Selskabet Skrifter*, 1908, No. 3), there can be only a finite number of Q's having the same value in the expansion. This important result has not yet been derived from the discussion of the continued fraction itself.

4. The first paper by Professor Dickson gave a survey of the main results in the theory of invariants arising in the theory of numbers. Special attention was given to the construction of formal modular invariants from the geometrical standpoint developed in the October number of the *Transactions*.

5. The second paper by Professor Dickson related to the theory of modular cubic and quartic curves for the interesting case in which the modulus is 2. Such a quartic curve has at most seven bitangents (and aside from special cases exactly seven) whose intersections are either singular points or points with indeterminate polars. In general, all such points are intersections of bitangents. The equivalence of two quartic curves can be decided from a knowledge of their real points, their singular points, and their points with indeterminate polars.

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Secretary of the Section.

MODULAR INVARIANT PROCESSES.

BY PROFESSOR O. E. GLENN.

(Read before the American Mathematical Society, September 8, 1914.)

Introduction.

LET $f = a_0x_1^n + \cdots$ be an ordinary algebraical quantic in m variables. Suppose that it is subjected to linear transformations whose coefficients are parameters representing positive residues of a prime number p. The result, $f' = a_0'x_1'^n + \cdots$, will be a quantic whose coefficients will be linear forms in the variables a_0, a_1, $\cdots$ with integral coefficients. Any function φ of the coefficients and variables

$$\varphi = \varphi(a_0, a_1, \cdots; x_1, x_2, \cdots)$$

which possesses the property

$$\varphi(a_0', a_1', \cdots; x_1', x_2', \cdots) \equiv \rho^k \varphi(a_0, a_1, \cdots; x_1, x_2, \cdots) \pmod{p}$$

identically in the a's and x's, ρ being the modulus of the transformation, is called a formal modular covariant of f,* or a formal covariant modulo p of f.

It is the purpose of this paper to develop some invariant processes which produce concomitants of this type; that is, processes which are characteristic of the invariant theory of modular transformations.

§ 1. *Modular Polars.*

It is easy to prove that with reference to m-ary transformations with integral coefficients modulo p the following set of functions is cogredient to the set of variables $x_1, x_2, \cdots, x_m$:

$$x_1^{p^t}, x_2^{p^t}, \cdots, x_m^{p^t} \qquad (t \text{ a positive integer}).$$

In fact if the transformations are

$$(1) \qquad x_i = \lambda_i x_1' + \mu_i x_2' + \cdots + \sigma_i x_m' \quad (i = 1, \cdots, m),$$

we have by the multinomial theorem

$$x_i^{p^t} \equiv \lambda_i^{p^t} x_1'^{p^t} + \mu_i^{p^t} x_2'^{p^t} + \cdots + \sigma_i^{p^t} x_m'^{p^t} \pmod{p}.$$

Hence by Fermat's theorem

$$(2) \qquad x_i^{p^t} \equiv \lambda_i x_1'^{p^t} + \mu_i x_2'^{p^t} + \cdots + \sigma_i x_m'^{p^t} \pmod{p},$$

which proves the statement.

In any formal-modular invariant function $\varphi(x_i)$ we may replace the variables x_i by $x_i + l x_i^{p^t}$ without disturbing the property of invariance. Hence follows the theorem.

* Hurwitz, *Archiv der Math. und Phys.*, ser. 3, vol. 5 (1903), p. 17. Dickson, Madison Colloquium Lectures, Lecture III, and Lecture IV, p. 68.

THEOREM 1: *The modular polar*

$$(3) \qquad E_m^{(t)} \equiv x_1^{p^t}\frac{\partial}{\partial x_1} + x_2^{p^t}\frac{\partial}{\partial x_2} + \cdots + x_m^{p^t}\frac{\partial}{\partial x_m} \pmod{p}$$

is an invariant operator.

Any other set of functions $f_1, f_2, \cdots, f_m$ which possesses the property of cogrediency with the variables will furnish a modular polar operator.

For illustration consider the modular polars of the quadratic form in m variables with arbitrary coefficients

$$q_m = \sum_{i,j=1}^{m} a_{ij}x_ix_j \quad (i \leqq j).$$

Operating with $E_m^{(1)}$, we obtain the polars

$$(4) \qquad \begin{aligned} E_m^{(1)}q_m &= \sum_{i<j=1}^{m} a_{ij}(x_ix_j^p + x_jx_i^p) + 2\sum_{i=1}^{m} a_{ii}x_i^{p+1}, \\ \tfrac{1}{2}E_m^{(1)2}q_m &= \sum_{i,j=1}^{m} a_{ij}x_i^px_j^p. \end{aligned}$$

These are both formal-modular covariants of q_m under (1). A direct extension shows that if $n \not\equiv 0 \pmod{p}$, an m-ary form f_n has n covariant polars $E_m^{(t)r}f_n$ $(r = 1, 2, \cdots, n)$ of degree 1 in the coefficients. If $p > n$, none of these polar covariants will vanish modulo p. Thus if $n = 3$, $p > 3$, $m = 2$, we have

$$E_2^{(t)} = x_1^{p^t}\frac{\partial}{\partial x_1} + x_2^{p^t}\frac{\partial}{\partial x_2}, \quad f = a_0x_1^3 + 3a_1x_1^2x_2 + \cdots$$

and

$$\begin{aligned} \tfrac{1}{3}E_2^{(t)}f &= a_0x_1^{p^t+2} + 2a_1x_1^{p^t+1}x_2 + a_2x_1^{p^t}x_2^2 + a_1x_1^2x_2^{p^t} \\ &\qquad + 2a_2x_1x_2^{p^t+1} + a_3x_3^{p^t+2}, \\ \tfrac{1}{6}E_2^{(t)2}f &= a_0x_1^{2p^t+1} + a_1x_1^{2p^t}x_2 + 2a_1x_1^{p^t+1}x_2^{p^t} + 2a_2x_1^{p^t}x_2^{p^t+1} \\ &\qquad + a_2x_1x_2^{2p^t} + a_3x_2^{2p^t+1}, \\ \tfrac{1}{6}E_2^{(t)3}f &= a_0x_1^{3p^t} + 3a_1x_1^{2p^t}x_2^{p^t} + 3a_2x_1^{p^t}x_2^{2p^t} + a_3x_2^{3p^t}. \end{aligned}$$

The analogy which this polar theory presents when compared with the algebraic polar theory is closer than it might appear to be at first sight. For a little consideration will show that it is immaterial whether we operate directly with

$E_m^{(t)}$, as in the illustrations above, or whether we operate with $E(y) = \left(y\dfrac{\partial}{\partial x}\right)$ the requisite number of times in succession and then set $y_i = x_i^{p^t}$ $(i = 1, \cdots, m)$. The two results will be identical modulo p. The essential difference between the present theory and the algebraic theory is that an algebraic polar becomes the original polarized form when $(y) = (x)$, whereas in the present theory (y) is expressible in terms of (x) in such a way as to give covariants other than the form itself.

§ 2. *Modular Aronhold Operators.*

Let us suppose that f is a form having $\mu + 1$ coefficients. As previously stated, the coefficients a_i' $(i = 0, \cdots, \mu)$ of the transformed of f under (1) are linear forms with integral coefficients in the variables $a_0, a_1, \cdots$, that is

$$(5)\quad a_i' \equiv \xi_i a_0 + \eta_i a_1 + \cdots + \tau_i a_\mu \pmod p \quad (i = 0, 1, \cdots, \mu),$$

where $\xi_i, \eta_i, \cdots$ take all values modulo p which are induced by the linear group (1). Thus (5) form a group induced by (1). Under group (5) the following are cogredient to a_0, $a_1, \cdots, a_\mu$:

$$a_0^{p^t},\ a_1^{p^t},\ \cdots,\ a_\mu^{p^t} \quad (t \text{ any positive integer}).$$

Hence (§ 1) the following operator, applied to any concomitant of f, gives a formal concomitant modulo p:

$$\delta_\mu^{(t)} = \left(a^{p^t}\frac{\partial}{\partial a}\right) \equiv a_0^{p^t}\frac{\partial}{\partial a_0} + a_1^{p^t}\frac{\partial}{\partial a_1} + \cdots + a_\mu^{p^t}\frac{\partial}{\partial a_\mu} \pmod p.$$

Consider a binary quadratic form

$$f = a_0x_1^2 + 2a_1x_1x_2 + a_2x_2^2.$$

Let $p = 3$. The algebraic concomitants are f and its discriminant D, and we have

$$\left.\begin{aligned} \delta_2^{(1)}f &\equiv a_0^3x_1^2 + 2a_1^3x_1x_2 + a_2^3x_2^2 \\ \delta_2^{(1)}D &\equiv a_0^3a_2 + a_1^4 + a_0a_2^3 \end{aligned}\right\} \pmod 3.$$

The latter are formal concomitants, modulo 3, of f. The modular Aronhold operator $\delta_\mu^{(t)}$ applied successively to a concomitant of f

$$\varphi(a_0, a_1, \cdots, a_m, x_1, \cdots)$$

gives a series of formal modular concomitants intermediate, in the sense of Boole, to φ and

$$\varphi_t = \varphi(a_0^{p^t}, a_1^{p^t}, \cdots, a_m^{p^t}, x_1, \cdots). \tag{6}$$

If $b_0, b_1, \cdots, b_\mu$ are the coefficients of a second m-ary quantic g of the same order as f, and φ an invariant function of f and g, then

$$\left(a^{p^t}\frac{\partial}{\partial b}\right), \quad \left(b^{p^t}\frac{\partial}{\partial a}\right),$$

applied to φ, give simultaneous formal modular concomitants.

§ 3. *Modular Transvectants.*

We define the modular transvectant o vo binary forms

$$f(x) = a_0x_1^m + ma_1x_1^{m-1}x_2 + \cdots, \quad \varphi(x) = b_0x_1^n + nb_1x_1^{n-1}x_2 + \cdots$$
$$m, n \not\equiv 0 \pmod{p},$$

as follows: Operate upon $f(x)\varphi(y)$ with

$$\Omega = \frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial x_2 \partial y_1}$$

r times, divide by $m\,!\,n\,!/(m-r)\,!\,(n-r)\,!$ and in the result set $y_i = x_i^{p^t}$ $(i = 1, 2)$. The result, which we abbreviate as $(f, \varphi)_{p^t}^r$, is the rth modular transvectant of f and φ. Thus, if $m = n = 2$,

$$(f, \varphi)'_p \equiv (a_0b_1 - a_1b_0)x_1^{p+1} + (a_0b_2 - a_1b_1)x_1x_2^p + (a_1b_1 - a_2b_0)x_1^px_2$$
$$+ (a_1b_2 - a_2b_1)x_2^{p+1} \pmod{p},$$
$$(f, (f, \varphi)'_p)'_p \equiv (a_0a_2 - a_1^2)[b_0x_1^{p^2+1} + b_1(x_1^{p^2}x_2 + x_1x_2^{p^2}) + b_2x_2^{p^2+1}]$$
$$\equiv \tfrac{1}{2}D\cdot E_2^{(2)}\varphi(x) \pmod{p}.$$

I proceed to the problem of finding a canonical formula for $(f, \varphi)_{p^t}^r$, from which several properties can be derived. A well-known form of Gordan's series* gives the expansion of

$$\frac{(m-r)\,!\,(n-r)\,!}{m!\,n!}\,\Omega^r f(x)\varphi(y)$$

as a power series in the argument (xy). In the Aronhold

* Grace and Young, Algebra of Invariants, p. 55.

symbolical notation ($f(x) = a_x^m$, $\varphi(y) = b_y^n$) this is

$$(ab)^r a_x^{m-r} b_y^{n-r} = \sum_{i=0}^{m-r} \frac{\binom{m-r}{i}\binom{n-r}{i}}{\binom{m+n-2r-i+1}{i}} (f, \varphi)^{i+r}_{y^{n-i-r}} (xy)^i.$$

The algebraical transvectant $(f, \varphi)^r$ is obtained from this by the change $y = x$, the right hand side reducing to the first term since $(xx) = 0$. The modular transvectant is obtained from this same expansion by the substitution $y = x^{p^t}$. It does not reduce to a single term on the right but becomes a polynomial in the universal formal modular covariant

$$L_t = (x^{p^t}x) = x_1^{p^t}x_2 - x_1x_2^{p^t}.$$

THEOREM 2:

$$(f, \varphi)^r_{p^t} \equiv \sum_{i=0}^{m-r} N_i \frac{\binom{m-r}{i}\binom{n-r}{i}}{\binom{m+n-2r-i+1}{i}} E_2^{(t)n-i-r} (f, \varphi)^{i+r} L_t^i,$$

where

$$N_i = (-1)^i (m-i-r)\,!/[m+n-2(i+r)]\,.\,!$$

In this expansion $E_2^{(t)}$ and L_t are modular and $(f, \varphi)^s$ is algebraic.

We may note that a modular transvectant of a form with itself, of odd index $(f, f)^{2k+1}_{p^t}$, does not vanish identically. But it is reducible in all cases and contains the factor L_t. The transvectants $(f, \varphi)^r_{p^t}$, $(\varphi, f)^r_{p^t}$ are entirely distinct, although they may be called conjugates owing to their symmetrical relationship.

A modular transvectant is a linear combination of modular polars of algebraic transvectants with the universal covariant L_t added to the system. It is known* that L_t is rationally expressible in terms of L_1 and $Q = L_2/L_1$. Moreover I have proved in another paper that Q is a covariant of L_1. Hence one method of procedure in constructing systems of concomitants of a quantic f (modulo p) is to construct the algebraical fundamental system of f and polarize it by the operators $E_2^{(t)}$, $\delta_n^{(t)}$. Then join L_1 to the polar system and form a second system consisting of the simultaneous fundamental system of f and L_1. The forms of the second system which are not

* Dickson, *Transactions*, vol. 12 (1911), p. 75.

found in the polar system are to be added to the polar system. That some forms of the second system will be polars is evident from the fact that, if F is any form whose order is not divisible by p,

$$(L_1, F) = x_1^p \frac{\partial F}{\partial x_1} + x_2^p \frac{\partial F}{\partial x_2} = E_2^{(1)} F \pmod{p}.$$

§ 4. *Concomitants of a Linear Form.*

Let $f = a_0 x_1 + a_1 x_2$; $p = 3$. The algebraical system of f is f itself. Polarizing this, we have

$$C = E_2^{(1)} f = a_0 x_1^3 + a_1 x_2^3, \quad C' = E_2^{(2)} f = a_0 x_1^9 + a_1 x_2^9,$$
$$D = \delta_1^{(1)} f = a_0^3 x_1 + a_1^3 x_2.$$

The modular system of L_1 is

$$L_1 = x_1^3 x_2 - x_1 x_2^3, \quad Q = x_1^6 + x_1^4 x_2^2 + x_1^2 x_2^4 + x_2^6.$$

The simultaneous system of f and L_1 is

$$(L_1, f^r)^r \quad (r = 1, \cdots, 4); \quad (Q, f^s)^s \quad (s = 1, \cdots, 6).$$

Of these, some belong to the polar system and some are reducible; as $(Q, f^2)^2 \equiv fC \pmod{3}$, etc. But

$$A = (L_1, f^4)^4 \equiv a_0^3 a_1 - a_0 a_1^3,$$
$$B = (Q, f^6)^6 \equiv a_0^6 + a_0^4 a_1^2 + a_0^2 a_1^4 + a_1^6,$$
$$E = (Q, f^3)^3 \equiv a_1(a_0^2 - a_1^2) x_1^3 - a_0^3 x_1^2 x_2 + a_1^3 x_1 x_2^2 + a_0(a_0^2 - a_1^2) x_2^3.$$

The polars

$$E_2^{(1)} D \equiv f^3, \quad E_2^{(1)} E \equiv D L_1, \quad \delta_1^{(1)} A \equiv 0, \quad \delta_1^{(1)} B \equiv A^2 \pmod{3}$$

are reducible. The polar C' is also reducible. In fact,

$$C' \equiv CQ - f L_1^2 \pmod{3}.$$

The complete set of irreducible concomitants, modulo 3, of the linear form f is

$$A, B, C, D, E, f, L_1, Q.$$

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