Give to z_2 any value a_2 in S_2, and φ will be analytic in z_1 alone. This holds for every choice of the fixed values assigned to $z_2, \cdots, z_n$. In a similar manner we find φ analytic in each remaining variable alone.

Now apply the theorem of Hartogs* which states that if a function of n complex variables is analytic in each one separately, it is analytic in all n variables taken together. Hence φ is analytic throughout $(S_1, \cdots, S_n)$.

HARVARD UNIVERSITY,
May, 1914.

CONCERNING A CERTAIN TOTALLY DISCONTINUOUS FUNCTION.

BY PROFESSOR K. P. WILLIAMS.

(Read before the American Mathematical Society, October 31, 1914.)

ONE of the most important properties of a continuous function is that it actually assumes every value between any two of its values. It is well known that a function can, however, possess this property without being continuous. An actual example to illustrate this seems to have been first given by Darboux in 1875. A function that is sometimes cited in this connection is due to Mansion.† The function that the latter gives actually takes all values between any two, but is discontinuous at the single point $x = 0$. Functions of this sort can be easily constructed by arbitrarily assigning the values at certain points, according to the function concept of Dirichlet. More interest would therefore attach to such a function if it is given by one and the same expression throughout its region of definition. The function given by Mansion does not, however, possess this property; for it contains the function $E(x)$, defined, as in number theory, as the integer equal to, or next smaller than x.

The purpose of this note is to give a function that takes every value between 0 and 1 inclusive, when x varies over the closed interval (0, 1), but which is discontinuous at every point. This function will, furthermore, be represented by one and the same analytical expression throughout its whole region of definition.

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* *Math. Ann.*, vol. 62 (1905), p. 1.

† "Continuité au sens analytique et continuité au sens vulgaire," in *Mathesis*, 1899.

Let $f(x)$ be equal to zero at the rational points of the interval (0, 1), and equal to 1 at the irrational points. We first obtain for this function an expression which is a modification of the one given by Hankel in his celebrated memoir on oscillating and discontinuous functions.*

Let

$$\varphi(x) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin (2n+1)\pi x}{2n+1};$$

then, as is well known,

$$\varphi(x) = 1, \text{ for } 0 < x < 1,$$

$$\varphi(x) = -1, \text{ for } -1 < x < 0,$$

while

$$\varphi(0) = \varphi(\pm 1) = 0.$$

This gives us for $f(x)$ the following expression:

$$f(x) = \prod_{n=1}^{\infty}\{\varphi(\sin n\pi x)\}^2.$$

The expression which Hankel gives for $f(x)$ defines it, in reality, only for the irrational points; so that its values at the rational points must be assigned.†

We next define the function $F_1(x)$ by the relation

$$F_1(x) = x + (1-2x)f(x).$$

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* *Math. Annalen*, vol. 20, pp. 63–112.

† Hankel puts

$$f(x) = \left(\sum_{n=1}^{\infty}\frac{1}{n^{\mu}}\right)\Big/ g(x),$$

where $\mu > 1$, and

$$g(x) = \sum_{n=1}^{\infty} 1/n^{\mu}[\varphi(\sin n\pi x)]^2.$$

He says that at the rational points $g(x)$ becomes infinite; thus making $f(x)$ equal to zero at those points. While it is true that at the rational points the denominators of some of the terms in $g(x)$ become zero, those terms do not behave in any sense as a function does at a pole. The terms abruptly take the form 1/0, and as this symbol is undefined, we cannot regard the series as defining $g(x)$ at the rational points. Other writers have given the function as Hankel gave it.

As Pringsheim has shown, we also have

$$f(x) = 1 - \lim_{m=\infty}\,[\lim_{n=\infty}(\cos m!\pi x)^{2n}].$$

Consequently we have

$$F_1(x) = 1 - x, \text{ for } x \text{ irrational and } 0 \leqq x \leqq 1,$$

$$F_1(x) = x, \text{ for } x \text{ rational and } 0 \leqq x \leqq 1.$$

The function $F_1(x)$ accordingly takes all values between 0 and 1 inclusive when x varies over the closed interval (0, 1). It is, in addition, discontinuous at every point, save the point $x = \frac{1}{2}$. We next modify the function so that $x = \frac{1}{2}$ is also a point of discontinuity.

Let

$$\bar{\varphi}(x) = \varphi(2x);$$

then, from the above values of $\varphi(x)$, and the fact that it is periodic, we obtain

$$\bar{\varphi}(0) = \bar{\varphi}(\tfrac{1}{2}) = \bar{\varphi}(1) = 0;$$

$$\bar{\varphi}(x) = 1, \text{ for } 0 < x < \tfrac{1}{2};\ \bar{\varphi}(x) = -1, \text{ for } \tfrac{1}{2} < x < 1.$$

Consider now the function

$$F_2(x) = \frac{(1-x)4^x}{2}[1 - \bar{\varphi}^2(x)] \cos 2\pi x,$$

where 4^x denotes the arithmetic root.

From the above table of values of $\varphi(x)$ we have at once

$$F_2(0) = \tfrac{1}{2};\ F_2(x) = 0,\ 0 < x < \tfrac{1}{2};\ F_2(\tfrac{1}{2}) = -\tfrac{1}{2},$$
$$F_2(x) = 0,\ \tfrac{1}{2} < x \leqq 1.$$

We construct finally the function

$$F(x) = F_1(x) + F_2(x).$$

It is apparent that $F(x)$ is obtained from $F_1(x)$ by merely interchanging the values at the two points $x = 0$ and $x = \frac{1}{2}$. From the properties of $F_1(x)$ it then follows that $F(x)$ takes all values between 0 and 1 inclusive, and is, furthermore, discontinuous at every point. We see, finally, that $F(x)$ can be represented by a single analytical expression throughout the interval (0, 1); for we have expressions for all the functions contained in it. We consequently have in $F(x)$ a function which possesses all the properties desired.

We shall note a few additional properties of the function we have obtained.

In addition to being single valued, $F(x)$ assumes a given value but once. We can thus regard it as giving a one-to-one transformation of the interval (0, 1) into itself, which is everywhere discontinuous. At every point save $x = \frac{1}{2}$ the function has no limit; that is, every point, except $x = \frac{1}{2}$, is a point of discontinuity of the second kind. It is also apparent that both the greatest and least values approached at a point are continuous functions.

INDIANA UNIVERSITY,
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PROOF OF THE CONVERGENCE OF POISSON'S INTEGRAL FOR NON-ABSOLUTELY INTEGRABLE FUNCTIONS.

BY DR. W. W. KÜSTERMANN.

IN the following pages I propose to give a proof of the

THEOREM: *If $f(x)$ is a real, periodic function, of period 2π, which in the interval $(0, 2\pi)$ has a proper or improper integral in the sense of Lebesgue, Harnack-Riemann, or Harnack-Lebesgue-Hobson,* then*

$$\lim_{r \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\alpha) \frac{1 - r^2}{1 + r^2 - 2r \cos (\alpha - x)} d\alpha = \lim_{t \to 0} \tfrac{1}{2}[f(x + t) + f(x - t)]$$

at every point x where the limit on the right hand side exists.

This theorem includes in particular the case where $f(x)$ remains finite—disposed of by Schwarz,† and the case where $f(x)$ becomes infinite at an infinite number of points, but has an *absolutely* convergent improper integral—discussed by Hobson and others.‡ Moreover, it goes farther, in that it

* For these definitions see Hobson, Theory of Functions of a Real Variable, Cambridge, 1907.

† Schwarz, Math. Abhd., vol. 2, pp. 144 and 175.

‡ Most completely by Hobson, Theory of Functions of a Real Variable, p. 719; cf. also Bôcher, *Ann. of Math.*, 2d ser., vol. 7, p. 81; Fatou, *Acta Math.*, vol. 30, p. 335; Picard, Traité d'Analyse, 2d ed., vol. 1, p. 268; Forsyth, Theory of Functions, 2d ed., p. 450.