SHORTER NOTICES.

Grundzüge der Differential- und Integralrechnung. Von Dr. GERHARD KOWALEWSKI, a. o. Professor der Mathematik an der Universität Bonn. Leipzig und Berlin, Teubner, 1909. 452 pp.

IT is the purpose of this book to present a rigorous treatment of the foundations of the calculus. The author therefore begins with a consideration of numbers. Assuming a knowledge of the rational numbers, he defines the irrational number by means of the Dedekind cut, and in the first three chapters, develops the properties of the real number system. In Chapter II (page 14) he defines a limit of a sequence in this way: "Steht eine Folge $u_1, u_2, u_3, \cdots$ zu einer Zahl u in der Beziehung, dass in *jeder* Umgebung von u fast alle Glieder der Folge liegen, so nennt man die Folge konvergent und u ihren Grenzwert." In this definition, he uses the words "fast alle" in the sense of "with a finite number of exceptions." In his notation $\lim u_n = u$ means that there exist u's of the set u_n in every neighborhood of u, with a finite number of exceptions. The equivalence of this definition with the usual form of ϵ statement is established later. Chapter IV deals with functions and variables. y is a function of x when to each x there corresponds a single y. The principal subjects contained in the chapter are the concepts of continuity, the definition and properties of the functions a^x, $\log x$, and finally the derivation of the $\lim_{n=\infty} (1 + 1/n)^n$. The development of this limit is effected with unusual freedom from artifice by means of the inequality

$$(1 + h)^n > 1 + nh \quad (1 + h > 0).$$

The sine and cosine are defined by means of the infinite series, and in preparation for this introduction Chapter VII gives a rather full treatment of the elementary properties of infinite series, concluding with the proof of the theorem on the differentiation of a power series.

Chapters IX and XI are devoted to the theory of maxima and minima of functions of one and two variables, the criteria for a maximum or a minimum for a function of a single variable being developed on the assumption that the derivative exists at the critical point. Chapter X is on the differentiation of functions of several variables and contains the usual definitions of partial derivatives, total differentials, etc.

For the purpose of introducing the inverse functions the author now proves the theorem that if $f(x)$ takes the same value only once on an interval it is monotonic and so possesses an inverse. Then with the additional hypothesis that $f'(x)$ exists and is not zero on the interval in consideration, he shows that the inverse has a derivative. He is now enabled to complete the differentiation of the elementary functions of trigonometry. In connection with the computation of π he gives an interesting pair of French verses by means of which the value of π may be written down to 30 decimal places. They are as follows:

Que j'aime à faire apprendre un nombre utile aux sages!
Immortel Archimède, artiste ingénieur,
Oui de ton jugement peut priser la valeur!
Pour moi ton problème eut de pareils avantages.

If in this we replace each word by the number of letters it contains we get as a result the value of π correct to 30 decimals, $\pi=3.141592653589793238462643383279\ldots$. The remainder of the chapter is devoted to a proof of the theorems on the inverses of systems of functions. Chapter XIII begins the study of integration and comprises the greater and more interesting portion of the book. After proving that every function $f(x)$ continuous on the interval $<a, b>$ (this notation means the end points are included) possesses an integral on this interval, he defines integrability, in general. And for the purpose he uses the following notation: Designate by $\mathfrak{Z}$ any decomposition of the interval $<a, b>$ into any finite number p of sub-intervals. Then any sequence of decompositions $\mathfrak{Z}_1', \mathfrak{Z}_2', \cdots, \mathfrak{Z}_n'$ such that $\lim \delta_n = 0$ (where δ_n is the maximal length of any subinterval of $\mathfrak{Z}_n'$) is called a characteristic sequence. Now form the sum $\mathfrak{S}_a^b(z) \equiv (x_1 - a)f(\xi_1) + (x_2 - x_1)f(\xi_2) + \cdots + (b - x_{p-1})f(\xi_p)$ (where ξ_i is any point in the interval (x_i, x_{i-1})). $\mathfrak{S}_a^b(z)$ is called an $\mathfrak{S}$-expression and any sequence of $\mathfrak{S}$-expressions is called a characteristic one if the corresponding $\mathfrak{S}$ sequence is also characteristic. Any function $f(x)$ is integrable in $<a, b>$ if every characteristic $\mathfrak{S}$-sequence is convergent. It follows if $f(x)$ is integrable in $< a, b >$ every characteristic $\mathfrak{S}$-sequence has the same limit which he calls $\int_a^b f(x)dx$, and that $f(x)$ is limited in $< a, b >$. Finally $\int_a^b f(x)dx$ exists if, and only if, $f(x)$ is limited in $< a, b >$ and

$$\int_a^{\underline{b}} f(x)dx = \int_{\overline{a}}^{b} f(x)dx.$$

Chapter XV deals with infinite series. In the consideration of uniform convergence attention is called to the definition given by Goursat in his Cours d'Analyse (Tome I, page 406): The series of continuous functions

$$u_0(x) + u_1(x) + \cdots + u_n(x) + \cdots$$

convergent in the interval $< a, b >$ is said to be uniformly convergent in this interval if to every positive ϵ there corresponds a positive integer n, independent of x, such that the remainder

$$R_n(x) = u_n(x) + u_{n+1}(x) + \cdots$$

remains in absolute value less than ϵ for all values of x in the interval. This definition is also to be found in Picard (Traité d'Analyse, second edition, page 211) and is that given by Darboux (*Annales Scientifiques de l'Ecole Normale Supérieure*, 1875, 11, series 2, 4, page 77). It requires that $|R_n(x)| < \epsilon$ not for every $n > m$ but only for one value of n. Kowalewski notes that if this definition is employed the theorem on the termwise integrability of a uniformly convergent series is not true. Osgood discussed the subject completely in his review of Goursat's Cours d'Analyse (BULLETIN, 1903, page 552) and calls attention to the fact noticed above by Kowalewski as having been found by Tannery in his Theory of Functions, 1886.

In Chapter XVII there is an extended treatment of the lengths of curves, together with examples of the determination of plane areas.

The treatment of double integrals, differentiation under the sign of integration, curve line integrals, Green's theorem, and the transformations of double integrals are studied quite fully in Chapter XVIII, which concludes the book. There is an appendix dealing with the elementary properties of determinants and concluding with functional determinants.

The typographical errors are neither numerous nor important. The following are perhaps the most noticeable:

Page 57, line 24, should read $\lim f(x_n, y_n) = f(x_0, y_0)$ instead of $\lim f(x_n, y_n) = \lim f(x_0, y_0)$.

Page 85. $\sqrt[n]{u_n}$ instead of $\sqrt[u]{u_n}$.

Page 210, line 9. $\left| \int_a^b f(x)dx - s_a^b(z) \right| \leqq \epsilon$ instead of

$$\left| \int_a^b f(\alpha)dx - s_a^b(z) \right| \leqq \epsilon.$$

Page 213, last line. $\int_a^b f(x)dx$ instead of $\int_a f(x)dx$.

Page 352, line 22. $S' \leqq S$ instead of $S' < S$.

Page 352, line 28. Lim $S(\bar{\mathfrak{Z}})$ instead of lim $S(\mathfrak{Z})$.

The book is admirable for its clearness, conciseness and rigorous style throughout. It has not been the author's design to develop the theorems with a minimum of hypothesis but rather to present them in those forms most usually occurring. There are no problems, but much of the text has been illustrated with well-selected examples.

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Les Fonctions Polyédriques et Modulaires. Par G. VIVANTI, Professeur à la Faculté des Sciences de Pavie. Ouvrage traduit par ARMAND CAHEN, Professeur au Lycée d'Evreux. Paris, Gauthier-Villars, 1910. vi+320 pp.

THE original Italian edition of this useful book appeared in 1906 and was reviewed in this BULLETIN, volume 14 (1908), page 144, by Professor J. I. Hutchinson. No important changes appear in the French translation. Although the work has the modest object of providing an easy introduction to the classic lectures of Klein on the icosahedron and to the treatise on elliptic modular functions by Klein and Fricke, this French translation is evidence of its great usefulness. Unfortunately the number of typographical errors is somewhat large, as was noted by Professor O. Perron in his review published in the *Archiv der Mathematik und Physik,* volume 18 (1911), page 259.

Since the scope and the contents of this work were so well presented in the review by Professor Hutchinson, to which