

These values we get from our graphical integration of the functions u and v , and they can be measured and carried over to the $\int u dr - \int v dr$ plane; or by having our $\int u dr - \int v dr$ plane on transparent paper, we can mark off the coordinates of the points R_0, R_1, \dots, R_n without the work of measuring these values. This second method also eliminates a small probable error in measurement.

Through the points R_0, R_1, \dots, R_n we draw a smooth curve, and this is our required curve $\theta = \theta_n$ in the $\int_0^{z_n} f(z) dz$ plane. In the same way we get the curves $\theta = \theta_0, \dots, \theta_{n-1}$. Through the points r_n on each curve $\theta = \theta_n$ ($n = 0, 1, \dots, n$) we draw a smooth curve, and have a net of small squares covering the Z -plane which is the graphical representation of the function $Z = \int_0^{z_n} f(z) dz$.

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THE UNIFICATION OF VECTORIAL NOTATIONS*

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THE unification of vectorial notations has taken several steps during the past year, but whether the steps be backward or forward, sideways or up in the air, would be difficult to say.

1. One step was forced. A report from the international committee on vector notations, appointed at Rome in 1908 with instructions to lay its recommendations before the congress at Cambridge in 1912, fell due. A member of that committee, though not in attendance at the congress, I am unable to state whether or not any report was made; but I believe that an extension of time until 1916 was asked and granted. So far as I am aware the committee apparently did not organize prior to the meeting in Cambridge last summer, and except for desultory publication on vectors by a few members of the committee, there had been no inside activity which could lead to a report. It does not appear therefore that much of a step in any direction during the past year or

* This essay may be considered as a continuation of one by the same title in this BULLETIN, May, 1910, p. 415.

the last four years can be attributed to the committee. It may be that something more definite will develop in 1916.

2. In the past four years there has been very great activity in the use of vector methods in Italy. Following the lead of Burali-Forti and Marcolongo, and using their notation, a large number of mathematicians have there printed a great variety of articles on vector analysis and particularly on its applications to geometry and physics. The publishing house of Mattei and Co. in Pavia sent out a circular last summer announcing the proximate appearance of a work on *Analyse vectorielle générale* consisting of three volumes: 1°. Transformations linéaires (published, 1912); 2°. Applications physico-mécaniques; 3°. Hydrodynamique.

The recent vectorial activity in Italy, and the appearance of such a general work as this, will do much toward fixing the notations of Burali-Forti and Marcolongo in Italian literature (although there are in Italy eminent students of Grassmann's methods, such as A. del Re). It may be recalled that their notations are: italic type for scalars, italic heavy type as \mathbf{a} and \mathbf{i} for vectors, the large cross as $\mathbf{a} \times \mathbf{i}$ for the scalar product of two vectors, the large caret as $\mathbf{a} \wedge \mathbf{i}$ for the vector product, and a great variety of non-algebraic symbolism connected with the linear vector function.

As these notations are different from those in vogue in Germany (system adopted in the German edition of the mathematical encyclopedia), different from those in use here in America (Gibbs's notations must have been pretty well popularized by Coffin's *Vector Analysis* if not by my edition of Gibbs's lectures), and different from the notations now introduced in the French edition of the mathematical encyclopedia (see below), there is no apparent gain in uniformity of notations attributable to these Italian activities,—although the attendant adoption and use of vectorial methods and ideas has made for a distinct advance of internationalization and uniformity in points of view. Something valuable has thus been accomplished.

3. In the *Bulletin of the International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics*, June, 1912, Dr. Macfarlane, president of the association, who believes that no satisfactory system of notations can be devised adventitiously, but only, if at all, by a study of fundamental principles, has published a thirty-five

page essay: "A system of notation for vector analysis; with a discussion of the underlying principles." As this essay accomplishes what its title professes, namely, a thorough analysis of those principles which its author, a most distinguished expert in the field, regards as fundamental, and the elaboration of a system of notation upon this analysis as foundation, the work deserves wide circulation among students of vector methods, and conscientious study by them. We hope that the place of publication may not prove a burial ground for the essay.

In the same place is found a tabulation by Shaw, secretary of the association, of the different notations used by different authors. This should be helpful for comparative study; it would have been even more useful had the table been enlarged so as to give in detail as to font, and so on, the fundamental notations of the different systems, even if some abridgment of those parts of the table which deal with derived and less important notations had been necessary. These two contributions by the president and secretary of the organization which, more than any other except the international committee, should be concerned with this problem of unification must occupy an important place in the literature of the problem.

4. To one, however, who believes that the principles which different authors regard as fundamental in vector analysis are even more varied and divergent than the notations actually employed, and who consequently believes that unification of notations can come about, if at all, only by the weight of usage, the most important contribution of the past year to the subject of vector notations appears to be Langevin's article in the French edition of the mathematical encyclopedia.* This is given such high rank in importance because

* Tome IV, vol. 5, fasc. 1, art. IV 16. *Notions géométriques fondamentales.* Exposé, d'après l'article allemand de M. Abraham (Milan), par P. Langevin (Paris). We may cite also another article appearing last year in the encyclopedia: Tome IV, vol. 2, fasc. 1, art. IV 4. *Fondements géométriques de la statique.* Exposé, d'après l'article allemand de H. E. Timerding (Brunswick), par Lucien Lévy (Paris). In this second contribution a certain amount of vector analysis is used and the notation, though differing in a detail or two, is in the main the same as Langevin's; it is unfortunate that there are any differences at all, and it almost seems as though French scientists, and the world at large, had a right to demand that uniformity should be enforced at least throughout Tome IV of the *Encyclopédie des Sciences Mathématiques*. If unification is not attainable on such a small scale, why talk of it at all?

vector analysis, hitherto almost unknown in French literature, has now entered authoritatively into France, because the adoption of any notation in the encyclopedia must have been accompanied by the maturest deliberation (especially in view of the international discussion now going on and of Langevin's membership on the international committee on vector notations), and because, if the notations introduced by him in this article shall be used throughout the appropriate parts of volumes four and five of the encyclopedia, there is likely to be a general decision in their favor throughout France.

In the first place Langevin distinguishes systematically between pure and pseudo-scalars, between polar and axial vectors. Pure scalars and polar vectors are those which do not depend in sign upon any assumption as to positive or negative directions of rotation; whereas pseudo-scalars and axial vectors change their sign when the conventions as to positive and negative direction for rotation are interchanged. The notations

$$a, \quad a, \quad \vec{a}, \quad \overset{\curvearrowright}{a}$$

represent respectively pure scalar, pseudo-scalar, polar vector, axial vector. That is, a point placed under a letter signifies that the symbol stands for a pseudo-scalar; a straight arrow above a letter, that it stands for a polar vector; a curved arrow, an axial vector.* The font of type has no significance as to the nature of the symbol.

That such a notation, dependent upon the affixing of diacritical marks, has its advantages must be clear. It does not immobilize alphabets or fonts; it may be carried out in black-board work and in manuscript. Indeed if Clarendons are used for vectors, as is now the nearly universal usage in works on physics, some such notation as a superposed arrow or an underscoring with a dash is necessary for work at the board,—unless one is content to let the physical significance of the symbol, which must of course be always borne in mind, suffice for characterization without any special symbolism. But whether compositors take kindly to these interlinear symbols, which upset the spacing between the lines, is doubtful; and

* It is particularly noteworthy that for the three unit vectors i, j, k no superposed arrow, straight or curved, is used; notationally they therefore rank as scalars.

perhaps the irregular spacing of the lines disfigures the page full as much as the introduction of special fonts which may be justified in.

For a unit vector a zero superscript is suggested; thus \vec{a}^0 is a unit vector in the direction of \vec{a} . The origin of this notation is implied in the remark "comme a^0 est en algèbre toujours égal à l'unité, nous proposons la notation $\vec{a}^0 \dots$." Such a reason as this for a notation is not scientific, of course, but merely alliterative; for the basis of the algebraic fact $a^0 = 1$ lies in the equation $a^0 a = a$, which depends on laws not satisfied by the usual products of vectors. It would have been equally proper to remark that since 1 is the arithmetical symbol for unity the notation \vec{a}^1 would be adopted for a unit vector along \vec{a} . In either case there arises a play upon the word unity, which is naturally somewhat of a convenience as a mnemonic device to correlate a notation and its significance, but which opens the way to confusion if it is considered as anything more profound than a bit of mnemonics. The notation \vec{a}^0 has really to be judged solely on the ground of convenience,* and as such it does not seem bad at all.

For the scalar product of two vectors Langevin uses $\vec{a} \vec{b}$ or $\vec{a} \cdot \vec{b}$, either juxtaposition or the interposed dot.† As near as I can estimate, the usage is about equally divided in the

* That Langevin considers his notation as based on anything more fundamental than convenience is extremely doubtful; but it is not so with all authors. For instance, there is the commendation which Burali-Forti and Marcolongo bestow on Grassmann for having once adopted (though he subsequently abandoned) the sign \times for the scalar product: "Son choix est très opportune, parce que le signe \times a toute les propriétés que possède le même signe en algèbre dans les produits de deux facteurs." See their *Éléments de Calcul vectorielle* (traduit par Lattès), p. 222. Now if any one will take the trouble to consult a list of postulates for ordinary algebraic multiplication, for instance Huntington's set (*Annals of Mathematics*, ser. 2, vol. 2, p. 8), he will find as the first property of the product $a \times b$ that the product is of the same class as the factors (law of closure), and for another property that if $a \times b = 0$, then either $a = 0$ or $b = 0$ (laws of cancellation). Perhaps these are properties not of the sign \times , but of the product; if so, what are properties of the sign \times as distinguished from properties of the product, and why?

† Lévy, loc. cit., uses the vertical bar between the letters, $\vec{a} | \vec{b}$, as the sign of the scalar product.

text; if the dot had consistently been used, the notation would have been Gibbs's; if it had consistently been omitted, it would have been Heaviside's. Parentheses and brackets have not been immobilized as in the German notation. This seems an advantage.* From some points of view, too, it is an advantage to have the alternative of use or non-use of the dot. Gibbs's adoption of the dot as the sign of the scalar product has always troubled some persons because they wished to be free to use the dot as a separatrix whenever convenient. This difficulty could readily be avoided by the adoption of a small circle, a sort of hollow dot, as the sign of the scalar product, $\vec{a} \cdot \vec{b}$. Prandtl put forth this suggestion in 1904,† and it seems queer that it has not had some vogue.‡

For the sign of the vector product Langevin uses the cross, $\vec{a} \times \vec{b}$, the large ungainly European cross with its arms at sixty degrees instead of ninety. When one follows Gibbs and introduces a cross, it looks best to use a rather small one, $\vec{a} \times \vec{b}$; the formulas become more compact and intelligible. It is, however, a matter for congratulation that Langevin has adopted for the scalar and vector products notations which are entirely familiar in the literature instead of devising some totally different affair like our Italian colleagues.

The symbolic operator ∇ is not introduced, but the terms grad, || div, and rot. There are, further, the monosyllabic symbols Pot, New, Lap, and Max, borrowed from Gibbs, and Lam, invented to represent the inverse of grad. So far as I am aware these integrating operators, proposed by Gibbs (with the exception of Lam), have never found much recog-

* Especially when as in Lorentz's Theory of Electrons (1909) the dot is also always used.

† *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 13, p. 442.

‡ Certainly Gibbs should not be accused of making a fetish of the dot; for when he was discussing with me the question of notation to be adopted in the publication of his *Vector Analysis* in 1901, he mentioned the disadvantages of immobilizing the separatrix dot and we considered a number of similar symbols as alternatives; if either of us had happened to mention the small circle, it might easily have been adopted instead of the Clarendon dot. Of course the dot may with perfect propriety be considered as a separatrix in Gibbs's system, because the fundamental product is the dyad; and yet it may be undesirable to immobilize the dot in a restricted sense, just as it is undesirable thus to immobilize parentheses.

§ Coffin employed this in his *Vector Analysis*, and anybody with an eye for typographical beauty must be inclined to follow him.

|| For Langevin grad $f = -\nabla f$, not ∇f . Each author has to choose between the two conventions as to sign, for usage is about equally divided.

tion, and it will be interesting to see if their use by Langevin serves to disseminate them to any appreciable extent. The abandonment of ∇ seems to me unfortunate,* but certain it is that grad, div, and rot are now in almost complete possession of the field and constitute the nearest approach to unification that we have.

Langevin does scarcely more than mention the linear vector function, and he has no particular notation for it; indeed he writes

$$\vec{\rho}' = \varphi \vec{\rho}, \quad \varphi = i\vec{\alpha} + j\vec{\beta} + k\vec{\gamma},$$

where according to his earlier agreement as to the meaning of juxtaposition φ would reduce merely to a scalar. Convenient as the linear vector function is in many problems, physicists seem little inclined to adopt it, whether in the Hamiltonian, Grassmannian, Gibbsian, or Cayleyan form, and Langevin probably does well not to enter upon it to any extent.

This somewhat full account of the vector notation introduced in the French edition of the encyclopedia seems justified in view of the facts, 1°, that it immobilizes no type, except the cross which is now not much used, and, 2°, that it is adapted to blackboard work without modification. If only a dot or very small circle were used consistently for the sign of the scalar product, the system would be so like Gibbs's that the complete adoption of Gibbs's methods of treating the linear vector function could be adjoined, if and when desired, without any change.

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* My reasons are fully explained in the reference cited at the beginning of this article.