a symmetric or under an alternating group. It may also be observed that if G is an imprimitive group of transformations of Sylow subgroups of order p^m, then G cannot involve an invariant subgroup whose order is of the form p^α, since its degree is of the form $1 + kp$. If the systems of imprimitivity of G are transformed according to a group involving smaller Sylow subgroups of the form p^α than those contained in G, it results from the theorem proved above that G contains other systems of imprimitivity which are transformed according to Sylow subgroups whose orders of the form p^α are equal to the orders of the corresponding Sylow subgroups of G. Hence the theorem:

If G is an imprimitive group of transformations of Sylow subgroups of order p^m and involves Sylow subgroups of order p^α, then G must have systems of imprimitivity which are transformed according to a group involving Sylow subgroups of order p^α.

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THEOREMS ON FUNCTIONAL EQUATIONS.

BY MR. A. R. SCHWEITZER.

(Read before the American Mathematical Society, April 27, 1912.)

1. In the Bulletin, volume 18 (1912), page 300, we have referred to Abel, *Crelle's Journal*, volume 2 (1827), page 389, in relation to the equation

$$\psi(x) - \psi(y) = \Omega^{-1}\{\phi(x, y)\}. \tag{1}$$

This reference suggests

$$\psi(x) - \psi(y) = \Omega^{-1}\{x\phi(y) - y\phi(x)\} \tag{2}$$

as a correlative of the functional equation* discussed by Abel, l. c., namely,

$$\psi(x) + \psi(y) = \Omega^{-1}\{x\phi(y) + y\phi(x)\}. \tag{2'}$$

Further special cases of the equation (1) are obtained by considering the generalizations of equation (2′) by Lottner,

* Cf. Cayley, Mathematical Papers, vol. IV, pp. 5–6.

Crelle's Journal, volume 46 (1854), pages 367, 368, etc. For example, we obtain

$$\psi(x) - \psi(y) = \Omega^{-1}\left\{\frac{x\phi(y) - y\phi(x)}{\theta(xy)}\right\} \tag{3}$$

and

$$\psi(x) - \psi(y) = \Omega^{-1}\left\{\frac{\Phi(x)\phi(y) - \Phi(y)\phi(x)}{\theta(xy)}\right\}. \tag{4}$$

In equations (2), (3), (4) let $\Omega^{-1}(x) = \psi(x)$; then the resulting equations are particular instances of the equation

$$\chi(x) - \chi(y) = \chi\{f(x, y)\},$$

which we have derived in the BULLETIN, l. c. Following the suggestions of Abel, l. c., page 386, and Lottner, l. c., page 368, we remark that equation (2) for $\Omega^{-1}(x) = \psi(x)$ contains a trigonometric subtraction formula, and equation (3) for $\Omega^{-1}(x) = \psi(x)$ contains an elliptic subtraction formula.

Abel, l. c., pages 388–389, has pointed out that the relation (2′) leads to an equation in $\phi(x)$ which in general is incapable of solution. The relation (2), on the contrary, gives

$$\begin{aligned} \phi(x) &= \sqrt{c^2 - (m - \alpha'^2)x^2} + \alpha' x, \\ \psi(x) &= a\alpha \int \frac{dx}{\sqrt{c^2 - (m - \alpha'^2)x^2}}, \\ \Omega^{-1}(x) &= \psi\left(\frac{x}{\alpha}\right) + \psi(0), \end{aligned} \tag{5}$$

where $\phi'(0) = \alpha'$, $\phi(0) = \alpha$, $\psi'(0) = a$, and c and m are arbitrary constants.

In connection with the discussion of Lottner, l. c., and the preceding equation (4) the functional equation of Lelieuvre* is possibly of interest.

2. Abel, in *Crelle's Journal*, volume 1 (1826), pages 11–15, has shown that if

$$f(x, y) = \phi^{-1}\{\phi(x) + \phi(y)\},$$

then

$$f\{x, f(y, z)\} = f\{y, f(x, z)\},$$

* *Bulletin des Sciences Mathématiques*, ser. (2), vol. 27 (1903), p. 31.

and in the BULLETIN, l. c., page 302, we have shown that if

$$f(x, y) = \phi^{-1}\{\phi(x) - \phi(y)\},$$

then

$$f\{y, f(x, z)\} = f\{z, f(x, y)\}.$$

The following theorems have a bearing on these two remarks.

THEOREM 1. *If* $\phi\{x, \phi(y, z)\} = \phi\{y, \phi(x, z)\}$ *and* $\phi\{f(y, z), z\} = y$ *and* $f\{\phi(y, z), y\} = z$, *then there exists a function* $\chi_\phi(x)$ *such that* $\phi(x, y) = \chi_\phi^{-1}\{\chi_\phi(x) + \chi_\phi(y)\}$ *and* $f(x, y) = \chi_\phi^{-1}\{\chi_\phi(x) - \chi_\phi(y)\}$.

In proof of this theorem we show first that

$$\phi\{x, \phi(y, z)\} = \phi\{z, \phi(x, y)\}.$$

By hypothesis,

$$\phi\{x, \phi(y, z)\} = \phi\{y, \phi(x, z)\}. \tag{6}$$

Hence writing $f(y, z)$ for y, we get

$$\phi\{x, \phi[f(y, z), z]\} = \phi\{f(y, z), \phi(x, z)\},$$

or since

$$\phi\{f(y, z), z\} = y$$

we obtain

$$\phi(x, y) = \phi\{f(y, z), \phi(x, z)\},$$

or by interchanging y and z,

$$\phi(x, z) = \phi\{f(z, y), \phi(x, y)\}.$$

In this relation we write $\phi(y, z)$ for z; then

$$\phi\{x, \phi(y, z)\} = \phi\{f[\phi(y, z), y], \phi(x, y)\},$$

that is, since

$$f\{\phi(y, z), y\} = z,$$

$$\phi\{x, \phi(y, z)\} = \phi\{z, \phi(x, y)\}. \tag{7}$$

Combining (6) and (7), we have by Abel's theorem, *Crelle's Journal*, volume 1, page 13, that there exists a function $\chi_\phi(x)$ such that

$$\chi_\phi\{\phi(x, y)\} = \chi_\phi(x) + \chi_\phi(y). \tag{8}$$

Therefore,

$$\chi_\phi\{\phi[f(x, y), y]\} = \chi_\phi\{f(x, y)\} + \chi_\phi(y)$$

or

$$\chi_\phi(x) = \chi_\phi\{f(x, y)\} + \chi_\phi(y),$$

that is,

(9) $$\chi_\phi\{f(x, y)\} = \chi_\phi(x) - \chi_\phi(y).$$

The relations (8) and (9) establish the preceding theorem.

THEOREM 2. *If* $f\{y, f(x, z)\} = f\{z, f(x, y)\}$ *and* $\phi\{f(y, z), z\} = y$ *and* $f\{\phi(y, z), y\} = z$, *then there exists a function* $\chi_f(x)$ *such that* $f(x, y) = \chi_f^{-1}\{\chi_f(x) - \chi_f(y)\}$ *and* $\phi(x, y) = \chi_f^{-1}\{\chi_f(x) + \chi_f(y)\}$.

Since

$$f\{\phi(y, z), y\} = z$$

we have

$$f\{\phi[f(y, z), z], f(y, z)\} = z.$$

But

$$\phi[f(y, z), z] = y.$$

Hence,

(10) $$f\{y, f(y, z)\} = z.$$

Now we consider

(11) $$f\{y, f(x, z)\} = f\{z, f(x, y)\}.$$

From (11) follows

$$f\{f(x, y), f(x, z)\} = f\{z, f[x, f(x, y)]\}$$

or by (10),

(12) $$f\{f(x, y), f(x, z)\} = f(z, y).$$

Hence by a theorem which we have proved, BULLETIN, l. c., there exists a function $\chi_f(x)$ such that

(13) $$\chi_f\{f(x, y)\} = \chi_f(x) - \chi_f(y).$$

Therefore,

$$\chi_f\{f[\phi(x, y), x]\} = \chi_f\{\phi(x, y)\} - \chi_f(x)$$

or

$$\chi_f(y) = \chi_f\{\phi(x, y)\} - \chi_f(x),$$

that is,

(14) $$\chi_f\{\phi(x, y)\} = \chi_f(x) + \chi_f(y).$$

Thus from (13) and (14) follows the proof of our second theorem.

3. Theorem 1 of § 2 concerned the *consistence* of the relations

(6) $$\phi\{x, \phi(y, z)\} = \phi\{y, \phi(x, z)\},$$

(6′) $$\phi\{f(y, z), z\} = y,$$

(6″) $$f\{\phi(y, z), y\} = z.$$

These relations, on the other hand, may be shown to be *completely independent*.* To do this, it suffices to exhibit eight systems, each of which satisfies or contradicts the preceding relations. Denoting by the symbol $(+ + -)$ that (6) is satisfied, (6′) is satisfied, (6″) is contradicted, and so on, we have as the required systems:

$(+ + +)$	$\phi(x, y) = x + y$	$f(x, y) = x - y$
$(+ + -)$	$= -x + y$	$= -x + y$
$(- + +)$	$= -x - y$	$= -x - y$
$(+ - +)$	$= -x + y$	$= x + y$
$(+ - -)$	$= x + y$	$= x + y$
$(- + -)$	$= x - y$	$= x + y$
$(- - +)$	$= x - y$	$= -x + y$
$(- - -)$	$= x - y$	$= x - y$

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DOUBLE CURVES OF SURFACES PROJECTED FROM SPACE OF FOUR DIMENSIONS.

BY DR. S. LEFSCHETZ.

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1. IN the study of a surface F in S_4, one of the problems is the determination of the genus π' of the double curve of a projection of F. Severi† has shown that a general F_m in S_4 has four fundamental projective characters, viz., the order m, the class n, the rank a of a hyperplane section, and the number t of trisecants through an arbitrary point O. He also gives an expression for the rank of the double curve in question, from which, knowing the number of pinch points J of F'_m in projection of F_m, we can obtain π'. It is not uninteresting

* Cf. E. H. Moore, The New Haven Mathematical Colloquium (1910), p. 82, §47.

† "Intorno ai punti doppi impropri . . . ," *Palermo Rendiconti*, vol. 15 (1901), p. 32.