2. Given three unrelated involutions of rays in the plane, with centers at A, B, and C, then for any point P in the plane the rays at A, B, and C that correspond in the involutions at those points to the rays PA, PB, and PC will not generally meet in a point. If, however, they do meet in a point P', then the points P and P' are said to be conjugate with respect to all three involutions. The locus of points that have conjugate points with respect to three involutions is found to be the general plane cubic. Professor Lehmer studies the curve from this point of view and connects the theory with the theory of the curve as developed by Schroeter in his Ebene Kurven dritter Ordnung (Leipzig, 1888).

C. A. NOBLE,
Secretary of the Section.

A NEW PROOF OF THE THEOREM OF WEIERSTRASS CONCERNING THE FACTORIZATION OF A POWER SERIES.

BY DR. W. D. MACMILLAN.

IN the BULLETIN for April, 1910, Bliss gives a simplified proof of the following theorem due to Weierstrass:

Let $f(y;\ x_1, \cdots, x_p)$ be a convergent power series in y and $x_1, \cdots, x_p$, such that $f(y;\ 0, \cdots, 0)$ begins with a term of degree n. Then $f(y;\ x_1, \cdots, x_p)$ is factorable in the form

$$f(y;\ x_1, \cdots, x_p) = [y^n + a_1 y^{n-1} + \cdots + a_n] \cdot g(y;\ x_1 \cdots, x_p),$$

where $a_1, \cdots, a_n$ are convergent power series in $x_1, \cdots, x_p$ vanishing for $x_1 = x_2 = \cdots = x_p = 0$, and g is a convergent power series in $y;\ x_1, \cdots, x_p$ which has a constant term different from zero.

Since $g(y;\ x_1, \cdots, x_p)$ has a constant term, we may denote its reciprocal by $\phi(y;\ x_1, \cdots, x_p)$ and state the theorem in the following form:

Let $f(y;\ x_1, \cdots, x_p)$ be a convergent power series in $y;\ x_1, \cdots, x_p$ such that $f(y;\ 0, \cdots, 0)$ begins with a term of degree n. Then a convergent power series $\phi(y;\ x_1, \cdots, x_p)$, having a constant term different from zero can be found such that the product

$f(y;\, x_1, \cdots, x_p)\cdot\phi(y;\, x_1, \cdots, x_p) = p^{(n)} = y^n + a_1 y^{n-1} + \cdots + a_n$ *is a polynomial in* y *of degree* n *and* $a_1, \cdots, a_n$ *are convergent power series in* $x_1, \cdots, x_p$, *vanishing for* $x_1 = x_2 = \cdots = x_p = 0$.

For the purpose of determining the coefficients of the series ϕ it will simplify the notation if we put $x_i = \xi_i \mu$. Then the series f, ϕ and $p^{(n)}$ can be written

$$
\begin{aligned}
f &= -y^n[1-b_0] + b_1\mu + b_2\mu^2 + \cdots, \\
\phi &= c_0 + c_1\mu + c_2\mu^2 + \cdots, \qquad (1)\\
p^{(n)} &= -y^n + p_1\mu + p_2\mu^2 + \cdots,
\end{aligned}
$$

where the b_k are known power series in y and are homogeneous in the ξ_i of degree k, b_0 containing terms in y only and vanishing for y equal to zero. The c_k are power series in y whose coefficients are to be determined, and the p_k are polynomials in y, also to be determined, of degree $n-1$.

On taking the product of f and ϕ we find

$$
\begin{aligned}
f\cdot\phi = {} & -c_0(1-b_0)y^n + [-(1-b_0)c_1y^n + b_1c_0]\mu + [-(1-b_0)c_2y^n \\
& + b_1c_1 + b_2c_0]\mu^2 + \cdots + \left[-(1-b_0)c_ky^n + \sum_{j=1}^{k} b_jc_{k-j}\right]\mu^k \qquad (2)\\
& + \cdots\cdots\cdots\cdots \\
= {} & -y^n + p_1\mu + p_2\mu^2 + \cdots + p_k\mu^k + \cdots.
\end{aligned}
$$

Comparing the coefficients of the various powers of μ we have

$$
\begin{aligned}
(1-b_0)c_0y^n &= y^n, \\
(1-b_0)c_1y^n &= b_1c_0 - p_1, \\
(1-b_0)c_2y^n &= b_1c_1 + b_2c_0 - p_2, \qquad (3)\\
&\cdots\cdots\cdots \\
(1-b_0)c_ky^n &= b_1c_{k-1} + b_2c_{k-2} + \cdots + b_kc_0 - p_k, \\
&\cdots\cdots\cdots
\end{aligned}
$$

These equations can be solved successively. From the first we have at once $c_0 = 1/(1-b_0)$. From the second it is seen that, since b_1 and c_0 are known power series in y, p_1 can be chosen uniquely so that $b_1c_0 - p_1$ shall be divisible by y^n. If we suppose p_1 so chosen, we can then write $b_1c_0 - p_1 = \beta_1 y^n$ and the solution for c_1 is $c_1 = \beta_1/(1-b_0)$. It is clear that c_1 and p_1

are homogeneous of degree 1 in the ξ_i. The third equation can be solved in the same manner, the polynomial p_2 being chosen so that the right member is divisible by y^n. Since this is possible in one and only one way, the polynomial p_2 and the power series c_2 are uniquely determined. If we suppose $c_1, \cdots, c_{k-1}$ to have been computed, then p_k can be determined uniquely so that

$$\sum_{j=1}^{k} b_j c_{k-j} - p_k$$

shall be divisible by y^n. After removing the factor y^n, c_k is determined by dividing the result by $1 - b_0$. One can thus compute as many coefficients as is desired, all the coefficients of a given order in μ being obtained in a single operation.

The proof of convergence is by dominant functions. Let us suppose that the series f converges for $|y| = 1/\sigma$, $|x_i| = 1/\rho_i$, and let us set

$$X = \frac{1}{(1-\rho_1 x_1)(1-\rho_2 x_2)\cdots(1-\rho_p x_p)} - 1.$$

Then a positive quantity M can be chosen so large that the series

$$F = -y^n + M\frac{\sigma^{n+1}y^{n+1}}{1-\sigma y} + M\frac{X}{1-\sigma y}$$

shall dominate the series f. By the above process a power series $\Phi(y; x_1, \cdots, x_p)$ and a polynomial

$$P^{(n)} = -y^n + A_1 y^{n-1} + \cdots + A_n$$

can be found such that the coefficients of Φ are all positive and greater than the moduli of the corresponding coefficients of the series ϕ; and the coefficients of the A_k, which are power series in $x_1, \cdots, x_p$ vanishing with these quantities, are all positive and greater than the moduli of the corresponding coefficients of the series a_k. Hence if the series Φ and the A_k converge so also do the series ϕ and the a_k.

Let us take

$$\Phi = \frac{1-\sigma y}{\omega - sy},$$

where $s = \sigma + M\sigma^{n+1}$, and ω is as yet undetermined. Then the equation $F \cdot \Phi = P^{(n)}$ becomes

$$\left[- y^n + M \frac{\sigma^{n+1} y^{n+1}}{1 - \sigma y} + M \frac{X}{1 - \sigma y} \right] \frac{1 - \sigma y}{\omega - s y}$$
$$= - y^n + A_1 y^{n-1} + \cdots + A_n,$$

which reduces readily to

$$(4) \qquad \frac{(1 - \omega)}{s} \left[\frac{y^n - \dfrac{MX}{1 - \omega}}{y - \dfrac{\omega}{s}} \right] = A_1 y^{n-1} + \cdots + A_n.$$

The left member of equation (4) will be a polynomial in y of degree $n - 1$ provided ω satisfies the relation

$$(5) \qquad \frac{MX}{1 - \omega} = \left(\frac{\omega}{s} \right)^n.$$

On putting $\omega = 1 - z$ we get

$$(6) \qquad z = \frac{s^n MX}{(1 - z)^n}.$$

Thus z is expansible as a power series in X the coefficients of which are all positive, and furthermore z vanishes with X. Hence

$$(7) \qquad \Phi = \frac{1 - \sigma y}{\omega - s y} = \frac{\sigma}{s} \left[1 + \frac{M \sigma^n + z}{1 - z - s y} \right]$$

is expansible as a power series in $y, x_1, \cdots, x_p$, the coefficients of which are all positive. The polynomial $P^{(n)}$ becomes

$$P^{(n)} = - y^n + \frac{1 - \omega}{s} \left[\frac{y^n - \left(\dfrac{\omega}{s} \right)^n}{y - \left(\dfrac{\omega}{s} \right)} \right] = - y^n + \frac{z}{s} y^{n-1}$$
$$+ \frac{z(1 - z)}{s^2} y^{n-2} + \cdots + \frac{z(1 - z)^{n-1}}{s^n}.$$

From equation (6) it is seen that the coefficients of this polynomial,

$$\frac{z(1-z)^k}{s^{k+1}} = \frac{s^{n-k-1}MX}{(1-z)^{n-k}} \qquad (k = 1, \cdots, n-1),$$

are expansible as power series in $x_1, \cdots, x_p$, with positive coefficients, vanishing for $x_1, = \cdots = x_p = 0$. Hence the theorem is proved.

UNIVERSITY OF CHICAGO,
August 14, 1910.

KOWALEWSKI'S DETERMINANTS.

Einführung in die Determinantentheorie einschliesslich der unendlichen und der Fredholmschen Determinanten. By GERHARD KOWALEWSKI. Leipzig, Veit & Comp., 1909. 540 pp.

A PRELIMINARY survey of the contents of this book may best be obtained by dividing it into three nearly equal parts. The first of these parts comprises Chapters I–X and deals with the pure theory together with the single application to systems of linear equations * — a subject both historically and logically so intimately connected with the theory as to be almost inseparable from it. In the second part — Chapters XI–XVI — certain applications to algebraic, analytic, and geometric problems are treated. The third part consists of Chapters XVII–XIX and deals with two extensions of the idea of determinants obtained by allowing the order of the determinant to increase indefinitely. Both the theory of these determinants (infinite determinants and Fredholm determinants) and the closely related theory of the corresponding generalizations of systems of linear equations (linear equations with an infinite number of variables and linear integral equations) are treated in these chapters. While the introduction of such subjects into a book on determinants is not wholly unprecedented — Mathews' revision of Scott's book containing a brief discussion of infinite determinants — it must still, in view of its extent, be regarded as an innovation. This feature of the book is to be welcomed and will doubtless be imitated by other writers of text-books on this subject.

* A few equations of higher degree (secular equation, etc.) which are intimately connected with the theory of determinants are also considered in this first part.