higher complex quantities of quaternion type. It possesses another real form which is like the system of ordinary quaternions and these two real forms are the only ones in which systems of higher complex quantities of quaternion type can appear.

The work is manifestly a labor of love. An interesting circumstance in connection is the fact that this first volume was brought out on the hundredth anniversary of the birthday of the author's father.

L. WAYLAND DOWLING.

LINEAR DIFFERENTIAL EQUATIONS.

Vorlesungen über lineare Differentialgleichungen. Von LUDWIG SCHLESINGER. Leipzig and Berlin, Teubner, 1908.

IT is over ten years ago that the author of the present "Vorlesungen" completed the publication of his well-known Handbuch der Theorie der linearen Differentialgleichungen. As every one familiar with the older book well knows, it was intended to be, as its name implied, a handbook containing a complete treatment of all that was at that time known about the subject. It seemed natural therefore, to expect under the title of "Vorlesungen" a briefer version of the same subject, adapted to the needs of the younger student and rendered more palatable for him by a proper selection of topics and by a more elementary treatment. And in a certain sense the "Vorlesungen" may indeed be considered as an introduction into the theory of linear differential equations, in so far at least as all of the most important results of the theory built up by Fuchs and his successors are discussed. But the method of treatment is so novel and the artistic unity of the book is preserved to such an extraordinary extent that we must look upon it as an important addition to analysis rather than as a treatise of more or less pedagogical merit.

It is well known that Riemann's discussion of the hypergeometric function furnished Fuchs with the fundamental ideas which led to the modern theory of linear differential equations, which theory may be said to date from Fuchs's paper of 1865. But we now know that Riemann himself had intended to construct a general theory of linear differential equations upon the same general principles which had led to such brilliant results in the theory of abelian functions. In the two fragments, published in 1876, after his death, he formulates a general problem which may briefly be stated as follows: Given m points $a_1, \cdots, a_m$ in the plane of the complex variable with each of which is associated one of the linear n-ary substitutions with constant coefficients $A_1, \cdots, A_m$; to determine a system of n functions of the complex variable x which shall have the given points $a_1, \cdots, a_m$, and no others, as branch points in such a way that when x describes a closed path in the positive direction around a_i, the functions y_κ shall undergo the linear substitution A_i. Riemann speaks of two different systems of functions which are nowhere "infinite of infinite order" which have the same branch points, the same fundamental substitutions and the same poles, as belonging to the same *class*.

It is easy enough to see that the solutions of a linear differential equation of the nth order are functions of this general character. But if the points a_i and the substitutions A_i and the poles are chosen arbitrarily, it is a problem of great difficulty to demonstrate the existence of a system of functions of the class defined by them. This is what is known as Riemann's problem and in its solution Schlesinger's Vorlesungen culminate. The method adopted consists essentially in studying the relations between the parameters which occur in the coefficients of the differential equation, the quantities which determine the linear substitutions A_i and the branch points a_i. These (transcendental) relations are shown to be of such a character that, the latter quantities being arbitrarily assigned, the former may be chosen in such a way that the corresponding differential equations have as their solutions a system of functions with the required properties.

As we have already indicated, the solution of Riemann's problem is the culminating feature of the book and appears only at the end of a long series of investigations in which all of the most essential properties of linear differential equations are studied. But all of these are presented in a novel way. Instead of considering the problems of integration in connection with a single linear differential equation of the nth order, Schlesinger follows Koenigsberger's example by considering a system of n differential equations of the first order. Such a system is characterized by its n^2 coefficients as well as by a system of n^2 solutions. It is owing to this fact that the entire theory appears as an application of the calculus of matrices of

n^2 functions which was originally developed by Volterra, and which certainly manifests itself here as a most important analytical instrument.

Let us consider the system of differential equations

$$(B) \qquad \frac{dy_\kappa}{dx} = \sum_{\lambda=1}^{n} y_\lambda a_{\lambda\kappa}(x) \qquad (\kappa = 1, 2, \cdots, n),$$

where the coefficients $a_{\lambda\kappa}$ are supposed to be real, finite and continuous functions of x in the interval $(p \cdots q)$. Let the values

$$(I) \qquad y_1 = y_1^{(0)}, \quad \cdots, \quad y_n = y_n^{(0)}$$

be arbitrarily prescribed for $x = x_0 = p$. In order to demonstrate the existence of a system of real continuous solutions of (B) which satisfy these initial conditions, the author makes use of the method of interpolation, which proceeds as follows, in close analogy with Riemann's definition of a definite integral: Let r be any value of x for which

$$p < r \leqq q.$$

Divide the interval $(p \cdots r)$ into m parts by interpolating $m - 1$ points $x_1, \cdots, x_{m-1}$, and let us put $x_0 = p$, $x_m = r$. In each of the m subintervals $(x_{\nu-1} \cdots x_\nu)$ obtained in this way choose an arbitrary value $\xi_{\nu-1}$,

$$x_{\nu-1} \leqq \xi_{\nu-1} < x_\nu,$$

and consider in place of the differential equations (B), the difference equations

$$\begin{aligned} y_\kappa^{(1)} - y_\kappa^{(0)} &= (x_1 - x_0) \sum_{\lambda=1}^{n} y_\lambda^{(0)} a_{\lambda\kappa}(\xi_0), \\ (C) \qquad y_\kappa^{(2)} - y_\kappa^{(1)} &= (x_2 - x_1) \sum_{\lambda=1}^{n} y_\lambda^{(1)} a_{\lambda\kappa}(\xi_1), \quad (\kappa = 1, 2, \cdots, n), \\ &\cdots\cdots\cdots \\ y_\kappa^{(m)} - y_\kappa^{(m-1)} &= (x_m - x_{m-1}) \sum_{\lambda=1}^{n} y_\lambda^{(m-1)} a_{\lambda\kappa}(\xi_{m-1}). \end{aligned}$$

The author shows in the first lecture that, under the assumptions made in regard to the continuity of the functions $a_{\lambda\kappa}$, the functions $y_\kappa^{(m)}$ defined by these difference equations approach definite limits y_κ as m becomes infinite, while each of the subintervals approaches the limit zero, these limits being independent of the particular method of subdivision employed, as well as of the choice made of the points $\xi_{\nu-1}$ in the various subintervals. Moreover the n functions obtained in this way are shown to be continuous functions of x which satisfy the system of differential equations (B) and the initial conditions (I).

In the second lecture we are told to consider not a single system of solutions (B), but n such systems corresponding to the n systems of initial conditions

$$\begin{array}{l} y_1 = y_{11}^{(0)},\ y_{21}^{(0)},\ \cdots,\ y_{n1}^{(0)}, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ y_n = y_{1n}^{(0)},\ y_{2n}^{(0)},\ \cdots,\ y_{nn}^{(0)}, \end{array} \qquad \text{for } x = x_0 = p,$$

where the determinant

$$|y_{i\kappa}^{(0)}|$$

is supposed to be different from zero. The n systems of solutions $(y_{11}, \cdots, y_{1n}), \cdots, (y_{n1}, \cdots, y_{nn})$ will be obtained by the limit process just indicated from the difference equations

$$(C') \qquad y_{i\kappa}^{(\nu)} - y_{i\kappa}^{(\nu-1)} = (x_\nu - x_{\nu-1}) \sum_{\lambda=1}^{n} y_{i\lambda}^{(\nu-1)} a_{\lambda\kappa}(\xi_{\nu-1}),$$
$$(i,\ \kappa = 1,\ 2,\ \cdots,\ n;\ \nu = 1,\ 2,\ \cdots,\ m)$$

which are merely the equations (C) written down for each separate set of initial conditions.

The equations (C') may be written in the form

$$(1) \qquad y_{i\kappa}^{(\nu)} = \sum_{\lambda=1}^{n} y_{i\kappa}^{(\nu-1)} \left[a_{\lambda\kappa}(\xi_{\nu-1})(x_\nu - x_{\nu-1}) + \delta_{\lambda\kappa}\right],$$

where $\delta_{ii} = 1, (i = 1, 2, \ldots n)$, and $\delta_{i\kappa} = 0$ for $i \neq \kappa$. But these may be looked upon as the equations for the multiplication of two square matrices, since the single relation between matrices

$$(a_{i\kappa})(b_{i\kappa}) = (c_{i\kappa})$$

is equivalent to the n^2 relations

$$c_{i\kappa} = \sum_{\lambda=1}^{n} a_{i\lambda}\, b_{\lambda\kappa}, \qquad (i, \kappa = 1, 2, \ldots n)$$

and since equations (1) are obviously of this form.

Let us follow the author in using the simple parenthesis as a symbol for a matrix as well as in its elementary significance, although that practice occasionally gives rise to a formidable collection of parentheses in a single equation. We suggest incidentally that this might easily be avoided, in the interests of clearness, by using some other symbol for a matrix, say a square bracket. Equations (1) may then be written as follows:

$$(2)\quad (y_{i\kappa}^{(\nu)}) = (y^{(\nu-1)}_{i\kappa})(a_{i\kappa}(\xi_{\nu-1})\cdot(x_\nu - x_{\nu-1}) + \delta_{i\kappa}) \quad (\nu = 1, \ldots, m),$$

which gives rise to the symbolic formula

$$(3)\qquad (y_{i\kappa}^{(m)}) = (y_{i\kappa}^{(0)})\prod_{\nu=1}^{m}(a_{i\kappa}(\xi_{\nu-1})\cdot(x_\nu - x_{\nu-1}) + \delta_{i\kappa}),$$

where the product upon the right member is, of course, a product of matrices taken in the proper order.

Let the matrix $(y_{i\kappa}^{(0)})$ of initial values be the unit matrix $(\delta_{i\kappa})$. The author now introduces the following symbol:

$$(\alpha)\qquad \widehat{\int_p^r} [a_{i\kappa}(x)dx + \delta_{i\kappa}]$$

to denote the matrix of the limits

$$\lim_{m \doteq \infty} \prod_{\nu=1}^{m} [a_{i\kappa}(\xi_{\nu-1})\cdot(x_\nu - x_{\nu-1}) + \delta_{i\kappa}] \quad (i, \kappa = 1, 2, \cdots, n).$$

The symbol (α) is to be read "integral matrix from p to r," and is used with great success throughout the book. The author tells us that he has chosen this symbol so as to emphasize its analogy with the ordinary integral, and at the same time to remind us of the initial letter of the word product, just as the integral sign reminds us of the first letter of the word sum.

Let us put $r = x$ and think of x as variable in the interval from p to q. The integral matrix

$$[\eta_{i\kappa}(x)] = \widehat{\int_p^r} [(a_{i\kappa}(x)dx + \delta_{i\kappa}]$$

will represent a matrix of n^2 functions such that

$$\eta_{i1}(x), \cdots, \eta_{in}(x) \qquad (i = 1, 2, \cdots, n)$$

constitute n systems of simultaneous solutions of system (B) which satisfy the initial conditions

$$\eta_{i\kappa}(p) = \delta_{i\kappa},$$

and the most general matrix of simultaneous solutions is obtained from $(\eta_{i\kappa})$ in the form

$$(y_{i\kappa}) = (c_{i\kappa})(\eta_{i\kappa}),$$

where the elements of the matrix $(c_{i\kappa})$ are arbitrary constants.

The equations (B) may be written

$$\left(\frac{dy_{i\kappa}}{dx}\right) = (y_{i\kappa})(a_{i\kappa}),$$

whence

$$(a_{i\kappa}) = (y_{i\kappa})^{-1}\left(\frac{dy_{i\kappa}}{dx}\right),$$

the determinant of the matrix $(y_{i\kappa})$ being different from zero in the whole interval $(p \cdots q)$. Thus the coefficients of (B) may be expressed in terms of n systems of simultaneous solutions. Schlesinger denotes this operation by the symbol $D_x(y_{i\kappa})$, so that

$$D_x(y_{i\kappa}) = (y_{i\kappa})^{-1}\left(\frac{dy_{i\kappa}}{dy}\right),$$

and speaks of this as the derivative matrix of $(y_{i\kappa})$ with respect to x. If the solutions of (B) are not subjected to the initial conditions

$$\eta_{i\kappa}(p) = \delta_{i\kappa}$$

nor any other specific conditions, he writes

$$(y_{i\kappa}) = \frown\!\!\!\!\!\int(a_{i\kappa}dx + \delta_{i\kappa}),$$

a notation which corresponds to the indefinite integral of ordinary analysis.

At the end of the second lecture we find the laws of combination of these new symbols of derivation and integration, which correspond closely to the familiar ones of the infinitesimal calculus, the principal difference being that symbolic multiplication of matrices takes the place of addition.

The third lecture introduces the integrating factors of Lagrange and Jacobi, and the idea of adjoint system, which is used in the familiar way for the purpose of showing how to integrate non-homogeneous systems by quadratures if the corresponding homogeneous systems have already been solved. This theory again finds an important application in enabling us to deduce, by successive approximation, a series for the solutions of the system (B), which is convergent in the whole domain of continuity of the coefficients.

The transition from functions of a real to functions of a complex variable is made by following the same general ideas that dominate the corresponding step in the theory of ordinary integrals. By separating all of the variables involved into their real and imaginary parts, i. e., by putting

$$y_\kappa = u_\kappa + \sqrt{-1}v_\kappa, \quad a_{i\kappa} = \alpha_{i\kappa} + \sqrt{-1}\beta_{i\kappa}, \quad x = \xi + \sqrt{-1}\eta,$$

a system of total linear differential equations is obtained from (B). Its integral matrix is shown to be independent of the path, if this path be restricted to a simply connected portion of the plane at every point of which the coefficients $a_{i\kappa}$ are analytic. It then follows easily that all of the values of an integral matrix for a given value of x, taken over a path not so restricted, may be obtained from one of them by multiplication with constant matrices. This leads to the notion of fundamental equation, the representation of the elements of a canonical fundamental system in the well-known form and all of the rest of the classical theory.

Schlesinger's solution of Riemann's problem has had to undergo some severe criticisms by Plemelj.* The discussion initiated by this attack is certainly worthy of careful attention on the part of all who might be tempted to apply these methods to similar problems. The reviewer has not been able to find the time to form a final opinion upon the main points involved in this controversy. It certainly does not invalidate the value of the work as a whole.

The following unimportant misprints may be noted: page 10, line 23, read $\mathfrak{x}_{\lambda-2}$ in the exponent in place of $\mathfrak{x}_{\lambda-1}$; page 40, line 13, read $u_{i\kappa}^{(\nu)}$ instead of $(u_{i\kappa}^{(\nu)})$; page 76, equation (12), read $a_{i\kappa}$ instead of $\alpha_{i\kappa}$.

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*See *Jahresbericht der Deutschen Mathematiker-Vereinigung*, January 31, 1909, and June 11, 1909.