THIRD REPORT ON RECENT PROGRESS IN THE THEORY OF GROUPS OF FINITE ORDER.

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§ 3. Group of Isomorphisms.

In view of the fundamental importance of the group of isomorphisms of any group it is desirable to have theorems by means of which the group of isomorphisms I of a given substitution group G can be readily determined. If any G of degree n contains n distinct subgroups of degree $n - 1$ which are composed of all the substitutions omitting fixed letters, then its substitutions transform these n subgroups just as they transform their own letters and hence G contains no substitution besides identity which is commutative with each one of these n subgroups of degree $n - 1$. When G is transitive it will have n such subgroups, provided it has one subgroup of degree $n - 1$. Any operator which transforms each of these n subgroups into itself must therefore be commutative with every operator of G. If G does not involve a subgroup of degree n which may correspond to such a subgroup of degree $n - 1$ in a holomorphism of G, then its I can be represented as a substitution group of degree n which involves G as an invariant subgroup.

If G is a transitive group and contains a subgroup of degree $n - 1$, its I may be represented as a transitive substitution group whose elements are the subgroups which may correspond to one of the largest subgroups of degree $n - 1$ in a holomorphism of G. If the degree of this transitive group exceeds n, it must be imprimitive and the m conjugate largest subgroups of degree $n - 1$ constitute one system of imprimitivity, while its other systems correspond to subgroups of degree and of index n under G. While these general theorems are frequently directly useful to determine the I of a given G, a number of recent more special theorems find wide application. Among these are the following:

If an abelian group G which involves operators whose orders exceed 2 is extended by means of an operator of order 2 which transforms each operator of G into its inverse, then the I of this extended group is the holomorph of G.* The I of the group obtained by extending the cyclic group of order $2m$ ($m > 2$) by means of an operator of order 4 which transforms each of its operators into its inverse is the holomorph of this cyclic group. The square of a complete group has the double holomorph of this group for its I. The necessary and sufficient condition that a holomorphism corresponds to an invariant operator under I is that the operators which correspond to themselves form an invariant subgroup and that the remaining operators correspond to themselves multiplied by invariant operators.†

If an operator of order 2 in I transforms an operator s of G into s_1s, then it transforms s_1 into its inverse. Hence every operator of order 2 in I transforms some operators of G into their inverses. Moreover, if such an operator transforms every operator of G except identity into a different operator, it must transform every operator of G into its inverse and hence G is an abelian group of odd order. Burnside considered the properties of G when I contains operators of order 3 which transform every operator of G except identity into a different operator. It may be remarked that the review of this note in the *Jahrbuch über die Fortschritte der Mathematik* ‡ is misleading, since it does not state that the operator of order 3 under consideration transforms all of the operators of G except identity into different operators.

By means of the preceding theorems it is easy to find the groups of isomorphisms of substitution groups of low degrees. This has been done for all the groups which can be represented on 7 or a smaller number of letters, as well as for the simple groups whose degrees do not exceed 14. Among the latter the group of order 7920 is especially interesting since it is both complete and simple. It is not difficult to see that the direct product of a complete group which contains only one subgroup of index 2 and the group of order 2 is simply isomorphic with its I. In particular, the direct product of the symmetric group whose degree is not 2 or 6 and the group of order 2 is simply isomorphic with its I.

The groups of isomorphisms of a number of special types of

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* *Amer. Jour. of Mathematics*, vol. 29 (1907), p. 4.

† *Transactions Amer. Math. Society*, vol. 4 (1903), p. 153.

‡ Vol. 34 (1905), p. 160.

groups of order p^m have recently been determined. Young * determined these groups for such groups of order p^m as contain cyclic subgroups of order p^{m-1}. In chapter 2 of Le Vavasseur's work cited above, he considers groups of isomorphisms of the groups of order p^4, and in chapter 3 he considers the groups of cogredient isomorphisms of a large number of the groups of order 2^5. In the current volume of the *Transactions* of this Society, Ranum generalizes the linear congruence groups of Jordan "by using different moduli for the different elements of the matrices, so that each element is a residue of its own particular modulus." He applies these generalized linear congruence groups to the group of isomorphisms of any abelian group and obtains a number of important new theorems. One of these affirms that the necessary and sufficient condition that the group of isomorphisms of an abelian group of order p^m be solvable is that all its invariants are distinct when $p > 3$, and that no three of the invariants are equal to each other when $p = 2$ or 3.†

The group of isomorphisms of every finite group is finite since its order cannot exceed $(g - 1)!$, g being the order of the group. There are only four groups for which the order of I has this maximal value; viz., the groups of orders, 1, 2, 3 and the non-cyclic group of order 4. As a rule the order of I is very much smaller, since the possible holomorphisms are greatly restricted by the properties of the operators. It has been observed that in a non-abelian group not more than three-fourths of the operators may correspond to their inverses and the groups which have this property have been considered. From this fact it follows that an abelian group may be defined by the property that more than three-fourths of its operators may correspond to their inverses in a holomorphism of the group.‡ Manning has considered the groups in which five-eighths or more of the operators may correspond to their inverses and in this connection proved a fundamental theorem in regard to the properties of groups involving two invariant subgroups which have only identity in common.§

The group of isomorphisms of the cyclic group of order $2 \cdot 3^m$ is the cyclic group of order $2 \cdot 3^{m-1}$. Hence it follows

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* Young, *Amer. Jour. of Mathematics*, vol. 25 (1903), p. 206.

† Ranum, *Transactions Amer. Math. Society*, vol. 8 (1907), p. 89.

‡ *Annals of Mathematics*, vol. 7 (1906), p. 59.

§ Manning, *Transactions Amer. Math. Society*, vol. 7 (1906), p. 223.

that each of the m successive groups of isomorphisms of this cyclic group as well as of the cyclic group of order 3^m is cyclic while the $(m+1)$th is identity. The only other groups which have the property that all their successive groups of isomorphisms are cyclic are the group of order 4 and those whose order is a prime number of the form $2 \cdot 3^m + 1$. From the fact that the order of the group of isomorphisms of every group whose order exceeds 2 is greater than unity it follows that we arrive at an infinite system of groups by forming the successive holomorphs of any group whose order exceeds 2. The characteristic operators of a group are the invariant operators of its holomorph. Since *an abelian group cannot have more than one characteristic operator besides identity*, it results that each of the successive holomorphs of an abelian group has either one or no invariant operator besides identity, as the abelian group has one or no characteristic operator in addition to identity. The question whether a non-abelian group can have more than one characteristic operator besides identity remains unsettled. This is also true of the question whether a non-abelian group can have an abelian group of isomorphisms. If such a group exists there must be a metabelian group of order p^α which has the same property.

The characteristic properties of a complete group are that it does not admit outer isomorphisms and none of its operators except identity is invariant under the group. A large number of well-known groups have the latter property without having also the former. It seems desirable to find groups which have the former property without having also the latter. In fact, the second part of the definition can only be justified by a proof of the existence of such a group. This proof is included in the proof that there exists a group of composite order which is both simple and complete since the direct product of such a group and the group of order 2 admits no outer isomorphisms but includes an invariant operator of order 2.* In view of the historic interest in the five-fold transitive groups of degrees 12 and 24 respectively it is of special interest to note that the latter is a complete group while the former has a group of twice its own order for its group of isomorphisms.

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* *Messenger of Mathematics*, vol. 37 (1907), p. 54.

§ 4. Substitution Groups.

In 1896 "Le grand prix des sciences mathématiques" was awarded to Maillet for his memoir entitled "Recherches sur la classe et l'ordre des groupes de substitutions." The memoir was not published until about six years afterwards * and hence its publication comes within the period covered by the present report. The memoir covers 120 pages and is divided into two parts. The first is devoted to the class of the primitive substitution groups which are simply isomorphic with either a symmetric or an alternating group, and it is followed by two notes in which hypersystems and primitive groups whose class is not less than four-fifth of the degree are considered. The second part, which the author considers the more important, is devoted to the limit of the order of the groups of degree n which do not involve the alternating group of this degree.

The term hypersystems is introduced with a view to exhibiting more clearly the connection between substitution groups and Lie's transformation groups, as well as to generalize some results previously published. Most of the results are extensions of theorems due to Jordan and Bochert. Some of these relate to the class of a primitive group which does not include the alternating group of the same degree. Manning considers the same question in several recent memoirs. † In most of the older work along this line the degree of such a group was assumed to be given and the smallest possible class of the primitive group was considered, while Manning assumes the minimum class and finds the maximum degree for this class.

The important theorem that every substitution group of prime degree p which contains more than one subgroup of order p is at least doubly transitive was first proved by means of group characteristics. Burnside has recently given a much simpler proof based upon a purely arithmetical property of the prime roots of unity. ‡ The enumeration of all the possible substitution groups of a given degree has been materially advanced during the period under consideration. Miss Martin's enumeration of the imprimitive groups of degree 15 was completed by the list published by Kuhn. This list is preceded

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* Maillet, *Mémoires présentés par divers Savants a l'Académie des Sciences de l'Institut national de France*, vol. 32 (1902).

† Manning, *Transactions Amer. Math. Society*, vol. 4 (1903), p. 351; vol. 6 (1905), p. 42; BULLETIN, vol. 13 (1906), p. 20.

‡ Burnside, *Quar. Jour. of Mathematics*, vol. 37 (1906), p. 215.

by a number of new theorems relating to the construction of such groups and groups which involve all the substitutions which are commutative with each of the substitutions of a transitive group.*

The investigations of Sylow in regard to groups of a prime degree which were noted in the preceding report have been extended by Frobenius, who proved that there are only four groups of degree p which contain exactly $p + 1$ subgroups of order p. This theorem was also proved by de Séguier without the use of group characters in a memoir which, like the one by Frobenius, contains a large number of other new theorems on substitution groups.† From the results of the last two paragraphs it follows that each of the unknown transitive groups of degree p may be assumed both to be multiply transitive and also to involve more than $p + 1$ subgroups of order p. Hence these groups contain transitive subgroups of lower degree and therefore belong to a category which has recently been studied by Manning.‡ The special case where $p = 2q + 1$, q being a prime, was considered by de Séguier in the memoirs just noted and he arrived at the result that there is no transitive group ot degree 23 in addition to the 7 which are well known. This result has been verified by the writer. It may be observed that the smallest possible number of transitive groups of degree $p > 5$ is 6. Jordan proved that this is also the actual number when p is either 47 or 59. Rietz proved that for every value of $p > 11$ of the given form there exists a simple group of composite order which can be represented as a simply transitive primitive group of degree $1 + kp$.§

A fundamental theorem relating to any transitive substitution group G of degree n has been stated as follows: If p^β is the highest power of p which divides n, each Sylow subgroup of order p^α in G has a transitive constitutent of degree p^β and all its other transitive constituents are of degree of $p^{\beta+\gamma}$ ($\gamma \geqq 0$). If $n = 2q^{\alpha_1}$, q being any odd prime, each of the Sylow subgroups whose order is a power of q has just two transitive constituents of degree q^{α_1}. If n is a power of a prime, a Sylow

* Kuhn, *Amer. Jour. of Mathematics*, vol. 26 (1904), p. 45.

† Frobenius, *Berliner Sitzungsberichte*, 1902, p. 351; de Séguier, *Comptes rendus*. vol. 137 (1901), p. 37, and *Liouville*, vol. 8 (1902), p. 253.

‡ *Transactions Amer. Math. Society*, vol. 7 (1906), p. 497.

§ Rietz, BULLETIN, vol. 11 (1905), p. 544.

subgroup of G whose order is a power of the same prime is transitive.*

The important applications of group characters to such problems as the determination of the least number of variables by means of which a given abstract group may be represented as a linear group, and of all the linear homogeneous substitution groups which are either simply or multiply isomorphic with a given group, have continued to attract many investigators to this field with a view either to extensions of the theory or to making it more accessible. Dickson, Burnside, and Schur have published valuable expository articles on the theory of group characters in which the subject is approached from different standpoints.† While Burnside makes considerable use of Hermitian forms in his fundamental theorems, Schur replaces these by simpler considerations so that he presupposes only an elementary knowledge of matrices from the theory of linear substitutions. Both of these authors employ methods which differ widely from those used by Frobenius, and Schur develops his theory under the heading "New foundation for the theory of group characters."

Frobenius proved various theorems relating to the group characters of the multiply transitive groups which do not involve the alternating group of the same degree and gave a number of historical data relating to these groups. Among these theorems are the following: A character of the symmetric group whose dimension does not exceed $\frac{1}{2}r$ is also a character of every r-fold transitive group of the same degree. The necessary and sufficient condition that a group is either two-fold or four-fold transitive is that it has the character $\alpha - 1$, or the characters $\alpha - 1$, $\frac{1}{2}\alpha(\alpha - 3) + \beta$, $\frac{1}{2}(\alpha - 1)(\alpha - 2) - \beta$ respectively, where $\alpha, \beta, \gamma, \cdots$ are the number of cycles of degrees $1, 2, 3, \cdots$ in a substitution.‡ By means of a fundamental theorem relating to the roots of unity Burnside establishes several theorems which are useful in calculating group characters, and by way of illustration he determines the characters of the simple group of order 504.§ Alasia || has also given a brief exposition of the characters of several groups.

* BULLETIN, vol. 9 (1903), p. 543.

† Dickson *Ann. of Math.*, vol. 4 (1902), p. 25; Burnside, *Proc. London Math. Society*, vol. 1 (1903), p. 117; Schur, *Berliner Sitzungsberichte*, 1905, p. 406.

‡ Frobenius, *Berliner Sitzungsberichte*, 1904, p. 558.

§ Burnside, *Proc. London Math. Society*, vol. 1 (1903), p. 112.

|| Alasia, *Rivista di Fisica, Matematica e Scienze naturali*, anno. 6 (1905), p. 1905.

Loewy published several important memoirs which are principally devoted to the study of the reducibility of the linear homogeneous substitution groups. Most of his theorems apply to groups of infinite order but he has also considered groups of finite order.* One of his theorems was extended by Dickson, who established its validity for much more extensive domains.† All the finite collineation groups in three variables have recently been determined by Blichfeldt, who employs more direct methods than those used by Jordan and Valentiner in their much earlier investigation of this problem. Blichfeldt gives also an enumeration of the principal imprimitive collineation groups in four variables, together with their generating substitutions, and a complete list of the primitive ones based upon various theorems which were developed by him principally in the *Transactions* of this Society.‡

Schur has given a complete solution of the problem of finding all the possible representations of a finite group as a linear fractional substitution group, and, together with Frobenius, he established the following fundamental theorem : A finite group of linear substitutions is equivalent to a real group when its substitutions transform into itself a quadratic form of a non-vanishing determinant, and only then ; two isomorphic groups of linear substitutions have the same irreducible components if any two corresponding substitutions have the same trace (Spur) and only then.§ The fundamental theorem proved by Jordan about thirty years ago, which establishes the fact that every finite linear homogeneous group on n variables contains an invariant abelian subgroup such that the order λ of the corresponding quotient group is less than a certain number depending only upon n, has been extended both by Blichfeldt and by Schur. The former has found further restrictions for λ in the article published in the *Transactions* to which we have just referred, as well as in an earlier article in the same journal, while the latter has considered the possible orders of the linear groups when the trace of each substitution is restricted to certain fields.

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* Loewy, *Transactions Amer. Math. Society*, vol. 4 (1903), pp. 44 and 171 ; ibid., vol. 6 (1906), p. 504 ; Verhandlungen des dritten internationalen Mathematiker-Kongresses (1904), p. 194.

† Dickson, *Transactions Amer. Math. Society*, vol. 4 (1903), p. 434.

‡ Blichfeldt, *Mathematische Annalen*, vol. 63 (1907), p. 552 ; ibid., vol. 60 (1905), p. 204 ; *Transactions Amer. Math. Society*, vol. 6 (1905), p. 230.

§ Frobenius and Schur, *Berliner Sitzungsberichte* (1906), p. 186 ; *Crelle*, vol. 127 (1904), p. 10.

In this case it is possible to find a number which is divisible by every possible group and which depends only upon the field and the number of variables.*

Shaw has investigated the algebras defined by finite groups. "By a group algebra is meant that linear algebra whose units are defined to be such that each unit e_i corresponds to an operator O_i of some given finite abstract group, and conversely, and such that to each equation of the group $O_iO_j = O_k$ corresponds an equation $e_ie_j = e_k$ of the algebra." In this investigation Shaw confines himself to the scalar continuous field and observes that if the coefficients are any numbers in such a field, every abelian group of the same order gives the same group algebra, a result which is not true for all fields of coefficients. He points out the ultimate connection between group algebras and the theory of group characters and group determinants, and cites Poincaré's fundamental theorems bearing on such algebras.†

In volume 23 of the *Mathematische und Naturwissenschaftliche Berichte aus Ungarn*, Visnya determines a necessary and sufficient condition that a finite group of linear substitutions is intransitive and he also considers all the possible Hermitian invariants of such a group. A sufficient condition for its intransitivity is due to Maschke and was published much earlier. Burnside has recently given some new criteria for the finiteness of the order of a group of linear substitutions. The coefficients in the substitutions of a group of homogeneous linear substitutions are generally complex numbers of the form $\alpha + \beta i$, where α and β are real numbers. If the group is of finite order, there is a finite number of coefficients and therefore there is a finite positive number M such that for each coefficient $|\alpha| < M$ and $|\beta| < M$. Similarly there must be another positive number m such that for each coefficient $|\alpha| = 0$ or $> m$ and $|\beta| = 0$ or $> m$. The existence of these two numbers is proved to be both a necessary and a sufficient condition that the group is finite. Another necessary and sufficient condition is expressed by the following theorem: "If, in a group of linear substitutions on a finite number n of symbols, the order of every substitution is equal to or less than a finite number m, then the group is of finite order." Burnside

* Schur, *Berliner Sitzungsberichte* (1905), p. 1.

† Shaw, *Transactions Amer. Math. Society*, vol. 5 (1904), p. 326.