

THE MAXIMUM VALUE OF A DETERMINANT.

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HADAMARD* has shown that the maximum value reached by the modulus of a determinant of order n the moduli of whose elements do not exceed unity is $n^{n/2}$. The result is fundamental in Fredholm's theory of integral equations.†

The problem, when the elements are required to be real, was studied by Professor Hathaway and myself in 1882. I do not think Mr. Hathaway's work was published. My own was published ‡ only in the abstract here reproduced.

“The elements of a determinant being restricted to a variation between the limits $-a$ and $+a$, it is found that for all determinants whose order is greater than 2, a numerical maximum will be obtained by putting the elements $-a$ in the principal diagonal and making all the other elements of the determinant $+a$. If we denote such a determinant of order n when $a = 1$ by D_n and the minor of an element in the i th row and k th column by A_{ik} , we have always $A_{ii} = D_{n-1}$, $A_{ik} = -D_{n-1}/(n-3)$, so that

$$\begin{aligned} D_n &= -\left(1 + \frac{n-1}{n-3}\right) \cdot D_{n-1} = +\left(1 + \frac{n-1}{n-3}\right) \left(1 + \frac{n-2}{n-4}\right) D_{n-2} \\ &= \pm \left(1 + \frac{n-1}{n-3}\right) \left(1 + \frac{n-2}{n-4}\right) \cdots \left(1 + \frac{5}{3}\right) \left(1 + \frac{4}{2}\right) \left(1 + \frac{3}{1}\right) D_3 \\ &= \pm (n-2)2^{n-1}, \text{ since } D_3 = 4. \end{aligned}$$

“The sign will be \pm according as n is $\begin{smallmatrix} \text{odd} \\ \text{even} \end{smallmatrix}$. The effect of a change in any element is to lessen the greatest factor leaving the rest the same.

“For the maximum cubic determinant $D_n^{(3)}a^n$, we have merely to make all the strata identical and equal to $D_n^{(2)}$. Its value is $\pm n! D_n^{(2)}a^n$.

“The four-dimensional determinant may be looked upon as a sort of determinant of plane determinants, the terms of the devel-

* *Bulletin des sciences math.*, 1893, p. 240. Pascal, I Determinanti, p. 240.

† *Acta Math.*, vol. 27 (1903), p. 365.

‡ Johns Hopkins University Circular, vol. 2, No. 20 (December, 1882), p. 22.

opment being cubic determinants of which the plane determinants are the strata. In this quasi-determinant the same rule of signs applies as in the ordinary determinant and we get for the maximum value accordingly

$$\pm D_n^{(2)} n! D_n^{(2)} a^n = \pm n! (D_n^{(2)})^2.$$

Similarly

$$D_n^{(2p)} = \pm (n!)^{p-1} (D_n^{(2)})^p, \quad D_n^{(2p+1)} = \pm (n! D_n^{(2)})^p.$$

“In any of these determinants we have at once a formula for the maximum number of positive terms in the development. This is for $D_n^{(q)}$

$$\frac{(n!)^{q-1} \pm D_n^{(q)}}{2}.$$

“We use the \pm sign according as $D_n^{(q)}$ is \pm .”

A simple example serves to show that the limit reached when the elements are real may be exceeded when the elements are complex. Thus

$$\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 4,$$

but

$$\begin{vmatrix} i & 1 & 1 \\ 1 & \frac{-1+i}{\sqrt{2}} & 1 \\ 1 & \frac{1+i}{\sqrt{2}} & -1 \end{vmatrix} = 2(\sqrt{2} + 1).$$

This explains why Hadamard's limit exceeds mine.

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July, 1907.