he supplements this theorem by considering the case when $p = 2$. The main results may be stated as follows: If the Sylow subgroups of order $2^m (m > 1)$ contained in any group G are either cyclic or contain a cyclic subgroup of order 2^{m-1} which includes only two invariant operators under one of these Sylow subgroups, then the number of operators of order 2 in G is of the form $1 + 4k$. When this condition is not satisfied the number of these operators is always of the form $3 + 4k$. When $m = 1$, G contains an invariant subgroup which is composed of all its operators of odd order, and the number of the subgroups of order 2 may have either of the two forms $1 + 4k$, $3 + 4k$. This is the only case where the form of the number of the subgroups of order 2 is not determined by the form of this number in a Sylow subgroup.

G. A. MILLER,
Secretary of the Section.

THE FOUNDATIONS OF MATHEMATICS.*

The Principles of Mathematics. By BERTRAND RUSSELL. Volume I. Cambridge. The University Press, 1903. xxix + 534 pp.

Essai sur les Fondements de la Géométrie. Par BERTRAND RUSSELL. Traduction par A. CADENAT, revue et annotée par l'auteur et par L. COUTURAT. Paris, Gauthier-Villars, 1901. x + 274 pp.

1. *The Problem.* — Pure mathematics has always been conceived in the minds of its votaries and by the world at large to be a science which makes up for whatever it lacks in human interest, and in the stimulus of close contact with the infinite variety of nature, by the sureness, the absolute accuracy, of its methods and results. Yet what has been accepted as sure and accurate in one generation has frequently required fundamental revision in the next. Euclid and his pupils could doubtless have complained of the lack of rigor and logical precision in his predecessors just as forcibly as some modern pupils of Weierstrass berate their scientific ancestors and companions.

* We may also refer our readers to the review by L. Couturat, *Bulletin des Sciences Mathématiques*, vol. 28, pp. 129–147 (1904). So large is the work of Russell that Couturat's review and our own supplement rather than overlap one another.

Euler, finding confusion in the theory of the infinite and infinitesimal, proceeded to explain away the difficulties, that others might be free from the prevailing errors. We cannot accept his reasoning to-day. At the beginning of the last century the state of infinite series was lamentable and Cauchy's memoir on the subject is said to have impressed itself on Laplace to such an extent that he postponed publishing his Mécanique Céleste until he became so hopeless of righting things that he gave up trying to do it. The righting has been accomplished in the present generation by Poincaré. Yet we very much doubt whether Laplace, before hearing of Cauchy's treatment, would have for a moment granted any possible inaccuracy in his own methods. Somewhat later Dirichlet treated the problem of determining a harmonic function from its boundary values and so careful a mathematician as H. Weber extended the method to the discussion of the equation $\Delta V + \lambda V = 0$ without any apparent qualms as to error. Nevertheless, now-a-days, the theoretical importance and the practical use of the principles of Dirichlet and Thomson are completely obscured for many by the too great emphasis laid upon the errors in the original demonstration of the principles.

We notice that the advance toward our present rigor has been made step by step by great men who, however, were no greater — one might almost say no more careful — than their fellows working in apparent unconsciousness of the impending trouble and perhaps even incredulous at first as to its reality. When will this revision stop? And whereunto will it finally lead? This is the problem of the ultimate foundation of mathematics. In attempting an answer one can learn only hesitancy from the past. The delicacy of the question is such that even the greatest mathematicians and philosophers of to-day have made what seem to be substantial slips of judgment and have shown on occasions an astounding ignorance of the essence of the problem which they were discussing. At times this has been due to the inevitable failings of individual intuition in dealing with matters that are still unsettled; but all too frequently it has been the result of a wholly unpardonable disregard of the work already accomplished by others. Even when guarding as much as may be against this latter sin, those who approach the depths of the subject upon which Russell has so courageously entered may well expect to hear the warning:

Procul, o procul este profani!

2. *The Solution.* — Says Russell: Pure mathematics is the class of all propositions of the form "p implies q," where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants. And logical constants are all notions definable in terms of the following: Implication, the relation of a term to a class of which it is a member, the notion of *such that*, the notion of relation, and such further notions as may be involved in the general notion of propositions of the above form. In addition to these, mathematics *uses* a notion which is not a constituent of the propositions which it considers, namely the notion of truth.

This is probably the first attempt to give a complete definition of mathematics solely in terms of the laws of thought and the other necessary paraphernalia of the thinking mind. Some there are who, under the influence of arithmetic tendencies, might be tempted to give a decidedly more superficial definition in terms of integers. Some might regard a complete definition as impossible. The fact that a definition such as the above may be given — and it is the purpose of Russell's Principles of Mathematics to demonstrate that the definition is not illusory nor too small nor too large — is attributable to two things: first, the more careful discrimination of what *pure* mathematics is; second, the extraordinary development of logic since Boole removed it from the trammels of medieval scholasticism.

He to whom the present highly developed state of the foundations of mathematics is chiefly due is Peano — one whose work unfortunately is very little known and still less appreciated in this country. True, Leibniz had long since done much and of recent years has been ably expounded by L. Couturat;* true it is, too, that Boole had freed us from Aristotelianism and that C. S. Peirce † and Schroeder had carried the technique of logic much farther; but they had never accomplished that intimate formal relation between logic and all mathematics which was the necessary precursor to a yet more intimate philosophic relation and which has been brought about by Peano aided by a large school of pupils and fellow-workers. The advance has been made largely by introducing into sym-

* La Logique de Leibniz, Paris, 1901.

† The fundamental importance of the logic of relations (see infra) was emphasized by C. S. Peirce in 1880–1884: but it is only beginning to have its full effect.

bolic logic such a simplification of notation as to relieve it of its unwieldiness and to allow its development into a powerful instrument without which one can hardly hope to get the best results in the treacherous though treasure-laden fields of the foundations of mathematics. Poincaré, to be sure, in his review of Hilbert's Foundations of Geometry * spurns this *pasigraphy,* characterizing it as disastrous in teaching, hurtful to mental development, and deadening for investigators, nipping their originality in the bud. However much we may agree in the first statements (see § 7, page 91), we had best be cautious in accepting such sweeping statements as the last, even from so great an authority — especially in view of the fact that, equipped with this pasigraphy, the Italian investigators, Peano and his pupil Pieri,† with some rights of priority, had given a more fundamental *logical* ‡ treatment of the subject on

* Translated in BULLETIN, vol. 10, p. 5 (Oct., 1903).

† "I principii della geometria di posizione," *Memorie della R. Accademia delle Scienze di Torino*, vol. 48, pp. 1–62. And, "Della geometria elementare come sistema ipotetico-deduttivo; Monografia del punto e moto," *ibid.*, vol. 49, pp. 173–222.

‡ While we appreciate and admire as much as anyone can the beauties of Hilbert's famous Grundlagen der Geometrie, we fail to see how the historical facts can justify what Poincaré says (*l. c.*, p. 23): "He has made the philosophy of mathematics take a long step in advance, comparable to those which were due to Lobachevsky, to Riemann, to Helmholtz, and to Lie." Poincaré makes the point that Hilbert regards his geometric elements as *mere things* and on this seems to rest a large part of the praise (*l. c.* bottom p. 21 and top p. 22). If this be so, it ought to be mentioned as a matter of history that Peano, in 1889, in his Principii di Geometria took precisely this stand, p. 24. In 1891–2, Vailati, *Rivista*, vol. 1, p. 127, vol. 2, p. 71, again formulated the principle in words. By 1897 the Italian school had gone as far beyond this point of view as to make it a *postulate* that points are classes — thus showing a twofold advance, once in recognizing the presence of a postulate, again in using the word class so as to bring the reasoning into form dependent upon precise logical processes alone. It has also been said that the idea of the independence of the axioms was due to Hilbert. As a matter of fact in 1894, Peano, "Sui fondamenti della geometria," *Rivista*, vol. 4, pp. 51 et seq., states the problem and, by actually setting up simple systems of elements, proves the independence of certain axioms from certain others. So that by 1899 the idea and method were both five years old at least. Again, in 1889, Peano laid it down as a principle that there should be as few undefined symbols as possible, and he used but few. In 1897–9 Pieri used but two for projective geometry and but two for metric geometry, whereas Hilbert was using a considerable number, seven or eight. (The idea of compatibility seems to have been first stated clearly by Hilbert.) There still remains in the Grundlagen der Geometrie matter enough for the amplest praise. The archimedean axiom, the theorems of Pascal and Desargues, the analysis of segments and areas, and a host of things are treated either for the first time or in a new way, and with consummate skill. We should say that it was in the technique rather than in the philosophy of geometry that Hilbert created an epoch.

which Poincaré was writing than is to be found in the work he was praising so highly. In the fields of arithmetic and algebra, too, Burali-Forti and Padoa, adherents of Peano, had reached a point far beyond the widest view of the chief of the German school that deals with the same subjects.* Furthermore, on this one point Poincaré may not be regarded as an authority; for his own work † in the field should be characterized as subjective rather than objective, speculative and suggestive rather than purely logical.‡ Anyone who is acquainted with the articles presented to the Philosophical Congress at Paris in 1900 by Peano, Burali-Forti, Padoa, and Pieri, cannot be convinced that these authors had become deadened, and the artificiality of their system is by no means so certain as it might be. Since then, our author, Russell, has simplified and improved the older work of C. S. Peirce on the theory of relations, adapting it to the system of Peano, and has produced a coherent treatment of the great problems underlying mathematics. In view of accomplished facts one inclines more readily to the praise given by Whitehead: "I believe the invention of the Peano and Russell symbolism forms an epoch in mathematical reasoning." §

3. *The Reason.* — It is not hard to detect the reason why mathematics has thus pushed its foundations back until they

* Compare the papers below referred to in the Bibliothèque du Congrès International with No. 2 of Hilbert's Mathematical Problems, BULLETIN, vol. 8, p. 447. We may refer also to Padoa, *L'Enseignement Mathématique*, vol. 5, p. 85. See also § 4 of the present review. For our readers, who may be working on the problem No. 2, we may note — what we unfortunately failed to note at the time of translating — namely, that a solution along the lines proposed by Hilbert seems logically impossible. A solution has long since been proposed in the article here referred to. There are those, however, who hold that Padoa has gone so far as to overshoot the mark. Hilbert has again taken up the matter much more searchingly than in 1900. It is to be regretted that his paper which was presented at Heidelberg, August, 1904, is not at hand for comparison.

† La science et l'hypothèse, Paris, 1903; and numerous scattered essays.

‡ That Poincaré seems frequently to have in mind the physical rather than the mathematical, the psychological rather than the logical point of view can be seen in several places in his review. On p. 8 he asserts that we know the axioms are non-contradictory "since geometry exists." And on p. 22 he seems to complain that the logical standpoint interests the author to the utter disregard of the psychological. It should be remembered that the first chief aim of the modern researches on the foundations of geometry is to be entirely rid of the psychological element — and this for the very reason that secondly we may decide just what that psychological element must be. This latter problem belongs rather to the philosopher and psychologist than to the mathematician.

§ *American Journal of Mathematics*, vol. 24, p. 367 (October, 1902).

have come to rest solely in logic. In the first place mathematical or other reasoning presupposes a mind capable of rational, that is, non-selfcontradictory ratiocinative processes. Now it always has been comfortably assumed that we can carry out such processes if only we are careful enough, that there is no need of formulating the laws of thought before beginning to reason, or even that a formulation and analysis of those laws is impossible.* Where then do the errors creep in? An examination of some typical cases shows that it is generally through lack of a sufficiently careful definition of the terms. This failure properly to define has led to interminable discussions which from the start could only lead either to nothing or to wrong results. In mathematics it is the absence of precise definition which brings in the erroneous statements concerning differentiation, continuity, and infinity, with a host of others. The perception of this difficulty was the origin of the principle of arithmetization and of epsilon proofs. In the end, however, after one has really mastered the principles of modern analysis he seldom needs the actual presence of epsilons to establish a theorem. Nevertheless it is a satisfaction to have this formal method to fall back on whenever challenged by one's own hesitancy or by that of others. In like manner, who has not at times during some long complicated or indirect logical demonstration felt the least bit uncertain; who would not be glad to have at his hand some formal method such as Peano's, based upon certain rudimentary propositions and concepts?

In truth it is a matter of more consequence than is sometimes thought, to have clearly in mind those processes which are definitely to be admitted as logical. The one process which stands out most definite in our consciousness is the syllogism. If a piece of reasoning can be put in the form of major premise, minor premise, conclusion, we are tolerably sure of its truth. But numerous proofs cannot be so constructed and it is one of the most frequent errors committed by the intuitive logicians to say that reasoning consists in a sequence of syllogisms. Perhaps the greatest advance made by Boole was the clear recognition of the necessity of asyllogistic reasoning.

The question then becomes of fundamental importance: What is at the bottom of our logic? When the matter is

---

* Duhamel, Des méthodes dans les sciences, Paris, 1875, vol. 1, p. 17.

looked into, it appears that we constantly use *propositions*, passing from certain propositions as hypothesis to certain others as conclusion. The laws of implication which govern the relation between hypothesis and conclusion constitute the logical theory or calculus of propositions. Casting about for other principles we come upon *classes* or sets of objects represented in ordinary speech by common nouns. The development of the interrelations of classes produces the logical calculus of classes. This calculus has a remarkable analogy with the calculus of propositions, but the relation is not quite dual. In the third place we perceive that *relations* are of the utmost importance. Every transformation, every function is a relation. In common language the verb does but express a relation between the subject and object. Thus there appears the necessity for a calculus of relations.* The complete logical calculus, as now used, is a combination of these three types. The whole number of laws of thought or logical premises which seem to be required for establishing the calculus in all the generality necessary for mathematics is small. In addition to these premises there are a certain number of elementary ideas or terms such as implication, and the notions of proposition, class and relation, which must be assumed as known. It is the discussion of these questions which are of a philosophical rather than mathematical nature, that fills the first Part of Russell's Principles.

We may grant, then, that logic is *necessary* to mathematics. It is affirmed to be *sufficient*. This in reality is the remarkable content of the definition given by the author. So immune are we from logical error that the necessity of logic might never force us to a critical examination of its principles ; but the affirmation of its sufficiency fully justifies and even renders imperative such an examination. Russell's entire volume is devoted to establishing this sufficiency. And although the subject is very new and many difficulties philosophical and mathematical are still outstanding, there can be little doubt that to an unexpectedly large extent the author is successful in his attempt and that in these Principles he has given a per-

* Peano and his immediate followers overlook the importance of this subject — so busy are they with other important questions. It is one of the lasting services of Russell, following very closely on the work done twenty years earlier by C. S. Peirce, to have recognized the necessity of this addition to Peano's system and to have supplied the deficiency. See his articles in the *Revue de Mathématiques*, vol. 7, nos. 2 and following (1901–1902).

manent set to the future philosophy of the questions which he handles.

4. *Some Notions.* — Owing to the wide-spread diversity of usage in the meaning of such fundamental notions as postulates, axioms, undefined symbols, definitions, consistency, independence (of postulates), irreducibility (of undefined symbols), completeness (of systems of postulates and undefined symbols), we think it best to enter upon some slight exposition * of these matters instead of taking up the critical discussion of some of the more abstruse problems which are treated by the author and which could scarcely be appreciated before such exposition.

Axiom is a word which has so long been used in so many vague ways that its use in pure mathematics had probably best be abandoned. The familiar definition: An axiom is a self-evident truth, means, if it means anything, that the proposition which we call an axiom has been approved of by us in the light of our experience and intuition. In this sense pure mathematics has no axioms: for mathematics is a formal subject over which formal and not material implication reigns.† The proper word to use for those statements which we posit would seem to be postulate. What self-evident truths can there be concerning objects which are not dependent on any definite interpretation but are merely marks to be operated upon in accordance with the rules of formal logic? Postulates, however, may be laid down at will so long as they are not contradictory. It is the postulates which give the objects their intellectual though not physical existence. Indeed before we can apply to the physical world any of the systems of logical geometry, for instance, we have the one great axiom: This system fits nature sufficiently for our purposes. To postulate such a statement would avail us naught. We must carefully consider the totality of our experience and decide whether the statement seems to represent a truth.

Definition is a term which has long been used by philosophers to stand for a process of analysis and exemplification

---

*See also E. V. Huntington on "Sets of independent postulates for the algebra of logic," *Transactions*, vol. 5, p. 288 (July, 1904).

†In regard to logic on which mathematics rests, we should incline to use the word axiom (if indeed we do not prefer to hold to premise) and not postulate. For here we are dealing with the actual (mental) world and not with a system of marks. The basis of rationality must go deeper than a mere set of marks and postulates. It is foundation of everything and must be more *real* than anything else.

which brings before the mind a real consciousness of the object defined. This sort of definition has to be used in dictionaries. In mathematics, however, no such vague process is permissible. Mathematical definition is simply the attributing of a name to some object whose existence has been established or momentarily postulated. It is the process of replacing a set of statements by a single name and is resorted to solely for convenience. In any science whose development has been perfected, definitions may be entirely done away with by those who are willing to sacrifice brevity. There can be little doubt that a large number of definitions might better be thus put out of the way.*

Although all definitions are thus merely nominal, there are three distinct aspects † of definition which are worth considering in detail in connection with the theory of integers. These may be characterized as (1) the particular definition, (2) the definition by postulates, (3) the definition by abstraction. They may be illustrated as follows: Suppose (1) that it is possible to find a logical class K of which the elements are, let us say, classes or propositions. Suppose further that by means of logical processes alone we may define operations on the elements like addition and multiplication of integers. Grant that there exists in the class K an element analogous to zero (in case K is a class of classes this would be the null-class; in case K were a class of propositions it would be the absurd). In fact suppose that we could set up a class K and a set of operations in K which have the properties of integers as we use them. We then might say from a formal point of view that the class K was the class of integers, that the elements of K were the integers themselves, and that the operations we had set up were the ordinary operations of arithmetic. This would be a satisfactory though very particular definition of the integers and would have the advantage that unless there were a contradiction inherent in our logic there could be no contradiction in our system of integers. Or (2) we may assume a certain set of elementary terms, known as undefined symbols, such as num-

---

* Peano: Bibliothèque du Congrès International de Philosophie à Paris, 1900, vol. 3 (1901).

† Compare Burali-Forti; Bibliothèque etc. "Sur les différentes définitions du nombre réel." We say *aspect* on account of a change in view which has been established since 1900. Also "Le classe finite" *Atti della Accademia reale di Scienze di Torino*, vol. 32, p. 34 (1896); and more recently "Sulla teoria generale delle grandezze e dei numeri," *ibid.*, vol. 39 (Jan., 1904).

ber, zero, and successor. These we could connect, as Peano and Padoa have done, by a system of postulates, and thus we should have a definition of number through postulates. In order to prove the non-contradictoriness of our postulates and indefinables, that is, the existence of our elements, we should have to set up some system which afforded one interpretation of the indefinables and of the postulates. As this must be done by going back to the laws of thought we finally get very near to where we started in the other sort of definition. The definition remains, however, slightly more general: for the integers thus defined are not merely one set of elements but any set which satisfies the postulates. Or (3) we may use the principle of abstraction on which Russell places a great deal of emphasis. We may say that two classes of objects, no matter what objects they be, have the same number when there exists a one-to-one relation between their elements.* Thus number becomes the common property of all similar classes, and is their only common property. The class of numbers becomes the class of all similar classes. Owing to Russell's development of the theory of relations this definition becomes also merely nominal and as it seems to be the most fundamental and philosophic it may be accepted as the best thus far given.

Although the use of postulates other than the premises of logic, and the use of undefined symbols other than those of logic seem needless and to be avoided in pure mathematics, the usage is so common that we may go on to say a few words concerning consistency, independence, irreducibility, and completeness — especially as these ideas are somewhat usable in the foundations of logic. To show the consistency of the system of postulates and undefined symbols it is evidently futile to attempt to develop the consequences of the postulates until no contradiction is reached (this method of stating the thing is sufficient to show wherein lies the futility) : for the most that can be accomplished in this way is to see that up to a certain point no contradiction has been reached. The method of proof consists merely in finding some system of entities known to exist and affording a possible interpretation of the undefined symbols and postulates. To make the proof really fundamental for the system of integers it seems necessary to go quite out of the

* Russell shows, Principles, p. 113, that this idea is not dependent on the general concept number, nor even on the concept unity. Two classes which can be placed in one-to-one correspondence are called *similar*.

field of mathematics into the domain of logic.* The method of showing the independence is merely to set up for each postulate one existent system of elements in which there are possible interpretations of our undefined symbols and which satisfies all the other postulates but not the particular one in question. This shows the independence of that one. If one of the undefined symbols used in the statement of the postulates can be given a nominal definition in terms of the others the system of indefinables is redundant. It was Padoa † who first made effective use of this idea. To show the irreducibility of the indefinables relative to the system of postulates it is necessary to set up a system of elements which satisfies all the postulates, which affords an interpretation of the undefined symbols, and which continues to satisfy these conditions when one of the undefined symbols is suitably altered : this must be done for each. The problem is quite similar to that of the independence of the postulates and is not difficult to solve in case the number of undefined symbols is small. All this difficulty is avoided in dealing with the different branches of mathematics when Russell's point of view — no new indefinables, no new postulates — is taken.

Huntington ‡ seems to have been the first to bring to effective use the idea of completeness. The problem is to show that if there are two sets M and M' of objects each § of which satisfies the postulates and affords interpretations of the indefinables, then the two sets of objects may be brought into one-to-one correspondence in such a way as to preserve the interpretation of the symbols. With the statement of this last idea we have arrived at the limit of present ideas concerning the interrelations of the notions at the base of mathematics as defined by postulates.

* See references given in footnote under § 2, p. 77. The consistency is far more important than the independence, irreducibility, or completeness : for these are merely a matter of elegance, whereas that determines whether or not all our reasoning upon the system in question is void.

† Bibliothèque etc. "Essai d'une théorie algébrique des nombres entiers, précédé d'une introduction logique à une théorie déductive quelconque." This remarkable essay should be read by every one. We may note that Padoa uses 'transformateurs' but introduces no theory of relations. In this respect Russell has introduced improvements.

‡ *Transactions*, vol. 3, pp. 264–282 (1902). See also Veblen, *Transactions*, vol. 5, p. 346 (1904).

§ Serious mistakes, resulting in definitions of no essential content, have been made by forgetting that the relations which connect the elements must be in correspondence, in addition to any correspondence between the elements. See also footnote under § 6, p. 87.

5. *Numbers.* — The analysis of number, cardinal or ordinal, finite or infinite, integral, rational, or real, with carefully drawn distinctions between the many allied ideas such as counting, quantity, magnitude, and distance, forms the content of Parts II–V., pages 111–370, of the Principles. To do anything like justice to this masterpiece of analysis in a field so strewn with difficulties would be impossible within the space at our command. The summaries given by the author at the close of each Part afford a clear review of the ideas which have been discussed and the points which have been won. Leaving out of account the advances which are made toward the precision of the terms which lie at the bottom of logic we can at best merely indicate some of the results which are of greatest interest to mathematicians.

It is shown that cardinal and ordinal integral numbers are inherently different, that finite cardinals and ordinals may be obtained in terms the one of the other but that this principle cannot be applied to the infinite. With the guidance of the principle of abstraction cardinal integer has been defined as a class of similar classes. This definition has the immediate advantage of giving finite and infinite cardinals at the same time. The finite may then be distinguished from the infinite by the fact that in the former the whole cannot be similar to its part, whereas in the latter it can. Another point which Russell establishes with the aid of Whitehead * is that by the use of logical addition the numerical addition of a finite or infinite number of finite or infinite cardinals may be and indeed (if we invoke the principle of abstraction) should be defined in such a manner that the order in which the numbers are added plays no part. This is a great victory for common sense and must appeal to everyone as a vindication of the school-child in his inherent notion that he has the same number of marbles whether he has five in one pocket and three in another or three in two pockets and two in a third, no matter which of his pockets these be. The principle of commutation and association of the terms in addition is entirely done away with, except in so far as mechanical difficulties prevent us from writing simultaneously a number of terms and the signs of addition connecting them. We may point to the fact that the work applies equally well to finite and infinite sums as an indication of its extreme generality

* *American Journal of Mathematics*, vol. 24 (1902).

and as evidence that at last we have a principle of addition distinctly above the plane of counting on one's fingers. In like manner the definition of multiplication is such as to be free from the laws of commutation and association of the factors and to apply equally to finite or infinite products of finite or infinite cardinals. Again a vindication of the school-child who rightly cannot see why it should make any difference whether he puts down four rows of three marks or three rows of four.

The discussion of the meaning of quantity and magnitude in Part III. and its connection with number we will not pause to consider, but we pass directly to the theory of order as developed in Part IV. The treatment of this subject is greatly simplified by the theory of relations. Order is shown to be an asymmetric transitive relation, an essential property of serial relations. It is clearly pointed out and it is important to notice that when a set of objects is given the relation is not necessarily included; whereas when the relation is given the field in which it operates must also be given. If recourse is had to the principle of abstraction the ordinal integer appears as "the common property of classes of serial relations which generate ordinally similar series." As the cardinals are classes of similar classes, so the ordinals are classes of like relations. The principle of induction is intimately associated with the system of ordinals rather than with the system of cardinals although for finite numbers the distinction is not so great as for infinites. We may say that the finite ordinal is that which can be reached by induction from 1. It appears that those who generate their system of numbers by a relation of succession or by counting — that is, by successive acts of attention — must in reality be coming at something which resembles ordinals much more nearly than cardinals. The difference between the theory of infinite cardinals and infinite ordinals brings to light the important fact that in mathematics we have two kinds of infinite: the cardinal, which has the property of being similar to a part of itself, and the ordinal, which cannot be reached by induction from 1.* The discussion naturally brings up the old question of extensive and intensive definition. The definition of an

* This would seem to render invalid the contention of Poincaré in his La science et l'hypothèse to the effect that the principle of induction is the essence of the infinite. We have seen that it is the essence of the finite. The difficulty seems to be that Poincaré has in mind the definition of the infinite as a growing variable. If this be so, the apparent contradiction resolves itself into a mere difference of definitions.

object is said to be extensive when the object is given by the enumeration of its parts; it is said to be intensive when the object is characterized by its properties. In the treatment of these questions and of transfinite cardinals and ordinals there is much which is instructive for the mathematician and the philosopher. The author points out with his customary frank desire to state no more than the truth that there still remain difficulties to solve. Thanks to his lucid and modest presentation there is no reason why he should not find adherents who will take up the work and attempt the solution in a spirit of hearty coöperation.

There is a school of creationists who, when they find that certain infinite processes lead to no rational limit nor yet to a number which becomes infinite, postulate the existence of a limit and thus obtain the irrational numbers. The author does not consider an *ipse dixit* like this to be a sufficiently good theorem of existence. He therefore considers infinite sets of rationals and by means of them he forms a set of things which he calls real numbers. A real number is neither a rational nor an irrational; it is a certain infinite set of rationals. The real numbers thus defined are shown to satisfy the notion of a continuum. According to the method followed, the continuum appears as an idea wholly ordinal in nature. With the aids thus prepared the author is able to give a very satisfactory account of the philosophy of the infinite and of the continuous. His treatment of the paradoxes of Zeno shows that the arguments of the ancient philosopher are by no means so far from right as might be imagined and that the contradictions are more apparent than real.

6. *Geometry and Mechanics.* — A short study of the properties of multiple series leads to a point from which it may be seen that: Geometry is the study of series of two or more dimensions.* In this manner the necessity of new postulates and new indefinables is avoided. The procedure is evidently reasonable. Mathematical geometry has long since been divested of all spatial relations between its elements. The above definition

* As the serial relation is emphasized rather than its domain (see discussion of order given above) the author avoids a definition which is null and which makes dimensions impossible. Compare discussion of completeness and footnote, § 4. For a fuller discussion of this important point we may refer to "The so-called foundations of geometry," by the present reviewer, in the *Archiv der Mathematik und Physik*, vol. 6, pp. 104–122 (1903). Toward the end of the discussion a change, which may cause some confusion, is made to the point of view of physical geometry. The first part, however, deals solely with purely mathematical geometry.

is but the culmination of the ideas of manifolds introduced by Grassmann and Riemann. As those who define geometry by postulates are forced to show the existence of their elements by having recourse to systems of numbers the question is quite pertinent: Why not begin with a purely nominal definition like the above and avoid the trouble of proofs of existence, of independence, and of irreducibility?

At this juncture it is interesting to compare the attitude taken in the Principles with that taken in the older Foundations of Geometry. It should be remembered that the author originally started with the study of the philosophy of dynamics and hence necessarily of geometry. To render the examination really searching the foundations of geometry had to be investigated. But, once started, the end was not so easily to be reached. Probing into the mysteries of infinity and continuity led to arithmetic in its wider sense. Trying to render precise the meaning of important words such as element, set, operation, conclusion, proof, etc., could but conduct to the study of logic, and the desire to be rid so far as possible from the contaminations of the personal element brought up at last at formal logic. And then the entire field had to be traversed in the forward direction with the necessity of constant acquisition of original results at every step! Surely the present work is a monument to patience, perseverance, and thoroughness.

In the essay on the foundations of geometry the author had not yet reached the logical stage — scarcely the arithmetic stage. He was content, as some still are, to analyse the ideas in the rough, to use a large number of indefinables, to state broad indefinite axioms instead of brief incisive postulates — in a word, to forego all the modern technique. The result was an extremely suggestive essay — one which still can be read with profit and by rights ought to be read, if only for the sake of contrast, in connection with the newer work. To-day we have mathematical geometry, then we had physical. If one wishes to read an excellent account of space from the physico-metaphysical point of view he has but to turn to the Russell of a few years since; if he would know the extreme point of modern mathematico-logical geometry he has merely to take up the Russell of to-day.

In Part VII. the analysis proceeds to mechanics. Here space is merely a certain three- (or n-) dimensional series; time, a simple series. There is a relation which connects part of space (the material points) with all time, that is, $a, b, c = R(a_0, b_0, c_0, t)$, where a, b, c are the coördinates of the material points. This relation R is so chosen as to allow for the impossibility of generating or destroying matter. The relation is also chosen so that if the relation between matter and time is known at two instants it is known at every instant. In this way is stated the causality in the universe. This seems very far off from the real world. It must delight the hearts of philosophers who believe in a pure idealism. It is found that arithmetic may be handled adequately with no help save from logic. This does not surprise us. Then geometry is put in the same category. Modern mathematicians have so accustomed us to look on merely the logical side of the subject that we are not troubled. Finally comes dynamics. Why not thermodynamics, electrodynamics, biodynamics, anything we please? There is no reason why not. There is in reality no place to stop, save when we have become tired of pure logic, if once we include geometry. As a matter of fact all our concepts whether of space, or matter, or electricity, or life, are but idealizations more or less well-defined, and, if we insist on subjecting the world to purely logical explanation, they all belong in the same class.

Upon this matter we may best quote Russell who, amid all his refinements, keeps a clear idea of their proper place in the system of all knowledge. He says: The laws of motion, like the axiom of parallels in regard to space, may be viewed either as parts of a definition of a class of possible material universes, or as empirically verified assertions concerning the actual material universe. But in no way can they be taken as *à priori* truths necessarily applicable to any possible material world. The *à priori* truths involved in dynamics are only those of logic; as a system of deductive reasoning, dynamics requires nothing further, while as a science of what exists, it requires experiment and observation. Those who have admitted a similar conclusion in geometry are not likely to question it here; but it is important to establish separately every instance of the principle that knowledge as to what exists is never derivable from general philosophical considerations, but is always and wholly empirical. *

* It would be interesting to discuss in how far this attitude is really in accord or out of accord with the apparently very different view of Poincaré (La science et l'hypothèse) that the question whether the parallel axiom is true or not true is devoid of sense owing to the fact that it is merely a *convenient* method of correlating experience and a *convention* can have neither truth nor falsity.

7. *Some Conclusions.*—There is one conclusion in logic which suggests itself almost inevitably at this point. For there are a considerable number of systems of logic current at present. Different authors have treated the subject differently — each choosing the system of indefinables and laws of thought which seemed best to him at the time. Now it is by no means true that these various systems of logic have been proved coextensive or even not mutually contradictory. If it should appear that they cannot be brought into harmonious relation one with another there will be some instructive, if bewildering, conclusions to draw. And as we have such complex entities as infinity and continua with which to deal it might not be regarded as surprising if some points were found to stand out permanently, so that logicians will permanently disagree. In fact at present there seems to be a grave logical difficulty in our logical system as developed by Russell. This trouble had been felt by Frege and a solution had been proposed by him; but it does not seem entirely satisfactory.* In view of the outstanding difficulties and the possible divergence of systems of logic held by equally good authorities, we come to the conclusion that it is dangerous to accept the naïve point of view of those who claim that a certain piece of reasoning depends on the operation of logic alone but who fail to state what those operations are and to use all the means possible to avoid the intrusion of extraneous ideas. They may not fall into error, but they are merely following in the footsteps of those who "knew" what infinity and continuity were.

From the pedagogic point of view we may also draw some conclusions. It is hardly necessary to trouble the student with the commutative and associative laws in multiplication of integers or with elaborate deductions of a number system before he is readily able to appreciate the needlessness of the former and the relation which the latter bears to the theory of finite and infinite cardinals and ordinals, the ideas of compactness and continuity, and the two kinds of infinity.

---

*In a long appendix, Russell gives a detailed exposition of the important work of Frege, which culminated in the Grundgesetze der Arithmetik, and he discusses this troublesome contradiction again from a different standpoint. It is this contradiction which Hilbert had in mind in his Heidelberg address referred to under § 2. He, therefore, attempts to recast the principles of logic and of arithmetic in such a manner as to render them sufficient for mathematical reasoning. We certainly hope that he has succeeded in doing so to the satisfaction of both mathematicians and philosophers.

A clear-cut physical conception that numbers possess order and may be associated with the points on a line is a workable idea which in practice is both necessary and sufficient for ordinary rigorous analysis. An inadequate vague idea regarded as a useful working hypothesis seems, on the whole, productive of more good and less harm than an inadequate definite idea regarded as final. In geometry and mechanics the physical attitude may be taken. Axioms, things deemed worthy of credence on the basis of experience, should take the place of postulates. This does not prevent, in fact it encourages, the statement of a large number of axioms without troubling too much as to their independence. At the same time these statements should include the essential idea of order and the useful idea of continuity and other ideas which are usually passed over.* In short we should use and train intuition to the utmost in connection with some logic; for pure logic alone is, as Poincaré states (§ 2), harmful to the earlier development of the mind.

From the mathematical standpoint we have learned that many of the objects which have been thought of as individual must be regarded as classes. We cannot define euclidean space, but we can define the class of all euclidean spaces.† The principle of abstraction, here involved, seems to arise from the necessity of taking the terms in a logical equation to represent the common attribute of all the objects which may in some way satisfy the equation. As during the progress of the discussion, we have introduced no new indefinables, no new postulates, no processes other than those of logic, there is no possibility of our arriving at contradictions except through the failure of our logical system to be logical; and behind this we cannot go. It remains merely to show the existence of the classes with which we have dealt; otherwise our work would be null. To quote freely from our author: The existence of zero is derived from the fact that the null-class is a member of zero; the

* Compare, for example, the series of articles by the reviewer on Spherical Geometry, *American Mathematical Monthly*, commencing January, 1904.

† This apparently considerably lowers the importance of the idea of *completeness* discussed in §4. For it appears as if the one-to-one correspondence between the different euclidean spaces were really of minor significance. This is but another instance of the fact that the elements themselves are unimportant — that it is the abstraction from them which is most fundamental. However, we believe that the idea of completeness is a new step, a step onward and toward a fuller description of the systems dealt with.

existence of 1, from the fact that zero is a unit-class (the null-class being its only member). By an evident induction we get all finite numbers. From the class of finite cardinals follows first the existence of the smallest of the infinite cardinals, and second, by considering them in the order of magnitude, the existence of ordinals and the smallest of the infinite ordinals. We may go on to obtain the rationals, compact enumerable series, continuous series. From the last we may see the existence of complex numbers, of the class of euclidean spaces, of projective spaces, of hyperbolic spaces, and of spaces with various metrical properties. Finally we may prove the existence of the class of dynamical worlds. Throughout this process no entities are employed but such as are definable in terms of the fundamental logical constants. Thus the chain of definitions and existence-theorems is complete, and the purely logical nature of mathematics is established throughout.

This is as far as we are conducted. But we are promised a second volume — may it be soon forthcoming — written with the collaboration of Whitehead. Herein will be contained actual chains of deduction leading from the premises of logic through arithmetic to geometry. Herein will also be found various original developments in which the notations of Peano and Russell have been found useful. For those who wish sooner to get at the Peano-Russell point of view in the matter, we append a bibliography, which while very incomplete may still be found useful in tracing the development of the ideas:

(1) Arithmetices principia nova methodo exposita, Turin, Bocca Frères, 1889.

(2) I principii di geometria logicamente esposti, Turin, 1889.

These two works by Peano are the starting point of the whole movement. They were written in the days when a careful explanation and translation of the symbolic method was in vogue and form a good starting point for study. The *Formulaire de Mathématiques*, edited by Peano, is rather hard to begin on. The *Rivista di Matematica*, now the *Revue de Mathématiques*, also edited by Peano, furnishes much easy and instructive reading matter. Logica matematica by Burali-Forti in the series of Manuali Hoepli may serve as a textbook. Omitting important memoirs by Burali-Forti on arithmetic and by Pieri on geometry which we have quoted in footnotes, we cite again

(3) Bibliothèque du congrès international de philosophie, volume 3 (1901).

The articles by Peano, Burali-Forti, Padoa, and Pieri show the point at which the Italian school had arrived in 1900. It is since that time that most of Russell's technical work has appeared. For the present state of the science, we would note a memoir by Whitehead:

(4) "On cardinal numbers," *American Journal of Mathematics,* volume 24 (1902), pages 367–394; and a still more recent paper by Burali-Forti, "Sulla teoria generale delle grandezze e dei numeri," *Atti della R. Accademia delle Scienze di Torino,* volume 39, (January, 1904).

EDWIN BIDWELL WILSON.

YALE UNIVERSITY,
July, 1904.

NOTES.

THE closing (October) number of volume 26 of the *American Journal of Mathematics* contains the following papers: "Invariants of a system of linear partial differential equations, and the theory of congruences of rays," by E. J. WILCZYNSKI; "On elements connected each to each by one or the other of two reciprocal relations," by C. DE POLIGNAC.

THE opening (October) number of volume 6 of the *Annals of Mathematics* contains the following papers: "On the subgroups of an abelian group," by G. A. MILLER; "Note on the continued product of the operators of any group of finite order," by W. B. FITE; "Reduction of an elliptic integral to Legendre's normal form," by N. R. WILSON; "The necessary and sufficient condition under which two linear homogeneous differential equations have integrals in common," by A. B. PIERCE; "A general method of evaluating determinants," by G. MACLOSKIE; "Application of groups to a complex problem in arrangements," by L. E. DICKSON; "On functions defined by an infinite series of analytic functions of a complex variable," by M. B. PORTER.

AT the Cambridge meeting of the British association for the advancement of science (cf. October BULLETIN, page 28), Professor A. R. FORSYTH presided over the subsection of pure mathematics, whose programme included the following papers: "A fragment of elementary mathematics," "Geometry of the complex variable," by Professor F. MORLEY; "Peano's