

All the terms in the product of these two series, after the second term, vanish. Putting the sum of the first two terms $= N = 0$, we have $a_0 - d(a_0 + 1) = 0$. If we assume $a_0 = -\frac{1}{9}$, then $d = 10$, and the two factor series become

$$U = -\frac{1}{9} - 11\frac{1}{9} - 112\frac{2}{9} - \dots$$

$$U' = 1 - 11 + 9 + 11 - 9 - 11 + 9 + 11 - \dots$$

Since $U \cdot U' = 0$, we have $U^p \cdot U'^p = 0$. As all the terms in U are of the same sign, it is easily seen that U^p is divergent for all positive integral values of p . U' and U^p are also divergent.

§ 8. If we assume $t = 2$, $s = 1$, $c = -1$, $a_0 = 1$, $b_0 = 1$, $N = 0$, then the condition that the sum of the product of (5) and (6) shall vanish becomes $d^2 + 1 = 0$ and (letting $i = \sqrt{-1}$) the factor series thus obtained are the two complex divergent series

$$1 + (1 + i) + i + 0 + 1 + (i + 1) + i + 0 + \dots$$

$$1 - (1 + i) + (i - 1) + (i + 1) - (i - 1) - (i + 1) + \dots$$

COLORADO COLLEGE, COLORADO SPRINGS,
April 12, 1902.

THREE SETS OF GENERATIONAL RELATIONS DEFINING THE ABSTRACT SIMPLE GROUP OF ORDER 504.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, October 25, 1902.)

1. CONSIDERABLE interest attaches to the simple group of order 504. The existence of this simple group was discovered by Professor Cole.* This was one of the facts that lead Professor Moore † to his investigation of the linear fractional group in the general Galois field, resulting in the discovery of a new doubly infinite system of simple groups.

* "On a certain simple group," *Mathematical Papers*, Chicago Congress of 1893.

† *BULLETIN*, December, 1893; *Mathematical Papers*, Congress of 1893.

The first set of generational relations follows readily from a general theorem quoted, with references, in § 2. From it is derived, in §§ 5-9, the second set of generational relations

$$A^7 = I, \quad B^2 = I, \quad (AB)^3 = I, \quad (A^3BA^5BA^3B)^2 = I.$$

The same method was followed by the writer* to determine the well-known generational relations for the simple linear fractional groups in the $GF[5]$ and $GF[2^2]$, each of order 60; and for that in the $GF[7]$, of order 168. In a "Note on the simple group of order 504," Burnside † established the above second set by a direct analysis. Other derivations of this second set are due to R. Fricke ‡ and to De Séguier.§

The simplest of the three is the (new) third set

$$C^9 = I, \quad D^2 = I, \quad (CD)^3 = I, \quad (C^3DC^5D)^2 = I.$$

First Set of Generational Relations, §§ 2-4.

2. THEOREM. || *The group of all linear fractional substitutions of determinant unity in the $GF[p^n]$ is simply isomorphic with the abstract group generated by the operators T and S_λ (λ running through the series of p^n marks of the field), subject to the generational relations*

$$(1) \quad T^2 = I, \quad S_0 = I, \quad S_\lambda S_\mu = S_{\lambda+\mu} \quad (\lambda, \mu \text{ any marks}),$$

$$(2) \quad S_\lambda T S_\mu T S_{\frac{\lambda-1}{\lambda\mu-1}} T S_{-(\lambda\mu-1)} T S_{\frac{\mu-1}{\lambda\mu-1}} T = I$$

(λ and μ any marks such that $\lambda\mu \neq 1$).

3. For $p^n = 2^3$, the group is a simple group of order 504. Let the $GF[2^3]$ be defined by the irreducible congruence

$$i^3 \equiv i + 1 \pmod{2},$$

so that the marks of the field are

$$0, \quad 1, \quad i, \quad i + 1, \quad i^2, \quad i^2 + 1, \quad i^2 + i, \quad i^2 + i + 1.$$

* Linear Groups, pp. 302-303.

† *Math. Annalen*, vol. 52 (1899), pp. 174-176.

‡ *Ibid.*, p. 335.

§ *Journal de Math.*, series 5, vol. 8 (1902), p. 267.

|| Found, but not yet published, by Professor Moore. A different proof for finite fields is due to the writer: Linear Groups, pp. 300-302.

The relations $S_\lambda S_\mu = S_{\lambda+\mu}$ all follow from

$$(3) \quad S_1^2 = S_i^2 = S_{i+1}^2 = S_{i^2}^2 = S_{i^2+1}^2 = S_{i^2+i}^2 = S_{i^2+i+1}^2 = I,$$

$$(4) \quad S_1 S_i = S_{i+1}, \quad S_1 S_{i^2} = S_{i^2+1}, \quad S_1 S_{i^2+i} = S_{i^2+i+1}, \quad S_{i^2} S_i = S_{i^2+i}.$$

Taking the inverse of every member of (4) and applying (3), we get

$$S_i S_1 = S_{i+1}, \quad S_{i^2} S_1 = S_{i^2+1}, \quad S_{i^2+i} S_1 = S_{i^2+i+1}, \quad S_i S_{i^2} = S_{i^2+i}.$$

To show that $S_i S_\mu = S_{i+\mu}$ for every mark μ , we note that

$$S_i S_{i+1} = S_i \cdot S_i S_1 = S_1, \quad S_i S_{i^2+1} = S_i \cdot S_{i^2} S_1 = S_{i^2+i} S_1 = S_{i^2+i+1}$$

$$S_i S_{i^2+i} = S_i \cdot S_i S_{i^2} = S_{i^2}, \quad S_i S_{i^2+i+1} = S_i \cdot S_{i^2+i} S_1 = S_{i^2} S_1 = S_{i^2+1}$$

Then $S_{i+1} S_\mu = S_{i+1+\mu}$ follows at once, since

$$S_{i+1} S_\mu = S_1 S_i S_\mu = S_1 S_{i+\mu} = S_{1+i+\mu}.$$

That $S_\lambda S_\mu = S_{\lambda+\mu}$ ($\lambda = i^2, i^2 + 1, i^2 + i, \text{ or } i^2 + i + 1$) follow quite similarly.

4. For $\lambda = 0$ or 1 or for $\mu = 0$ or 1, relation (2) reduces to *

$$(5) \quad (TS_1)^3 = I.$$

Among the remaining relations (2) occur the following six :

$$(6) \quad S_i TS_i TS_{i^2+i} TS_{i^2+1} TS_{i^2+i} T = I,$$

$$(7) \quad S_{i^2} TS_{i^2} TS_i TS_{i^2+i+1} TS_i T = I,$$

$$(8) \quad S_{i+1} TS_{i+1} TS_{i^2+1} TS_{i^2} TS_{i^2+1} T = I,$$

$$(9) \quad S_{i^2+1} TS_{i^2+1} TS_{i^2+i+1} TS_{i^2+i} TS_{i^2+i+1} T = I,$$

$$(10) \quad S_{i^2+i} TS_{i^2+i} TS_{i^2} TS_{i+1} TS_{i^2} T = I,$$

$$(11) \quad S_{i^2+i+1} TS_{i^2+i+1} TS_{i+1} TS_i TS_{i+1} T = I.$$

From these follow all the relations (2) in which neither λ nor μ is 0 or 1. For, in (6)–(11), each subscript $i, i + 1, i^2, i^2 + 1, i^2 + i, i^2 + i + 1$ is followed immediately by all of the others, excepting its reciprocal, provided the first subscript be regarded

* Thus, for $\lambda = 0$, (2) becomes $TS_\mu TS_1 TS_1 TS_{-\mu+1} T = I$. Transforming both members by TS_μ , we get $TS_1 TS_1 TS_1 = I$.

as following the last ; while from (2) results an equivalent relation by using for the initial λ, μ any two consecutive subscripts. We have therefore the following

THEOREM.* *The abstract simple group of order 504 is generated by the operators $T, S_1, S_i, S_{i+1}, S_{i^2}, S_{i^2+1}, S_{i^2+i}, S_{i^2+i+1}$ subject to the generational relations $T^2 = I$ and (3), (4), (5), (6), (7), (8), (9), (10), (11).*

Second Set of Generational Relations, §§ 5-9.

5. Transforming both members of (8) by TS_{i+1} , we get

$$(12) \quad S_{i+1} TS_{i+1} TS_{i+1} TS_{i+1} = S_{i^2+i} TS_{i^2} TS_{i^2+i}.$$

Denoting each product by P , we form PTP and get

$$S_{i+1}(TS_{i+1})^7 = S_{i^2+i} TS_{i^2} TS_{i^2+i} TS_{i^2+i} TS_{i^2} TS_{i^2+i}.$$

In view of (10), the second member equals

$$S_{i^2+i} TS_{i^2} T \cdot TS_{i^2} TS_{i+1} \cdot S_{i^2+i} = S_{i+1}.$$

Hence

$$(13) \quad (TS_{i+1})^7 = I.$$

Set $A \equiv TS_{i+1}, B \equiv S_i$. By (3), (4), (5), (13),

$$(14) \quad A^7 = I, B^2 = I, (AB)^3 = I.$$

6. From (10) and (7), we obtain respectively

$$S_{i^2} TS_{i^2} = S_{i+1} TS_{i+1} TS_i TS_{i+1} TS_{i+1} = A^{-2} B A^2,$$

$$S_{i^2+i+1} = TS_i TS_{i^2} TS_{i^2} TS_i T.$$

* Note added November 10. *The relations (8), (9), (10) may be omitted. Indeed, we may write (6) thus :*

$$S_i TS_i TS_{i^2+i+1} S_1 TS_{i^2+1} TS_{i^2+i} T = I.$$

Replacing $TS_i TS_{i^2+i+1}$ by $S_{i^2} TS_{i^2} TS_i T$, as may be done by (7), and then $TS_i T$ by $S_1 TS_1$, as may be done by (5), we obtain (10), with its subscripts permuted cyclically. Again, we may write (6) thus :

$$S_i TS_{i+1} S_1 TS_{i^2+i} TS_{i^2+1} TS_{i^2+i} T = I.$$

Replacing $S_i TS_{i+1}$ by $TS_{i+1} TS_{i^2+i+1} TS_{i^2+i+1} T$, in virtue of (11), and then $TS_i T$ by $S_1 TS_1$, in virtue of (5), we obtain a relation immediately equivalent to (9). Finally, we may write (7) thus :

$$S_{i^2} TS_{i^2} TS_1 \cdot S_{i+1} TS_{i^2+i+1} TS_i T = I.$$

Replacing $S_{i+1} TS_{i^2+i+1} T$ by $TS_i TS_{i+1} TS_{i^2+i+1}$, in virtue of (11), and then $TS_i T$ by $S_1 TS_1$, we obtain (8).

But $ABA^{-1} = TS_i T$. Hence

$$(15) \quad S_{i^2+i+1} = ABA^{-3}BA^3BA^{-1},$$

$$(16) \quad S_{i^2+1} = B \cdot ABA^{-3}BA^3BA^{-1} = A^{-1}BA^3BA^3BA^{-1},$$

since $BAB = A^{-1}BA^{-1}$ by (14). Since S_{i^2+1} is of period 2,

$$(17) \quad (A^3BA^5BA^3B)^2 = I.$$

7. We next express all the generators of the group in terms of A and B . Transforming (8) by TS_{i^2+1} , we get

$$S_{i^2+1}A^2TS_1S_2TS_2T = I.$$

Replacing S_{i^2+1} and S_2TS_2 by their values and TS_1 by AB ,

$$(18') \quad T = A^{-1}BA^3BA^3BA^2BA^5BA^2.$$

Since $BA^2B = A^{-1}BA^{-2}BA^{-1}$ by (14), we get

$$\begin{aligned} T &= A^{-1}BA^3BA^2BA^{-2}BA^4BA^2 \\ &= A^{-1}BA^2BA^{-2}BA^{-3}BA^4BA^2. \end{aligned}$$

Replacing $BA^{-3}BA^4BA^2$, of period 2 by (17), by its inverse, we get

$$(18) \quad T = A^{-1}BA^2BA^3BA^3BA^3B.$$

8. The S_μ are given by (15), (16) and the relations

$$S_1 = TAB, S_{i+1} = T^{-1}A, S_{i^2+i} = TABS_{i^2+i+1}, S_{i^2} = S_{i^2+i}B.$$

Some of these relations may be simplified. Thus,

$$\begin{aligned} S_{i^2+i} &= A^{-1}BA^2BA^3BA^3BA^3B \cdot AB \cdot ABA^{-3}BA^3BA^{-1} \\ &= A^{-1}BA^2BA^3BA^3BA^{-1}BA^3BA^{-1} && [\text{by (14)}] \\ &= A^{-1}BA^2BA^3BA^4BA^4BA^{-1} && [BA^{-1}B = ABA] \\ &= A^{-2}BA^{-2}BA^2BA^4BA^4BA^{-1} \\ & && [BA^2B = A^{-1}BA^{-2}BA^{-1}] \\ &= A^{-2}BA^{-2}B \cdot BA^3BA^3BA^5 \cdot A^{-1} && [\text{by (17)}] \\ &= A^{-2}BABA^3BA^4 \\ &= A^{-3}BA^2BA^{-3} && [BAB = A^{-1}BA^{-1}]. \end{aligned}$$

Taking the inverse for S_{i^2+i} , we have the relations

$$(19) \quad \begin{cases} S_i = B, S_1 = TAB, & S_{i^2+i} = A^3BA^5BA^3, \\ S_{i+1} = T^{-1}A, & S_{i^2} = A^3BA^5BA^3B. \end{cases}$$

9. THEOREM. *The simple group of order 504 is generated by two operators A and B subject to the generational relations (14) and (17).*

It remains to show that the operators defined by (15), (16), (18) and (19) satisfy the relations $T^2 = I$, (3)–(11), in view of relations (14) and (17) alone. We derive at once the relations

$$\begin{aligned} S_i^2 &= I, S_{i^2+i+1}^2 = I, S_{i^2+1}^2 = I, S_{i^2}^2 = I, S_{i^2}S_i = S_{i^2+i}, \\ S_{i^2+i}^2 &= A^3BA^5 \cdot BA^{-1}B \cdot A^5BA^3 \\ &= A^3BA^{-1}BA^{-1}BA^3 = A^7 = I. \end{aligned}$$

Since $BA^2B = A^{-1}BA^{-2}BA^{-1}$ by (14), we get

$$\begin{aligned} T^2 &= (A^5BA^5BA^2BA^3BA^3B)^2 \\ &= A^5BA^5BA^2 \cdot A^2BA^4BA^4 \cdot A^5BA^2BA^3BA^3B \\ &= A^5BA^5 \cdot BA^4BA^4BA^2B \cdot A^2BA^3BA^3B && \text{[by (17)]} \\ &= A^5BA^5 \cdot A^5BA^3BA^3 \cdot A^2BA^3BA^3B && \text{[by (17)]} \\ &= (A^5BA^3BA^3B)^2 = I && \text{[by (17)].} \end{aligned}$$

That $S_{i+1} \equiv T^{-1}A$ has period 2 is shown as follows:

$$\begin{aligned} ATAT &= BA^2BA^3BA^3BA^5BA^3BA^3BA^3B \\ &= BA^2 \cdot A^2BA^4BA^4 \cdot A^3BA^3BA^3B = I \\ & && \text{[by (17)].} \end{aligned} \tag{18}$$

We may show that $S_1^2 = I$ by indicating the square of

$$(20) \quad T^{-1}AB = BA^4BA^4BA^4BA^5BA^2B$$

and replacing $BA^{-1}B$ by ABA six times in succession. We have therefore derived all the relations (3) and $T^2 = I$. Next,

$$\begin{aligned} S_1S_i &= TAB \cdot B = T^{-1}A = S_{i+1}; \\ S_1 &= TAB = A^{-1}BA^3BA^3BA^2BA^5BA^3B = S_{i^2+1}S_{i^2}, \end{aligned}$$

upon employing for T formula (18') which follows from (18), (17), (14). Hence $S_1 S_{i^2} = S_{i^2+1}$. Again,

$$S_1 S_{i^2+i} = S_1 S_{i^2} S_i = S_{i^2+1} S_i = (S_i S_{i^2+1})^{-1} = S_{i^2+i+1}^{-1} = S_{i^2+i+1}.$$

Hence relations (4) have all been derived.

It remains to derive relations (5)–(11). From what precedes, we may make use of relations (1). Since $TS_1 = AB$, (5) follows from (14). We next verify that

$$(21) \quad S_{i^2} TS_{i^2} = A^{-2} BA^2.$$

Employing $S_{i^2} = S_{i^2}^{-1} = BA^4 BA^2 BA^4$ and the value of T from (18), we obtain as the condition for (21),

$$A^2 BA^4 BA^2 BA^3 BA^2 BA^3 BA^5 BA^2 B = I.$$

Replacing the first two products $BA^2 B$ by $A^{-1} BA^{-2} BA^{-1}$, we get

$$A^2 BA^3 BA^{-2} B A B A^{-2} BA^2 BA^5 BA^2 B = I.$$

Replacing BAB by $A^{-1} BA^{-1}$ and then $BA^4 BA^4 BA^2$ by its inverse, in view of (17), we get

$$A^2 B A B A^3 B A B A^2 B = I,$$

a relation following readily from (14). As in §6, (10) and (7) follow from (21), (15), (16), and

$$(22) \quad ABA^{-1} = AS_{i+1} S_i S_{i+1} A^{-1} = TS_i T.$$

From (18') and (21) follows (8) as in §7. Note that (10) also follows from (8), and $A^7 = I$, as in §5.

Employing (22), relation (6) may be written

$$B \cdot ABA^{-1} \cdot S_{i^2+i} \cdot TS_1 \cdot S_{i^2} TS_{i^2} \cdot S_i T = I.$$

Applying (14), (19) and (21), the relation becomes

$$A^{-1} BA^{-2} \cdot A^3 BA^5 BA^3 \cdot AB \cdot A^{-2} BA^2 \cdot BT = I.$$

By (14) we may replace $BA^5 B$ and $BA^{-2} B$ by $ABA^2 BA$:

$$A^{-1} BA \cdot ABA^2 BA \cdot A^4 \cdot ABA^2 BA \cdot A^2 BT = I.$$

Replacing $BA^6 B$ by ABA , we get, by (18),

$$A^{-1} BA^2 BA^3 BA^3 BA^3 BT = T^2 = I.$$

Transforming (11) by T , we get

$$TS_{i+1} \cdot S_{i^2} TS_{i^2} \cdot S_{i+1} T \cdot S_{i+1} T \cdot S_i \cdot TS_{i+1} = I.$$

Setting $TS_{i+1} = A$ and $S_i = B$, and applying (21), it becomes an identity.

To verify (9), we write it in the form

$$S_{i^2+i} \cdot TS_1 \cdot S_{i^2} TS_{i^2} \cdot S_{i+1} T \cdot S_i \cdot S_{i^2} TS_{i^2} \cdot S_{i+1} T = I.$$

Applying (16), (21), $TS_1 = AB$, $S_{i+1} T = A^{-1}$, $S_i = B$, it becomes

$$A^{-1}BA^3BABABA^{-2}BA = I,$$

which is an identity since $BABAB = A^{-1}$ by (14).

Third Set of Generational Relations, §§10-12.

10. Consider the two operators

$$C = S_i T, \quad D = S_{i+1}.$$

From (6) and (8) we obtain respectively

$$(S_i T)^4 = S_{i^2} TS_{i^2+i} TS_{i^2} T, \quad TS_{i^2+i} TS_{i^2} T = S_{i+1} TS_{i+1} TS_{i^2+i}.$$

Hence, applying also (11), we derive

$$(S_i T)^9 = (S_{i^2+i+1} TS_{i+1} TS_{i^2+i})^2 S_i T = I.$$

Also, $DC \equiv S_1 T$ is of period 3 by (5). Hence

$$(23) \quad D^2 = I, \quad C^9 = I, \quad (DC)^3 = I.$$

11. Noting that $C^{-1}DC = TS_{i+1} T$, we may write (11) thus :

$$S_{i^2+i+1} TS_{i^2+i} \cdot DC \cdot D \cdot C^{-1} \cdot C^{-1} DC = I,$$

$$S_{i^2+i+1} TS_{i^2+i} = C^{-1} DC^2 DC^{-1} D = C^{-1} DC^3 DC.$$

Multiplying the latter on the left by S_{i+1} and on the right by $S_i T$, we get

$$S_{i^2} TS_{i^2} T = DC^{-1} DC^3 DC^2 = CDC^4 DC^2.$$

Applying this result in (7), we get

$$S_{i^2+i+1} T = C^6 DC^5 DC^7.$$

We now obtain the formulæ *

* It may be shown that $S_{i^2+i+1} = DC^7 DC^3 DC^7$.

$$(24) \quad S_{i^2+i} = S_{i^2+i+1} T \cdot TS_1 = C^6 DC^5 DC^6 D,$$

$$(25) \quad S_{i^2+1} = S_{i^2+i} S_{i+1} = C^6 DC^5 DC^6,$$

$$(26) \quad S_{i^2} T = S_{i^2+i}^{-1} S_i T = DC^3 DC^4 DC^4.$$

Since S_{i^2+1} is of period 2, it follows from (25) that

$$(27) \quad (DC^5 DC^3)^2 = I.$$

We may write (6) in the form

$$C^2 \cdot S_{i^2} T \cdot TS_i \cdot TS_{i^2} \cdot S_1 T \cdot S_{i^2+i} \cdot T = I.$$

Applying (26) twice and (24) once, this relation gives

$$C^2 DC^3 DC^4 DC^{-1} DC^5 DC^4 DC^5 DC^6 D = T.$$

Replacing $DC^{-1}D$ by CDC in view of (23) and then $C^3 DC^5 D$ by $DC^4 DC^6$, in view of (27), we get

$$(28) \quad T = C^6 DC^3 DC^4 DC^5 DC^6 D.$$

From the definitions of A and B at the end of § 5, we get

$$(29) \quad A = TD = C^6 DC^3 DC^4 DC^5 DC^6,$$

$$(30) \quad B = CT^{-1} = CDC^3 DC^4 DC^5 DC^6 DC^3.$$

12. THEOREM. *The simple group of order 504 is generated by two operators C and D , subject to the generational relations (23) and (28).*

In view of the theorem of § 9, it suffices to show that, if A and B are defined by (29) and (30), relations (14) and (17) follow from relations (23) and (27). First,

$$(31) \quad BA = CT^{-1} \cdot TD = CD,$$

so that BA is of period 3 in view of (23). To show that $B^2 = I$, we replace $DC^6 DC^4$ by its inverse $C^5 DC^3 D$ in the indicated square of B and get

$$B^2 = CDC^3 DC^4 \cdot DCD \cdot C^6 DC^4 DC^5 DC^6 DC^3.$$

Replacing DCD by $C^{-1}DC^{-1}$ and, in the result, $DC^3 DC^5$ by $C^4 DC^6 D$, we get

$$B^2 = CDC^7 DCD C^5 DC^6 DC^3.$$

Making similar replacements, we get $B^2 = C^6 DC^9 DC^3 = I$.

In the indicated product AA , we replace DC^5DC^3 by C^6DC^4D , then DCD by $C^{-1}DC^{-1}$, then C^6DC^4D by DC^5DC^3 , then $DC^{-1}D$ by CDC twice, and get

$$(32) \quad A^2 = C^7DC^8.$$

It follows at once that

$$A^6 = C^7DC^6DC^6DC^8.$$

It will equal the expression for A^{-1} given by (29) if

$$C^4DC^6DC^6DC^5 = DC^4DC^5DC^6D.$$

Replacing C^4DC^6D by DC^3DC^5 , the condition becomes

$$I = C^4DC^7DCDC^5DC^6D.$$

Replacing DCD by $C^{-1}DC$ and then C^6DC^4D by DC^5DC^3 , the condition reduces to an identity in view of $D^2 = I$, $C^9 = I$.

Finally, to derive (17), we note that

$$BA^3BA^5BA^3 = BA \cdot A^2 \cdot BA \cdot A^4 \cdot BA \cdot A^2,$$

and substitute the values of BA and A^2 given by (31) and (32). The result $CDC^5DC^4DC^8$ is evidently of period 2 by (23).

Generators of the Linear Fractional Group of Order 504.

13. For the first set of generational relations, we may take

$$T: z' = \frac{1}{z}; \quad S_\mu: z' = z + \mu.$$

Hence, for the second set, we may take

$$A: z' = \frac{1}{z} + i + 1; \quad B: z' = z + i.$$

For this second set, Burnside employs

$$A': z' = iz; \quad B': z' = \frac{z + i^2}{i^2z + 1}.$$

We readily obtain the relations

$$A' = TS_{i+1}TS_iTS_iTS_iTS_{i+1} = S_{i^2+i+1}TS_{i^2+i}TS_{i+1}TS_{i+1},$$

$$B' = S_{i+1}TS_{i+1}TS_{i^2+i+1}TS_{i^2+i}.$$

For the third set, we may evidently take

$$C: z' = 1 \div (z + i); \quad D: z' = z + i + 1.$$

Note that $S_{i^2+1}T$ is of period 7; while

$$BA^3: z' = \frac{z+i+1}{z}$$

is of period 9; also, as a check on (27), that

$$DC^5DC^3: z' = \frac{z+i^2}{iz+1}.$$

THE UNIVERSITY OF CHICAGO,
October 19, 1902.

GENERATIONAL RELATIONS DEFINING THE ABSTRACT SIMPLE GROUP OF ORDER 660.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, October 25, 1902.)

THE abstract group of order 660, simply isomorphic to the group of linear fractional substitutions of determinant unity taken modulo 11, may be generated by two operators S and T subject to the following relations: *

$$(1) \quad S^{11} = I, \quad T^2 = I, \quad (ST)^3 = I,$$

$$(3 \ 3 \ 3 \ 3 \ 3), \quad (7 \ 7 \ 7 \ 7 \ 7), \quad (2 \ 2 \ 4 \ 8 \ 4), \quad (4 \ 4 \ 9 \ 7 \ 9),$$

$$(5 \ 5 \ 2 \ 9 \ 2), \quad (6 \ 6 \ 8 \ 9 \ 8), \quad (8 \ 8 \ 5 \ 3 \ 5), \quad (9 \ 9 \ 10 \ 8 \ 10),$$

$$(2 \ 3 \ 9 \ 6 \ 7), \quad (2 \ 7 \ 6 \ 9 \ 3), \quad (2 \ 8 \ 3 \ 7 \ 10), \quad (2 \ 10 \ 7 \ 3 \ 8),$$

$$(3 \ 6 \ 4 \ 5 \ 10), \quad (3 \ 10 \ 5 \ 4 \ 6), \quad (4 \ 7 \ 5 \ 6 \ 10), \quad (4 \ 10 \ 6 \ 5 \ 7),$$

where the symbol $(a \ b \ c \ d \ e)$ denotes the relation

$$S^a T S^b T S^c T S^d T S^e T = I.$$

These relations are very redundant. We proceed to reduce

* For a proof of the general theorem due to Moore, see Dickson, *Linear Groups*, § 278, Corollary. When the field is of order a prime p , we may set $S_i = S$, $S_i^2 = S^i$. By § 279, we need retain only the 16 relations (a, b, c, d, e) given above, since the others follow by a cyclic permutation of a, b, c, d, e . It may be shown that $S^2T, S^3T, S^4T, S^5T, S^6T, S^7T, S^8T, S^9T$ have the respective periods 12, 5, 5, 6, 6, 5, 5, 12, results not used here.