

## THE NINTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE ninth summer meeting of the AMERICAN MATHEMATICAL SOCIETY was held at Northwestern University, Evanston, Ill., on Tuesday and Wednesday, September 2-3, 1902. Morning and afternoon sessions were held on each day in Room 7 of the University Hall. The President of the Society, Professor Eliakim Hastings Moore, presided at the first session, and was relieved at the later sessions by Professor T. S. Fiske and Professor H. S. White.

The attendance numbered about fifty-five, including the following thirty-seven members of the society :

Professor Oskar Bolza, Mr. A. R. Crathorne, Professor L. E. Dickson, Professor L. W. Dowling, Professor John Eiesland, Dr. William Findlay, Professor T. S. Fiske, Dr. W. B. Fite, Dr. J. W. Glover, Professor G. W. Greenwood, Professor A. S. Hathaway, Professor T. F. Holgate, Dr. Edward Kasner, Dr. H. G. Keppel, Professor Kurt Laves, Dr. D. N. Lehmer, Professor Heinrich Maschke, Professor J. A. Miller, Professor E. H. Moore, Professor Frank Morley, Dr. F. R. Moulton, Professor H. B. Newson, Professor Alexander Pell, Professor D. A. Rothrock, Miss I. M. Schottenfels, Professor G. T. Sellew, Professor J. B. Shaw, Professor E. B. Skinner, Professor Ormond Stone, Professor E. J. Townsend, Professor H. W. Tyler, Professor C. A. Waldo, Professor H. S. White, Professor W. H. Williams, Professor J. W. A. Young, Mr. J. W. Young, Professor Alexander Ziwet.

The Council announced the election of the following persons to membership in the Society : Professor T. J. I'a. Bromwich, Queen's College, Galway, Ireland ; Mr. J. S. Brown, New York City ; Professor G. C. Edwards, University of California, Berkeley, Cal. Eight applications for membership were received.

The Council decided that the name of the new section, authorized at the last meeting, should be the San Francisco Section ; and the programme committee of the Section was approved. In accordance with a motion adopted by the Society, the Council appointed a special committee consisting of Professors Tyler (chairman), Fiske, Osgood, Young and Ziwet to

report on standard definitions of requirements in mathematical subjects for admission to colleges and scientific schools. The committee will cooperate with the corresponding committees of the Society for the promotion of engineering education, the National educational association and other interested bodies.

Besides the scientific sessions the members gathered together on various social occasions. In particular may be mentioned a dinner at the old academy building of the University, followed by a visit to the Observatory on invitation from the director, Professor Hough. Before adjourning, the Society by a rising vote expressed to Northwestern University and to the local members of the Society its appreciation of their hospitable efforts in providing for the meeting and the entertainment of the members.

The large attendance and the character of the programme is worthy of comment in view of the fact that this summer meeting is the first one held west of Buffalo.

The following papers were read at this meeting :

(1) DR. F. R. MOULTON : "A method of constructing general expressions for the elements of the planetary orbits which are valid for a finite time."

(2) PROFESSOR A. S. HATHAWAY : "The quaternion treatment of the problem of three bodies."

(3) DR. J. V. COLLINS : "A general notation for vector analysis."

(4) PROFESSOR L. E. DICKSON : "Definitions of a linear associative algebra by independent postulates."

(5) PROFESSOR L. E. DICKSON : "Definitions of a field by independent postulates."

(6) DR. E. V. HUNTINGTON : "Definitions of a field by sets of independent postulates."

(7) DR. OTTO DUNKEL : "Regular singular points of a system of homogeneous linear differential equations of the first order."

(8) PROFESSOR OSKAR BOLZA : "Some instructive examples in the calculus of variations."

(9) PROFESSOR J. B. SHAW : "On linear associative algebras."

(10) DR. W. B. FITE : "Concerning the commutator subgroups of groups whose orders are powers of primes."

(11) MR. J. W. YOUNG : "On a certain group of linear substitutions and the functions belonging to it" (preliminary report).

(12) PROFESSOR JACOB WESTLUND: "On the class number of the cyclotomic field  $k(e^{2\pi i}/p^m)$ ."

(13) PROFESSOR L. E. DICKSON: "Announcement of new simple groups" (preliminary communication).

(14) PROFESSOR ALFRED LOEWY: "Ueber die Reducibilität der Gruppen linearer homogener Substitutionen."

(15) PROFESSOR H. S. WHITE: "A special twisted cubic with rectilinear directrix."

(16) PROFESSOR F. MORLEY: "Orthocentric properties of the plane  $n$ -line."

(17) DR. VIRGIL SNYDER: "Forms of sextic scrolls of genus greater than 1."

(18) PROFESSOR PETER FIELD: "Quintic curves for which  $p = 1$ ."

(19) DR. C. J. KEYSER: "Concerning the line and plane geometries of point four-space, and allied theories."

(20) PROFESSOR ARNOLD EMCH: "A linkage for  $u = e^{i\theta}z$ ."

(21) DR. EDWARD KASNER: "The bilinear point-line connexion in space: an extension of Clebsch's connexion."

(22) MR. E. A. HOOK: "Multiple points on Lissajous's curves in two and three dimensions."

(23) DR. L. P. EISENHART: "Infinitesimal deformation of the skew helicoid."

(24) PROFESSOR JOHN EIESLAND: "Null systems in space of five dimensions."

(25) PROFESSOR E. D. ROE: "Note on a partial differential equation."

(26) DR. SAUL EPSTEEN: "On integrability by quadratures."

(27) MR. W. B. FORD: "On the possibility of differentiating term by term the developments for an arbitrary function of one real variable in terms of Bessel functions."

(28) PROFESSOR T. F. HOLGATE: "Apolar triads on a straight line and on a conic."

(29) PROFESSOR H. B. NEWSON: "List of continuous groups of collineations in space."

(30) MISS HELEN BREWSTER: "The group of collineations leaving a quadric surface invariant."

(31) PROFESSOR ALEXANDER PELL: "On the generalized Beltrami problem."

(32) PROFESSOR L. HEFFTER: "Ueber Curvenintegrale in  $m$ -dimensionalem Raum."

The papers of Dr. Dunkel and Mr. Hook were presented to the Society through Professor Bôcher, those of Professor Loewy and Professor Heffter through Professor Moore, that of Dr. Epsteen through Professor Lovett, and Miss Brewster's through Professor Newson. In the absence of the authors, Dr. Huntington's paper was read by Professor Dickson, Professor Westlund's and Professor Emch's by Professor Moore, Professor Loewy's by Professor Maschke, Dr. Snyder's and Professor Field's by Dr. Fite, and Miss Brewster's by Professor Newson. The papers of Dr. Collins, Dr. Dunkel, Dr. Keyser, Mr. Hook, Dr. Eisenhart, Professor Roe, Mr. Ford, and Professor Heffter were read by title.

Professor Bolza's paper was published in the October number of the BULLETIN. The papers of Dr. Fite and Dr. Epsteen will appear in later numbers of the BULLETIN. Abstracts of the other papers presented are given below. The abstracts are numbered to correspond to the titles in the list above.

1. In constructing expressions for the elements of the planetary orbits three things are desirable: (*a*) that they shall, if they are series, converge for at least a finite interval of time; (*b*) that it shall be possible practically to find their coefficients; and (*c*) that the time for which they are valid shall be very long. The method of developing the elements as power series in the masses can be proved to give results which converge for a finite interval of time, but these series have the practical inconvenience of containing secular terms even of the first order. The thought has been that these secular terms may be the expansions of periodic terms. Following out this idea, Lagrange has succeeded by quite arbitrary and up to the present unjustified processes in avoiding the secular terms of the first order in the masses and in the eccentricities. Secular terms of higher orders appear, and even if they did not, his results in no way prove the stability of the solar system, as is frequently stated. Dr. Moulton inquires whether these secular terms may not be avoided by processes which can be proved to be valid. The answer is in the affirmative, although the method of treatment differs from that of Lagrange. The series which are used fulfill condition (*a*); they fulfill (*b*) about as well as those of the standard methods; and they are probably valid for a much longer time than those now in use.

2. Professor Hathaway employs the quaternion analysis in the problem of three bodies, with the hope that the new and concise forms of that analysis may suggest important developments. Let  $\alpha, \beta, \gamma$  be the vector sides  $BC, CA, AB$  of a fixed triangle of reference, and put

$$S \cdot \beta\gamma = 1/a, \quad S \cdot \gamma\alpha = 1/b, \quad S \cdot \alpha\beta = 1/c, \quad UV\alpha\beta = k.$$

Then if  $P, Q, R$  be the positions of masses  $a, b, c$  at any time  $t$ , there is a definite strain  $\phi$  such that

$$\phi k = 0, \quad \phi ABC = PQR$$

whose differential equation is

$$(1) \quad \ddot{\phi} = \phi \left( \frac{bcaSa}{T^3\phi\alpha} + \frac{ca\beta S\beta}{T^3\phi\beta} + \frac{ab\gamma S\gamma}{T^3\phi\gamma} \right) = \phi\pi, \text{ say,}$$

or in conjugate form,  $\ddot{\phi}' = \pi\phi'$ . The area and energy integrals are

$$(2) \quad \dot{\phi}\phi' - \phi\dot{\phi}' = V\epsilon, \quad \frac{1}{2}S_1\dot{\phi}'\dot{\phi} + S_1\phi'\phi\pi = C,$$

where  $\epsilon$  is the perpendicular to the invariant plane, and  $C$  is the sum of the kinetic and potential energies.\* Forms are reduced when expressed in terms of  $\psi = \phi'\phi$ , and  $\pi$  is already reduced since  $T^2\phi\rho = -S \cdot \rho\psi\rho$ .

We have  $\dot{\phi}'\phi - \phi'\dot{\phi} = xVk$ , which gives by differentiation  $\pi\psi - \psi\pi = xVk$ , a reduced differential equation for  $x$ . Also by differentiating  $\psi$  twice we find  $2\dot{\phi}'\dot{\phi} = \Phi$ , say, in reduced form, so that the identity

$$[k(\psi + xVk)k\psi k(\psi - xVk) + 4S_2\psi \cdot k\Phi]^2 = 0$$

or

$$(3) \quad x^4 + 2x^2(S \cdot k\psi k\Phi + S_2\psi) + 4xS_1 \cdot k\psi k\psi k\Phi + S_2\psi^2 + 4S_2 \cdot \psi\Phi - 2S_1 \cdot k\psi k\psi k\psi k\Phi = 0$$

is a reduced equation for  $x$  itself. This agrees with Lagrange,  $x$  being to a constant factor his  $\rho$ ; but does not agree with

\* The cubic in any  $\phi$  is  $\phi^3 - S_1\phi \cdot \phi^2 + S_2\phi \cdot \phi - S_3\phi = 0$ . Thus  $S_r$  is an operator on a strain which gives a characteristic scalar. Products under  $S_r$  are cyclically commutative and replaceable by their conjugates.  $S_1$  is distributive over a sum, and  $S_3$  over a product. In the present case  $S_3 = 0$ , and  $S_2$  has the distributive property of  $S_3$  for coplanar factors.

Dziobek [Planetary motions, page 56 (19)]. I have verified, directly and indirectly, that (3) is a particular integral of the differential equation for  $x$  in connection with (1), the general integral being found by replacing  $x$  by  $x + C$  in (3). This settles a question which Dziobek says was undetermined (Planetary motions, page 57).

By differentiating  $\psi$  three times and substituting, we find the reduced form of (1)

$$(4) \quad \ddot{\psi} = 2(\dot{\psi}\pi + \pi\dot{\psi}) + \pi\dot{\psi} + \psi\pi + x(Vk\pi - \pi Vk).$$

This is equivalent to three ordinary equations of third order between the mutual distances of the bodies, found by operating with  $S_1$ ,  $S_1 \cdot \psi$ ,  $S_1 \cdot \pi$ . The first two are straight differentials of the reduced forms of (2), so that with these integrations the final forms are

$$(5) \quad S_1 \ddot{\psi} + 2S_1 \cdot \psi\pi = 4C,$$

$$(6) \quad S_1 \cdot \psi\dot{\psi} = 2S_1 \cdot \psi^2\pi + \frac{1}{2}S_1 \cdot \dot{\psi}^2 - x^2 + 2T^2\epsilon,$$

$$(7) \quad S_1 \cdot \pi\ddot{\psi} = 2S_1 \cdot \pi\dot{\psi} + 4S_1 \cdot \pi^2\dot{\psi}.$$

Dziobek does not note that the area integral (6) is one of the integrals of (4).

3. The different branches of vector analysis are very much alike in several respects. They deal with the same fundamental concept, the vector, and employ largely the same derived concepts, the scalar and vector products and their combinations. Apparently then there is no reason why a single consistent notation should not be used in all the branches of the subject. Instead of this we seem to be tending in the opposite direction, as is evidenced in Professor Gibbs's new book.

Professor Gibbs has made certain improvements in the notation which will doubtless stand. He avoids Hamilton's cumbersome  $T$  and  $U$  and puts his signs of multiplication *between* instead of to one side of the factors, thus admitting of a new sign of operation being put on either side. This Professor Gibbs was led to do doubtless through his study of dyadics. However, instead of retaining Hamilton's letters or Grassmann's notation for the scalar and vector products, he has introduced two new marks, the dot and the cross, to denote them.

Now all interests can be easily harmonized in the following way:—Let  $a^*b$ ,  $a^ab$ ,  $a^2b$ ,  $ab$  denote respectively  $Sab$ ,  $Vab$ , quaternion  $ab$ , dyadic  $ab$ . Also let  $a^*b^*c$  be denoted by  $[abc]$ , as is done by Gibbs, whenever this notation is more convenient. In this way are retained Hamilton's notation, and the good points in Gibbs's and Grassmann's notations. There is another very good reason why Gibbs's dot and cross should be given up and the letters retained. It is that the use of letters admits of an easy extension of the notation to new products. That there are such besides the quaternion the author has demonstrated by constructing them and putting them to practical use.

4. In the usual theory of higher complex members, the coördinates are either real numbers or else ordinary complex quantities. To avoid the resulting double phraseology and to attain an evident generalization of the theory, Professor Dickson considers systems of complex numbers whose coördinates belong to an arbitrary field  $F$ . The usual definition rests upon a multiplication table for  $n$  units. In essence, a system of complex numbers depends upon  $n^3$  constant marks  $\gamma_{iks}$  of  $F$  subject to the three conditions

$$\sum_{s=1}^n \gamma_{iks} \gamma_{slt} = \sum_{s=1}^n \gamma_{kls} \gamma_{ist} \quad (i, k, l, t = 1, 2, \dots, n),$$

$$\Delta_a \equiv \left| \sum_{i=1}^n \gamma_{iks} a_i \right| \text{ not zero for every } a_1, \dots, a_n,$$

$$\Delta_a' \equiv \left| \sum_{k=1}^n \gamma_{iks} a_k \right| \text{ not zero for every } a_1, \dots, a_n.$$

These conditions are shown to be independent by using systems with  $n = 2$ .

For the second definition the elements are  $A = (a_1, a_2, \dots, a_n)$ , where  $a_1, \dots, a_n$  are marks of  $F$ . Addition is defined thus:

$$A + B = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n).$$

A second rule of combination of elements is defined by four properties: (1) For any two elements  $A$  and  $B$  of the system,  $A \cdot B$  is an element whose coördinates are bilinear functions of the coördinates of  $A$  and  $B$ , with fixed coefficients belonging to  $F$ . (2)  $(A \cdot B) \cdot C = A \cdot (B \cdot C)$ , if  $A \cdot B$ , etc., belong to the

system. (3) There exists in the system an element  $I$  such that  $A \cdot I = A$  for every element  $A$ . (4) There exists in the system at least one element  $A$  such that  $A \cdot Z \neq 0$  for any element  $Z \neq 0$ . The elements are shown to form a system of complex numbers according to the usual definition. The four properties are shown to be independent. The paper will appear in the *Transactions*.

5. Professor Dickson's second paper gives a definition of a field by means of nine independent postulates and a second definition by means of eleven independent postulates. Of various possible sets of postulates, these two sets (especially the second) appear to present most naturally the elementary properties necessarily holding for a field and in a manner most suitable for actually testing for the field property.

A set of elements with two rules of combination denoted by  $\circ$  and  $\square$  forms a field if the following properties hold: (1) If  $a$  and  $b$  belong to the set, then also  $a \circ b$  belongs to the set. (2)  $a \circ b = b \circ a$ , whenever  $a \circ b$  and  $b \circ a$  belong to the set. (3)  $(a \circ b) \circ c = a \circ (b \circ c)$ , whenever  $a \circ b$ ,  $b \circ c$ ,  $(a \circ b) \circ c$ , and  $a \circ (b \circ c)$  belong to the set. (4) For any two elements  $a$  and  $b$  of the set, there exists in the set an element  $x$  such that  $(a \circ x) \circ b = b$ . (5), (6), (7) are the same as (1), (2), (3) with  $\circ$  replaced by  $\square$ . (8) For any two elements  $a$  and  $b$  of the set, such that  $c \square a \neq a$  for at least one element  $c$ , there exists in the set an element  $x$  such that  $(a \square x) \square b = b$ . (9)  $a \square (b \circ c) = (a \square b) \circ (a \square c)$ , whenever  $a \square b$ ,  $a \square c$ , etc., belong to the set.

The second set employs postulates (1), (2), (3), (5), (6), (7), (9) and the following four: (4') There exists in the set an element  $z$  such that  $z \circ b = b$  for every element  $b$ . (4'') If elements  $z$  of character (4') exist, then for some such element  $z$  and for every element  $a$ , there exists an element  $x$  for which  $a \circ x = z$ . (8') There exists an element  $u$  such that  $u \square b = b$  for every element  $b$ . (8'') If elements  $u$  of character (8') exist, then for some such element  $u$  and for every element  $a$  such that  $c \square a \neq a$  for at least one element  $c$ , there exists an element  $x$  for which  $a \square x = u$ .

This paper will appear in the *Transactions*.

6. Dr. Huntington's paper presents several sets of postulates which define a field, that is, an assemblage in which all the



rational operations of algebra can be performed. The simplest of these sets contains the following seven postulates, where the symbols  $\oplus$  and  $\odot$  indicate abstract operations exemplified by addition and multiplication of rational numbers :

1) If  $a, b$  and either  $a \oplus b$  or  $b \oplus a$  belong to the field, then  $a \oplus b = b \oplus a$ .

2) If  $a, b, c, a \oplus b, b \oplus c$  and either  $(a \oplus b) \oplus c$  or  $a \oplus (b \oplus c)$  belong to the field, then  $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ .

3) For every two elements  $a$  and  $b$  ( $a = b$  or  $a \neq b$ ) there is an element  $x$  in the field such that  $a \oplus x = b$ .

4), 5) The same as 1) and 2), when  $\oplus$  is replaced by  $\odot$ .

6) For every two elements  $a$  and  $b$  ( $a = b$  or  $a \neq b$ ), provided  $a \oplus a \neq a$ , there is an element  $y$  in the field such that  $a \odot y = b$ .

7) If  $a, b, c, b \oplus c, a \odot b, a \odot c$  and either  $a \odot (b \oplus c)$  or  $(a \odot b) \oplus (a \odot c)$  belong to the field, then  $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$ .

The independence of the postulates of each set is established by the now familiar method of Peano and Hilbert. The paper will appear in the *Transactions*.

7. In this paper, which will be published in the *Proceedings the American Academy of Arts and Sciences*, Mr. Dunkel considers a system of differential equations

$$\frac{dy_i}{dx} = \sum_{j=1}^{j=n} \left( \frac{\mu_{i,j}}{x} + a_{i,j} \right) y_j \quad (i = 1, 2, \dots, n)$$

in which the  $\mu_{i,j}$ 's are constants and the  $a_{i,j}$ 's are functions of the real independent variable  $x$ , not necessarily analytic, but continuous in the interval  $0 < x \leq b$ . The absolute values,  $|a_{i,j}|$ , multiplied by certain positive integral powers of  $\log x$  are required to be integrable up to the point  $x = 0$ ; these powers vary according to the set of solutions being developed, and they may in some cases be zero. When the  $a_{i,j}$ 's satisfy these conditions, the point  $x = 0$  is called a regular singular point. Use is made of a theorem on elementary divisors to reduce the system of equations to a canonical form. Solutions of the canonical system are developed by the method of successive approximations; and from these the form of the solutions of the original system is then determined. The results thus obtained are applied to the single differential equation

$$\frac{d^n y}{dx^n} + \left(\frac{\mu_1}{x} + p_1\right) \frac{d^{n-1} y}{dx^{n-1}} + \left(\frac{\mu_2}{x} + p_2\right) \frac{1}{x} \frac{d^{n-2} y}{dx^{n-2}} + \cdots \\ + \left(\frac{\mu_n}{x} + p_n\right) \frac{1}{x^{n-1}} y = 0,$$

in which the  $\mu$ 's are constants, and the  $p$ 's are functions of the real independent variable  $x$  satisfying the same requirements that were imposed upon the functions  $a_{i,j}$ .

Let  $r_\kappa$  be a root of multiplicity  $e_\kappa$  of the indicial equation. The multiplicities of all the roots of this equation whose real parts are the same as that of  $r_\kappa$  are considered, and  $e_\kappa$  is chosen as any one of these multiplicities as great as any other in this set. It is then required that the absolute values of the  $p$ 's after being multiplied by  $(\log x)^{e_\kappa - 1}$  be integrable up to  $x = 0$ ; and with this requirement it is found that we can develop the following  $e_\kappa$  solutions and their first  $n - 1$  derivatives corresponding to the root  $r_\kappa$ :

$$y^{\kappa\lambda} = x^{r_\kappa} (\log x)^{e_\kappa - \lambda} E_n^{\kappa\lambda}, \quad (\lambda = 1, 2, \dots, e_\kappa), \\ \frac{d^i y^{\kappa\lambda}}{dx^i} = x^{r_\kappa - i} (\log x)^{e_\kappa - \lambda} E_{n-i}^{\kappa\lambda},$$

where the functions  $E_{n-i}^{\kappa\lambda}$  are continuous in an interval  $0 \leq x \leq c$ , and

$$E_n^{\kappa\lambda}|_{x=0} = 1, \quad E_{n-i}^{\kappa\lambda}|_{x=0} = r_\kappa(r_\kappa - 1) \cdots (r_\kappa - i + 1).$$

In the case  $n = 2$  these results not only yield the theorems which were established by Professor Bôcher in the *Transactions* for January, 1900, but for the case of equal exponents go slightly further.

9. Professor Shaw's paper gives the reduction of all numbers of linear associative algebras to typical forms, and a new basis of classification of such algebras. It makes use of a previously discovered form involving units of the type  $\lambda_{ij}$  such

$$\text{that} \quad \lambda_{ijk} \cdot \lambda_{i'j'k'} = 0 \quad \text{if } j \neq i' \\ = \lambda_{ij'k+k'} \quad \text{if } j = i'.$$

The units are also subject to restrictions as to the size of the num-

ber  $k$ , represented by the third subscript. The method is given of deriving all algebras to which belong numbers of which the most general number satisfies a given characteristic equation. The place of Peirce's, Scheffer's and Molien's investigations is also shown, with an examination of the boundaries of these investigations and an extension and generalization of their results. The paper is a conclusion to one presented before the Chicago Section, January, 1902.

11. The moduli of periodicity for the normal integrals of the first kind on the Riemann surface for  $w^4 = (z - \kappa_1)^2(z - \kappa_2)^2(z - \kappa_3)(z - \kappa_4)^3$  depend on a single parameter  $\omega$  which undergoes a linear substitution by every monodromy of the branch points. Mr. Young's paper considers the group  $\Gamma$  of all such substitutions, which is found to be generated by the two,  $\omega' = \omega + \sqrt{2}$  and  $\omega' = -1/\omega$ . All functions automorphic for the group are shown to be rationally expressible in terms of the anharmonic ratio of the branch points.

The properties of  $\Gamma$  are studied in detail, the investigation following the lines laid down by Klein in his treatment of the (elliptic) modular functions, and results analogous to practically all of Klein's results are obtained. The most noteworthy difference is found in the fact that  $\Gamma$  contains an infinite series of subgroups of genus  $p = 0$ . These subgroups are of index  $2n$  and are isomorphic with the dihedron group of order  $2n$ ; two sets of generators are determined for each.

The paper closes with a discussion of the quotient groups of  $\Gamma$  with the "principal" congruence subgroups, the latter consisting of all substitutions of  $\Gamma$  that are congruent to identity (mod.  $q$ ). When  $q$  is an odd prime, the quotient group is shown to be of order  $\frac{1}{2}q(q^2 - 1)$  or  $q(q^2 - 1)$  according as  $q$  is of the form  $8h \pm 1$  or  $8h \pm 3$ . In the former case the quotient group is simply isomorphic with Klein's  $G_{\frac{1}{2}q(q^2-1)}$ ; in the latter case it contains a subgroup of half its order which is simply isomorphic with  $G_{\frac{1}{2}q(q^2-1)}$ .

12. Dr. Westlund's paper, which is an extension of a paper presented at the meeting of the Chicago section in January last, treats of the relation between the class numbers of the cyclotomic number field  $k\left(e^{\frac{2\pi i}{p^n}}\right)$ , when  $p$  is any odd prime, and its subfield  $k\left(e^{\frac{2\pi i}{p^{n-1}}}\right)$ . The method used is similar to that used by Weber for the case  $p = 2$ .

13. Professor Dickson's third paper is an addition to his article, "Theory of linear groups in an arbitrary field," *Transactions*, volume 2 (1901), pages 363-394. By certain modifications, the methods of §§ 9-10 may be made to apply to the there excluded case of modulus 2, provided  $n > 1$ . Hence there results a new system of simple groups of order  $2^{6n}(2^{6n}-1)(2^{2n}-1)$ , for any integer  $n > 1$ . There remains for subsequent investigation the case  $n = 1$ , the order being then 12096. For  $n = 2$  and  $n = 3$  the orders of the new simple groups are respectively  $2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$ ,  $2^{18} \cdot 3^5 \cdot 7^2 \cdot 19 \cdot 73$ .

14. The paper of Professor Loewy concerns the reducibility of groups of linear homogeneous substitutions, and is in abstract as follows: A group  $G$  of linear homogeneous substitutions in  $n$  variables is styled reducible if there exist  $m < n$  linear homogeneous functions of the variables with constant coefficients, such that under the group they are transformed in a linear homogeneous way. The paper considers the following fundamental theorem: To every reducible group  $G$  of linear homogeneous substitutions there belongs a definite finite number of irreducible groups of linear homogeneous substitutions, which are uniquely determined apart from the order, in case one considers similar groups as identical. This theorem is applied in the first place to the theory of groups of a finite number of linear homogeneous substitutions and then to the theory of those infinite groups introduced by the author in volume 53, page 225, of the *Mathematische Annalen* and designated as groups of the type of a finite group. Further, there is introduced the more general notion of reducibility or irreducibility of a group  $G$  of linear homogeneous substitutions with respect to a realm of rationality or field (Körper)  $\Omega$  to which all the coefficients of the various substitutions of the group belong, and the fundamental theorem is treated from this standpoint. In conclusion, the following theorem is proved: In order to exhibit all the finite groups belonging to a given field  $\Omega$ , it is sufficient to exhibit all the finite groups which are irreducible with respect to the field  $\Omega$ .

15. A twisted cubic has three osculating planes through every point. There are always two points in which these three planes are perpendicular to one another. If there are more than two such points, there must be an infinite number,

and their locus is then called the directrix of such a specialized cubic curve. For the case where the curve is a parabola, *i. e.*, if the plane at infinity osculates the curve, O. Böklen and W. F. Meyer have shown that its directrix is a straight line, and have discussed the conditions necessary for its existence. Professor White's paper is an examination of the same problem for cubical ellipses and hyperbolas. By analogy with plane conics one would look for a curvilinear directrix, if any; but it is proven that it must be straight. The conditions for its existence are found in the identical vanishing of a ternary quadric, giving three conditions of which, however, only two are independent. Meyer's twisted cubics with a directrix are defined by three conditions, while this new class is subject to only two; the latter, however, does not include the former class, but the two together comprise all possible twisted cubics with directrix.

16. Professor Morley's memoir is a continuation of one published in the *Transactions*, volume 1, page 97. What corresponds, for a plane  $n$ -line, to the elementary facts that for a 3-line the perpendiculars meet at a point, and for a 4-line the 4 such points lie on a line?

First, four lines determine uniquely a hypocycloid of class 3,  $\Delta^3$ ; if from the center of the  $\Delta^3$  touching each 4 of 5 lines a perpendicular be drawn to the fifth line, these five perpendiculars meet at a point; for 6 lines the 6 such points lie on a line. So there is a quite simple curve  $\Delta^{2n-1}$  uniquely determined by  $2n$  lines; if from the center of the  $\Delta^{2n-1}$  touching each  $2n$  of  $2n + 1$  lines a perpendicular be drawn to the line omitted, these perpendiculars meet at a point and for  $2n + 2$  lines the  $2n + 2$  such points lie on a line. The curve  $\Delta$  is for metrical purposes very convenient, much more so than Clifford's  $n$ -fold parabola, also uniquely determined by  $2n$  lines. Second, if from the nine-point center, or center of Feuerbach's circle, of each 3 of 4 lines, a perpendicular be drawn on the line left out, these perpendiculars meet at a point. And the center of a circle arising from  $n$  lines has the analogous property. Thus we have for an even number of lines an orthocenter and a directrix; for an odd number, two and therefore a line of orthocenters.

17. Dr. Snyder has applied the same methods which he employed in the study of unicursal and elliptic scrolls of order six to obtain the forms and equations of those of genus greater

than 1. There are eleven types of genus 2, five of genus 3 and two of genus 4. The scrolls cannot have a triple conic nor more than one double conic as part of the nodal curve. Added to the paper is a scheme showing that the maximum genus of a scroll of order  $2n$  is  $(n - 1)^2$ , and of a scroll of order  $2n - 1$  is  $n(n - 1)$ .

18. Professor Field obtains the general equation of a quintic curve having five fixed double points. It follows from the form of this equation that if a conic and cubic be taken and breaks made at two of their points of intersection, there result quintic curves. The curves are classified according to the sequence of the double points. Forty-six forms are obtained.

19. Dr. Keyser's paper is devoted primarily to a systematic introduction to the line and plane geometries of four-space, and secondarily to a correlation of these with kindred doctrines, such as the circle geometry of ordinary space and the theory of the four-space point conceived as a plenum of circular cones. The point and lineoid (three-space) theories of four-space, in so far as needed, are assumed. By reason of the locus and envelope phases of the line and plane, one has to do with four fundamental element aspects. These are strictly four distinct elements, giving rise to four distinct geometric theories which, like the elements, fall into reciprocal pairs. In a first chapter, by means of an arbitrary finite pentaedron of reference, *i. e.*, the configuration determined by any five non-collineoidal points or any five non-copunctal lineoids, four appropriate homogeneous coördinate systems are established, and their relationships disclosed in connection with the homographic and dualistic transformations. Each of these systems is composed of ten variables whose nine ratios, being connected by five (equivalent in general to three independent) quadratic identities, reduce to the necessary and sufficient number six of independent functions. These identities, being essentially like those which appear in the circle theory of ordinary space, are of order five, whence it appears that four arbitrary lines (planes) determine uniquely a fifth and so give rise to a configuration of lines (planes) analogous to the "pentacycle" of Stephanos. A second chapter which is concerned with the solution of preliminary elementary problems in terms of the adopted coördinate systems, is followed by a third devoted to a somewhat detailed exposition of

the above-mentioned correlation. The sphere (of ordinary space) may be conceived as an orthosphere, *i. e.*, a sphere regarded as the assemblage of spheres orthogonal to it; or it may be regarded as the envelope of all the orthospheres containing it. Calling the sphere an orthosphere or a sphere according as it is conceived in the former or in the latter way, and pairing the spheres and orthospheres (of ordinary space) respectively with the points and lineoids of four-space, it is seen that to the union of point and lineoid corresponds orthogonality of sphere and orthosphere — a principle which facilitates the parallelization of a theory of either space with its correspondent in the other. Among such readily deducible correspondences may be mentioned: the circle (and its geometry) has four aspects matching the four element aspects (and corresponding theory phases) mentioned above; the geometry of circles orthogonal to a sphere is analytically identical with the line (plane) geometry of a lineoid (four-space point); the theory of circles in involution with a given circle is parallel to the theory of lines (planes) having a point (or lineoid) in common with a given plane (line); and the geometry of circles in bi-involution with a given circle corresponds proposition with the four-space geometry of coplanar lines or of collinear planes.

20. In a previous paper, read before the society (Chicago) June 1, 1902, and to be published in the *Transactions*, Profes-

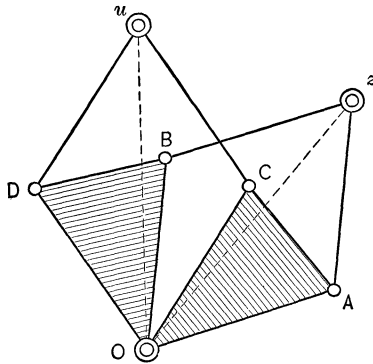


FIG. 1.

or Emch has shown that all algebraic transformations of complex variables may be realized by linkages. In this paper he describes a very simple linkage realizing the rotation  $u = e^{i\theta} \cdot z$ ,

where  $\theta$  is constant. In Fig. 1, all links with the exception of  $AC$  and  $BD$  are equal. If  $AC = BD$  are chosen in such a manner that

$$\angle AOC = \angle BOD = \theta,$$

then it can easily be proved that  $\angle zOu = \theta$ , no matter how the linkage may be distorted. Thus, if  $z$  is moved into any part of the complex plane, it is clear that the point  $u$  represents the complex number  $u = e^{i\theta} \cdot z$ .

Taking  $\theta = 2\pi/n$ ,  $n - 2$  equal cells may be successively combined in exactly the same manner, as  $OCuD$  is attached to  $OAzB$ , thus forming a closed linkage of  $n$  cells. The vertices  $z, u, u_1, \dots$  of this linkage form a regular variable polygon of  $n$  sides. If

$$z = \sqrt[n]{w} = r^{1/n} e^{i\phi/n} \quad (k = 0, 1, 2, \dots, n - 1; w = re^{i\phi}),$$

then the other vertices of the polygon represent the remaining roots. This compound linkage has been devised to show in a simple case the realization of many valued functions by linkages, which in general presents considerable difficulties.

21. The plane connex of Clebsch is defined by equating to zero a double ternary form containing a set of point coördinates and a set of line coördinates. The analogous space connex, defined by forms involving the coördinates of a point and of a plane, has been studied by Krause (*Mathematische Annalen*, 1879), Sintsof and others. The paper of Dr. Kasner gives another extension to space by introducing forms involving a set of point coördinates and a set of line coördinates. The vanishing of such a quaternary-senary form  $a_x^n A_r^m$ , where  $x$  or  $x_1 : x_2 : x_3 : x_4$  represents the point and  $P$  or  $P_{12} : P_{13} : P_{14} : P_{34} : P_{42} : P_{23}$  the line, defines the general *point-line connex* of space. After a survey of the general case, of the connection with certain types of differential equations, and of the configurations defined by the intersection of two or more connexes, the author considers in detail the case where each set of variables is involved linearly,  $n = 1, m = 1$ .

To each point  $x$  there corresponds, by means of this bilinear connex, a linear line complex, and to each line  $P$  there corresponds a plane. The study of the connex consists in the discussion of these two transformations, which are both linear.



To enumerate a few of the results: The locus of the point whose corresponding linear complex is special is a quadric surface. There are five points, situated on this quadric, each possessing the property that the corresponding complex is special and has for its directrix a line passing through the point. There are two fundamental lines  $P$ , *i. e.*, such that the corresponding plane is indeterminate. The connexion is defined geometrically by means of the (1, 1) correspondence which may be set up between the lines of a linear congruence and the points of a quadric surface. The paper concludes with the discussion of certain invariants and covariants and the establishment of normal forms with respect to both cogredient and digredient transformations.

22. Mr. Hook's paper, which will be published in the *Annals of Mathematics*, deals with curves given by either two or three equations of the form

$$x_i = \cos m_i(t + t_i) \quad (i = 1, 2 \text{ or } 1, 2, 3),$$

$m_i$  and  $t_i$  being constants,  $x_i$  the cartesian coördinate, and  $t$  a variable parameter. The plane curves are classified into open and closed curves. The three-dimensional curves fall into four types based on the number of open or closed curve projections they possess on the three coördinate planes. This classification applies only to the case in which the ratios of the quantities  $m_i$  are commensurable. Transcendental curves are not considered at all. For each type of plane curves a formula is found for the number of real double points. On the space curves both double and quadruple points are possible and formulæ giving the numbers of these on each type of curve are found. A number of interesting theorems in regard to the relative positions of the multiple points on the curves are established. It is found that in all cases the tangents at the multiple points are real and distinct.

23. Dr. Eisenhart has applied the methods of Weingarten to the study of the infinitesimal deformation of the skew helicoid and has found that the solution is complete, for the characteristic partial differential equation of the second order can be integrated. The cartesian coördinates of the characteristic surfaces, which correspond to the helicoid with orthogonality of linear elements, are given by quadratures. They constitute a

general class of surfaces of which the surfaces of revolution form a subclass, corresponding to a constant value of one of the two arbitrary functions which enter into the general integral of the characteristic equation. When these surfaces are surfaces of revolution, and only in this case, their lines of curvature correspond to the asymptotic lines on the helicoid.

The associate surfaces of the deformation are given without quadrature and are found to be the general moulded surfaces of Monge. Moreover, when any moulded surface is given, the characteristic function which determines the corresponding deformation can be found at once. Surfaces of revolution are moulded surfaces and it is shown that when the associate surface is a surface of revolution the characteristic surface also is a surface of revolution, and conversely.

24. Professor Eiesland's paper contains a study of the null system

$$dx_5 + x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4 = 0,$$

all the lines of which may be means of Lie's transformation

$$x_1 = \frac{1}{2}P_1, \quad x_2 = X, \quad x_3 = \frac{1}{2}P_2, \quad x_4 = X_4, \quad x_5 + x_1 x_2 + x_3 x_4 = X_3$$

be transformed into  $\infty^5$  parabolas lying in planes parallel to the  $X_3$ -axis. The relations between the  $\infty^7$  lines of the null system and these  $\infty^5$  parabolas have been discussed. As an interesting application of this theory, a class of surfaces in ordinary space was found whose asymptotic lines are known. In the case of certain two-dimensional translation surfaces in the space  $M_5$ , a class of surfaces in ordinary space was obtained which Darboux arrived at in his *Leçons*, volume I, page 141, by an entirely different method.

In the last part of the paper the question of invariance of the null system in  $n$ -dimensional space is discussed; of all the euclidean motions  $\left(\frac{n-1}{2}\right)^2$  rotations and a translation along an axis transform all the lines of the null system into themselves. For  $n = 5$  the following theorems have been proved: 1° There exist  $\infty^{21}$  projective transformations which leave the null system invariant. 2° There exist  $\infty^{21}$  contact transformations leaving the differential equations

$$\frac{d^3 X_3}{dx_1^3} = 0, \quad \frac{d^2 X_2}{dx_1^2} = 0$$

invariant. 3° There exist  $\infty^{10}$  projective transformations which leave invariant the complex of lines defined by the differential equations

$$\begin{aligned} dx_5 + x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4 &= 0, \\ dx_1 dx_2 + dx_3 dx_4 &= 0. \end{aligned}$$

This complex has been called an asymptotic complex on account of the rôle it plays in the transformation of surfaces in  $M_5$  into surfaces of ordinary space on which the asymptotic lines are known; in fact, the lines of this complex have been shown, when transformed by Lie's transformation to become linear tangents along the asymptotic curves of the surface. 4° There exist only one rotation and a translation along the  $X_5$ -axis leaving an asymptotic complex invariant.

25. In Professor Roe's paper an elementary proof is given that any function  $\phi(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n)$  which satisfies the partial differential equation

$$\left( y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} + \dots + y_n \frac{\partial}{\partial x_n} \right) \phi = 0,$$

where the  $y$ 's are independent of the  $x$ 's, is a function of any set of  $n - 1$  independent determinants of the  $x$ 's and  $y$ 's of the form

$$(xy) = x_i y_{i+1} - x_{i+1} y_i.$$

The paper will be published in the *Annals of Mathematics*.

27. Mr. Ford's paper is concerned with the determination of sufficient conditions for a function  $f(x)$  of the real variable  $x$  in order that each of the two following developments, occurring in mathematical physics, for  $f(x)$  in terms of Bessel's function  $J_\nu(x)$  may be differentiated term by term:

$$\text{I.} \quad f(x) = 2 \sum_1^\infty \frac{J_\nu(\lambda_n x)}{J_\nu^2(\lambda_n)} \int_0^1 x f(x) J_\nu(\lambda_n x) dx,$$

where  $\lambda_n$  is one of the positive roots of the equation  $J_\nu(x) = 0$ ;

$$\text{II.} \quad \begin{aligned} f(x) &= (2\nu + 2) \int_0^1 f(x) x^{\nu+1} dx \\ &+ 2 \sum_1^\infty \frac{J_\nu(\lambda'_n x)}{J_\nu^2(\lambda'_n)} \int_0^1 x f(x) J_\nu(\lambda'_n x) dx, \end{aligned}$$

where  $\lambda'_n$  is one of the positive roots of the equation  $xJ'_v(x) - vJ(x) = 0$ .

The results obtained are summarized in the following :

**Theorem:** Each of the series I and II converges when  $a' < x < b'$  ( $0 < a' < b' < 1$ ) to the limit  $f(x)$ , and when differentiated term by term the resulting series converge for the same values of  $x$  to the limit  $f'(x)$ , provided that  $v > -\frac{1}{2}$  and that the function  $\phi(x) = x^{-v} f(x)$  satisfies the following conditions :

**Condition A:**  $\phi(x)$  is continuous, or is made up of a finite number of a continuous portions throughout the interval  $0 \leq x \leq 1$ .

**Condition B:**  $\phi(x)$  possesses a derivative  $\phi'(x)$  which is continuous throughout the interval  $a' \leq x \leq b'$ , and in the intervals  $0 \leq x < a'$ ,  $b' < x \leq 1$  is either continuous or is made up of a finite number of continuous portions. Also, the function

$\frac{\phi'(x)}{x}$  is finite in the neighborhood of the point  $x = 0$ .

**Condition C:**  $\phi(x)$  possesses a second derivative  $\phi''(x)$  which is finite throughout the interval  $a' - \epsilon \leq x \leq b' + \epsilon$  ( $\epsilon$  arbitrarily small and positive).

**Condition D:**  $\phi(1) = 0$ .

Moreover, when  $-1 < v \leq -\frac{1}{2}$  the above theorem is true if we require also that the function  $|x^v f(x)|$  be adapted to integration in the neighborhood at the right of the point  $x = 0$ .

28. Defining two homogeneous binary forms  $f(x, y)$  and  $\phi(x, y)$ , each of the  $n$ th order, as apolar when they are so related that the invariant  $\left( \frac{\partial f}{\partial x} \frac{\partial \phi}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial \phi}{\partial x} \right)^n$  vanishes identically, Professor Holgate finds a linear construction of the third element in one of two apolar cubic forms when its two remaining elements and the three elements of the other form are given. If  $A_1, A_2, A_3$  are the three root points of  $f^3(x, y) = 0$  and  $B_1, B_2, B_3$  the three root points of  $\phi^3(x, y) = 0$ , then if  $B'''$  is the harmonic conjugate of  $B_1$  with respect to  $A_1$  and  $A_2$ , and  $C'''$  the conjugate of  $A_3$  in the involution determined by the pairs  $A_1, A_2$ ;  $B_1, B'''$ ; similarly if  $B'$  and  $B''$  are the harmonic conjugates of  $B_1$  relative to the pairs  $A_2, A_3$  and  $A_3, A_1$ , respectively, and  $C'$  and  $C''$  the conjugates of  $A_1$  and  $A_2$  in the involutions  $A_2, A_3$ ;  $B_1, B'$ , and  $A_3, A_1$ ;  $B_1, B''$ , respectively; then  $B', C'$ ;  $B'', C''$ ;  $B''', C'''$  is an involution such that any pair together with  $B_1$  constitutes a triad apolar to  $A_1, A_2, A_3$ . Hence  $B_2$  and  $B_3$  need only be a pair in this involution, and if either of

these points is given the other can be readily found by linear constructions.

29. In the theory of continuous groups of transformations the collineation groups are of prime importance. In Lie's works are to be found complete lists of continuous groups of collineations in the plane, but nowhere does he give a list of all such groups in space of three dimensions. Professor Newson's paper contains a list of 490 continuous groups of collineations in space; this list is thought to be complete. A synthetic method was employed in determining these groups which is wholly independent of Lie's methods. These 490 groups are classified according to the thirteen well-known types of collineations in space and each is designated by a convenient symbol. This table is a natural extension of the table of groups of collineations in the plane given in Professor Newson's memoir in the *American Journal of Mathematics*, Volume 24, Number 2.

30. The group of collineations leaving a quadric surface  $F$  invariant is one of the most important subgroups of the general projective group. Miss Brewster's paper contains an exhaustive study of the  $\infty^6$  collineations in the group  $G_6(F)$ . These are included under the types I, III, VI, IX, X and XI (Professor Newson's classification) and one special collineation of type XII. These collineations combine to form twenty-five continuous groups and seven mixed groups. These groups are classified according to their types, their invariant figures determined and the structure of each group discussed with reference to continuous groups of other types and singular transformations. A table is given at the end showing each group, its symbol, its invariant figure and the corresponding group in Lie's table on pages 251-254 of the third volume of "Theorie der Transformationsgruppen."

31. Dr. Pell applied a method given by Darboux (Leçons, etc., volume I, page 53 ff.) to determine the surfaces whose geodesics can be represented upon the plane as two-parameter curves of the form

$$f_1(x, y) + \mu f_2(x, y) + \nu f_3(x, y) = 0.$$

The problem has been treated by Beltrami's method by Dr. Stecker in the *Transactions* of 1901-1902.

32. The paper of Professor Heffter, which will appear in the *Transactions*, extends to space of  $m$  dimensions considerations concerning curvilinear integrals previously developed by him in the *Göttinger Nachrichten* of February 8, 1902, for the case  $m = 2$  of the plane. As the curve  $C$  of integration is taken the general rectifiable curve; in case the curve  $C$  lies within a closed continuous region of continuity of the integrand function, the usual definition of the integral receives a certain modification, in that the range of values of the limitand sum is enlarged; this modification is seen to simplify the development of the theory as regards the so-called fundamental theorem of the integral calculus, the theory of the curvilinear integral as function of the upper limit in case it is independent of the path  $C$ , and the theorem of Cauchy.

EDWARD KASNER,  
*Assistant Secretary.*

COLUMBIA UNIVERSITY.

---

THE MEETING OF SECTION A OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, PITTSBURGH, PA., JUNE 28-JULY 3, 1902.

BOTH in attendance and in the number and character of the papers presented, the Pittsburgh meeting of Section A of the American Association for the Advancement of Science, was very successful. The Secretary has not preserved a memorandum of the members who were present, but he estimates that forty or more attended the different sessions. The Section met on Monday afternoon, June 30, the Vice-President, Professor G. W. Hough, of Northwestern University, presiding, to hear the address of the retiring Vice-President, Professor James McMahan, of Cornell University. In the absence of Professor McMahan, his address, the subject of which was "Some recent applications of function theory to physical problems," was read by Professor R. S. Woodward, of Columbia University. This address was published in *Science* for July 25.

Amongst the papers on the programme of the Section were three important reports: I, "A report on quaternions," by Professor Alexander Macfarlane, which will be published in the *Proceedings* of the Association. II, "A second report on