$$\Delta(\varphi\psi) = \varphi\Delta\psi + \psi\Delta\varphi + 2\Omega,$$

where

$$\Omega = \frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y} + \frac{\partial\varphi}{\partial z}\frac{\partial\psi}{\partial z}.$$

As is well known, Ω vanishes only when the two families $\varphi = \text{const}.$ $\psi = \text{const}$, are orthogonal. Therefore

IX. *If a pair of potential surfaces $\varphi = 0$, $\psi = 0$ combined form a potential surface, the families $\varphi = const.$, $\psi = const.$ are orthogonal.*

The projective generalization of this result is

IX′. *If a surface apolar to a conic decomposes into two surfaces apolar to the same conic, then the tangent planes to the latter surfaces at any point in their intersection cut the plane of the conic in a pair of lines conjugate with respect to the conic.*

A MODERN ENGLISH CALCULUS.

An Elementary Treatise on the Calculus, with Illustrations from Geometry, Mechanics and Physics. By GEORGE A. GIBSON, M.A., F.R.S.E., Professor of Mathematics in the Glasgow and West of Scotland Technical College. London, Macmillan & Co., 1901. 12mo, pp. xix + 459.

IN the year 1891 Harnack's Elements of the Differential and Integral Calculus, which appeared in Leipzig in 1881, was translated into English. This book gave the first systematic presentation in the English language of the leading principles of modern analysis in their relation to the foundations of the infinitesimal calculus. While not wholly free from errors, and sometimes difficult to read, owing to inadequate exposition of details, the book is nevertheless conceived in the spirit of modern mathematics and it lays stress on those principles of analysis which are essential for a rigorous development of the calculus.

The first book of English origin to treat the calculus from a modern standpoint was Lamb's Infinitesimal Calculus,* published in 1897. This is an excellent treatise and any later work on the calculus, of modern tendencies, must have many points of contact with it.

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* A notice of this book by the present writer appeared in *Science*, new series, vol. 7 (1898), No. 176, p. 678.

The year that has just closed has witnessed the appearance of a new text-book on this subject by Professor Gibson, who, like Professor Lamb, is a teacher in a technical school and was led to write the book by the desire to afford to technical students the training in mathematics which will be useful to them in their profession. The author's conception of what is a useful training for such students differs widely from the ideas that prevail among many teachers in technical schools, who think they have done their whole duty by their class when they have taught them to perform the formal work of differentiation and integration, and to compute centers of gravity and moments of inertia from the formulas. From the start Professor Gibson lays stress on the conceptions and the methods of the calculus. The notion of the function is set forth, not in a line or two, but in over a page of text, well illustrated by examples, and ending with a formal definition which is both clear and accurate. The meaning of the symbol $f(x)$ is fully explained. In Chapters II and III the subject of graphs is treated at length. These chapters will be welcome to teachers of the calculus as a convenient place to which to refer students whose previous training in analytic geometry has been irregular and who wish to acquire in a brief space of time that part of the subject which is essential for beginning the study of the calculus.

Chapter IV is entitled Rates; Limits. The idea of the rate of change of one variable with respect to another is set forth in such a manner that the beginner cannot help feeling the necessity of the last step—the introduction of the limit as the simplest and most natural way of *defining* the actual rate or velocity, when the change of the dependent variable is not a uniform one. A general explanation of a limit follows and then two formal definitions of a limit are given. The first is accurate and easily intelligible to the beginner. The second is the ε definition. But the author is judicious in his use of ε proofs, none occurring in the early chapters and only a few at places in the later chapters where no other proof is so simple. The treatment of the conception of a function's becoming infinite is too brief, and the expression "converges to ∞" is to be deprecated, no matter how carefully the meaning which the author attaches to it may be defined. If the student is taught to read the notation: $x = \infty$ as "when x becomes infinite," the very language that he uses conduces to the formation of right ideas. This is the more important since such ideas are not formed from definitions and precept, but from practice and example. For the same reason it is desirable to read $\underset{x=a}{\mathrm{L}} f(x)$, not as "limit of $f(x)$ for x equal to a," but as "limit of $f(x)$ when x approaches a." The chapter, taken as a whole, is a strong one and one that every beginner in the calculus will do well to read. The examples are a particularly noteworthy feature

Chapter V deals with continuity and certain special limits, in particular

$$\lim_{m=\infty} \left(1 + \frac{1}{m}\right)^m.$$

In these five chapters, occupying 100 pages, not only the fundamental conceptions of the calculus are treated, but all of the special limits are evaluated which are needed in the differentiation of the elementary functions. Chapters VI and VII, 48 pages, are devoted to differentiation, Chapter VIII, 12 pages, to physical applications. This arrangement has the advantage of being systematic and makes it easy for the student at any later time to turn to the particular topic to which he wishes to refer. It has the disadvantage, however, of not presenting the subject from its most attractive and from its most accessible side. The reviewer believes that it is preferable to begin with the problem, already treated to some extent in analytic geometry, of finding the tangent to a curve and then, after explaining the notion of the function and the method of representing it by a graph, to take up a few simple differentiations and to illustrate the use of the derivative by numerous problems in velocities, maxima and minima, and graphs. Consider to how great an extent elementary mathematics is formal. School algebra consists largely at least as at present taught, in learning rules and applying them to examples that are precise copies of illustrative ones worked out in the text-book, and it thus sinks almost to the level of Latin composition. The "originals" of geometry, and many parts of analytic geometry, especially loci problems, afford real training in mathematical reasoning. But the formal tendency is still predominant when the student begins calculus. Problems in velocities and maxima and minima present at the outset a means of making the student think about what he is doing, provided that they are so chosen that he must formulate his data. Thus the problem, "A point moves in such a manner that the space traversed is given by the formula $s = 120 - 16t^2$; find its velocity at the end of two seconds," can have no higher value than to encourage a dull student to begin work on the exercises assigned. On the other hand the problem, "A vessel is anchored in three fathoms of water and the cable passes over a sheave in the bowsprit, which is six feet above the water. If the cable is hauled in at the rate of a foot a second, how fast is the vessel moving through the water when there are five fathoms of cable out?" —given out after the class has had some practice on easier problems, but not preceded by a similar problem worked for the student, is of value in making the student think for himself. The same is true of problems in maxima and minima, those that require the student to put into equations data that are given him without formulas affording effective training in the principles to which the calculus owes its origin. In their essential features these problems are of the nature of physical applications. They train the student to use his calculus from the start without involving the technical difficulties of many of the problems which Professor Perry considers in his book, "Calculus for Engineers." We are in hearty sympathy with Professor Perry as regards the objects which he has in view, for we believe that calculus should be brought home to the student and that he should through many and varied problems be brought to feel that calculus stands in vital relation to the phenomena of every day life. No more fatal criticism on a course in calculus can be made than that which is contained in the remark of the student who says that he does not see what calculus is for. Professor Perry's book contains much that can be so far freed from technical difficulties as to become available for the general student, and the same is true of the books of Autenheimer * and Nernst and Schoenflies.† As a means to the same end, too, it is desirable to introduce at an early stage the integral as the limit of a sum. The writer is accustomed to spend about fifteen lectures of the first half of the first course in calculus on this subject and to have a great many problems, not merely in volumes, centers of gravity and moments of inertia, but in fluid pressures, attractions, and work and energy worked out by the students.‡ His experience has been that this topic is an

* "Lehrbuch der Differential- und Integralrechnung," 4th ed., Weimar, 1895.

† "Kurzgefasstes Lehrbuch der Differential- und Integralrechnung," on which is based Young and Linebarger's "The Elements of the Differential and Integral Calculus," New York, 1900.

‡ A good collection of problems of the class which the writer has in mind is found in Professor Byerly's "Problems in Differential Calculus," Boston, Ginn & Co. This collection also includes problems in the integral calculus.

especially instructive one. Right ideas regarding the fundamental conceptions of the calculus and a correct understanding of its principles are acquired, not through a lengthy exposition at the outset when the mind of the student is unprepared for what he is to receive, but by the teacher's constantly training him all through the course to form right habits of thought in these matters. A valuable aid in this direction may be found in these introductory chapters of Professor Gibson's book, when used for collateral reading, and in many passages of the later chapters in which the author warns the student against a false conception or trains him to use the rigorous methods of modern analysis.

Chapter IX begins with Rolle's theorem and the theorems of mean value. The whole modern treatment of the differential and integral calculus rests to a large degree on these important theorems and it is indicative of the modern tendencies of the work before us that these theorems not only are given a leading position, but also are applied systematically throughout the remainder of the book. Thus the error made in the older treatises on the calculus in the proof of the theorem that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ is pointed out in Chapter XI and it is shown by means of a simple example that the assumption made in these proofs to the effect that

$$\lim_{h=0} [\lim_{k=0} f(h, k)] \quad \text{and} \quad \lim_{k=0} [\lim_{h=0} f(h, k)]$$

have the same value is unwarranted; for, if for example $f(h, k) = (h + 2k)/(h + k)$, the first of these double limits has the value 1 and the second the value 2. Then follows a rigorous proof of the theorem.*

Chapter X is entitled: Derived and Integral Curves, Integral Function, Derivatives of Arcs and Volume of a Surface of Revolution, Polar Formulæ, Infinitesimals.

Chapter XI, Partial Differentiation, contains a treatment

* This is the proof that is now commonly given in continental textbooks on the calculus. It is often ascribed to Schwarz; but it appears in substance in Serret's Cours de calcul différentiel und intégral, 1st ed., 1868, pp. 76–78, where the statement is made that it is due to Ossian Bonnet. Schwarz refers to this passage in his memoir. In Bonnet's proof the continuity of both the derivatives of the second order which enter is employed, and it is necessary only to incorporate the requirement of the continuity of these derivatives into the statement of the theorem to make Bonnet's proof, which is based on the theorems of mean value, rigorous. What Schwarz actually did was to show (a) that the theorem as stated by Serret without any requirement regarding the continuity of the two derivatives of the second order is false; (b) that it is enough to require the continuity of *one* of these derivatives.

of this subject which in many respects is admirable. In the theory of partial derivatives and differentials two theorems are fundamental.

Theorem A. If u is a function of the independent variables $x, y, \cdots$ and if each of these is in turn a function of the independent variables $r, s, \cdots$, then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \cdots$$

with similar equations for $\partial u/\partial s$, etc., provided that the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \cdots$ on the one hand, and on the other the partial derivatives $\frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \cdots, \frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \cdots$ are continuous.

Theorem B. If the differential of u, du, is defined by the formula*

$$du = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \cdots$$

when $x, y, \cdots$ are the independent variables, then

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \cdots,$$

no matter whether $x, y, \cdots$ are the independent variables or whether $x, y, \cdots$ are functions of $r, s, \cdots$ and these latter are regarded as the independent variables, provided merely that the first partial derivatives that enter are continuous.

The second of these theorems is indispensable in establishing the legitimacy of the ordinary definition of du, both for functions of several variables and for those of a single variable, a fact generally overlooked by English writers on the calculus, not excepting the present author.

In presenting the subject to beginners it is well to commence with the case that the number of variables in the group $r, s, \cdots$ is one,—call the variable t,—and then the formula of Theorem A becomes

* The author denotes increments Δx, Δy, Δu, etc., throughout the book by δx, δy, δu, etc. This notation is unfortunate inasmuch as it is used by mathematical writers for things not increments much oftener than is the large Δ, and the student is likely sooner or later to get confused ideas as to what meaning is to be attached to δu.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \cdots.$$

This Professor Gibson does and his presentation is clear and accurate. But in the general case Theorem B is entirely omitted and only a special case of Theorem A appears in the course of an example that is worked out in the text. The few pages still needed for a fairly complete treatment of the subject of partial differentiation would have been a welcome addition to the chapter before us.

Chapter XII, Applications to the Theory of Equations, will command the attention of all who are engaged in teaching calculus, mechanics, or any of the other physical sciences to which calculus is applied. Newton's method of approximating to the roots of an equation is set forth, the steps are illustrated graphically and a check for the error is obtained. One of the equations, for example, to the solution of which the method is applied is $x + \sin x - \frac{1}{3}\pi = 0$. The method of successive approximations follows. The expansion of a root of the equation $x = \varphi(x)$ into a series is taken up and applied to finding the roots of the equation $x = \tan x$. Numerous examples are given for the student to work out; these include, in particular, the determination of the constant a of the catenary from the equation

$$l = a(e^{\frac{c}{2a}} - e^{-\frac{c}{2a}}),$$

l and c being given and l being only slightly greater than c; *e. g.*, $l = 105$, $c = 100$. There is an excellent paragraph on Proportional Parts, with a simple check for the error. The chapter closes with a paragraph on Small Corrections.

In Chapter XIII, pages 262–297, the formal methods of integration are treated at some length. In connection with this chapter the student should learn to use some good table of integrals.*

Chapter XIV, pages 298–323, deals with properties of definite integrals. The theorems of mean value are given and application of the first law is made to the approximate computation of certain definite integrals, one of the examples being to show that, if $n > 2$,

$$.5 < \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1 - x^n}} < .524.$$

* Cf., for example, B. O. Peirce: A Short Table of Integrals, Ginn & Co., Boston, which in the revised edition of 1899 contains 569 formulas of integration and in addition many other useful formulas and tables.

The chapter closes with an account of Amsler's planimeter.

It is not until Chapter XV, pages 324–351, that the integral as the limit of a sum is introduced. The definite integral was defined (page 262) as the difference between two values of the indefinite integral

$$\int_a^b f(x)dx = [D_x^{-1} f(x)]_a^b,$$

and the existence of an indefinite integral was shown by reference to the area under the curve $y = f(x)$. While this is not the customary definition of a definite integral, the discrepancy so far as the student is concerned is less startling than would appear at first sight, since the foundation on which he builds his integral calculus is, with Professor Gibson as with others, the area under the curve $y = f(x)$. Nevertheless, of the problems of the integral calculus which can be treated either by determining a function u so that it will satisfy the conditions

$$\frac{du}{dx} = f(x), \qquad u|_{x=x_0} = 0,$$

or by means of the limit of a sum, it seems to us undesirable to solve very many by the former method. For, the latter method is of fundamental importance in the calculus; it is instructive, and it is not beyond the powers of the beginner at an early stage. Moreover, it is the only method that there is for the problems of the calculus that lead to multiple integrals and it is precisely the training that the simple integrals afford which the student needs in order to deal readily with these problems. The chapter gives methods for the approximate computation of definite integrals (the Trapezoidal Rule, Simpson's Rule). It closes with double integrals

Chapter XVI, pages 352–374, is entitled Curvature, Evolutes, and includes a treatment of the cycloid.

In Chapter XVII, Infinite Series, the text of which occupies but twelve pages, the usual tests for convergence are obtained, the reasoning on which they rest being set forth with admirable clearness. The ordinary sufficient condition that a series of continuous functions should yield a continuous function is established and applied to power series. We note with satisfaction a rigorous proof on page 388 that, if a power series vanishes for all values of its argument within a certain interval, each of its coefficients must vanish. The chapter, though somewhat brief, is well done, and its presence is characteristic of the modern tendencies of Professor Gibson's work.*

Chapter XVIII, pages 390–407, contains an excellent presentation of Taylor's Theorem and concludes with a good collection of examples. A paragraph is devoted to the differentiation and the integration of series. In the statement of the tests (Theorems I and II, pages 399, 400) the condition that the terms of the series be *continuous* functions should be included.†

A chapter on Taylor's Theorem for a Function of Two or More Variables; Applications, and one on Differential Equations conclude the book. In the former chapter is contained a rigorous treatment of Indeterminate Forms such as is now usual in continental treatises on the calculus. The latter is, in the language of the preface, "designed to illustrate the types of equations most frequently met with in dynamics, physics, and mechanical and electrical engineering."

In the foregoing account and critique of the book before us the reviewer is conscious of having laid special stress on the author's development of the fundamental principles of the calculus and he fears that some readers of the BULLETIN may have received the impression that Professor Gibson's work is an imitation of certain continental treatises on the calculus which, while containing much that is of interest to the specialist in the theory of functions of a real variable, appear to the general student to be difficult and arid. This is, however, by no means the case. The style is simple and the presentation is such as to appeal to the student who has appreciation for the exact sciences. The choice of material is exceptionally good and covers a wide range. The problems appended to each chapter for the student to solve are numerous and well selected.

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* On p. 382 near the top: "The series (the exponential series, §49) is therefore convergent for every finite (positive) value of x." The restriction *finite* is incorrect, for x is a *constant* with reference to the convergence of the series and there is no such thing in the calculus as an *infinite constant*. One of the excellent features of Professor Gibson's book is that the author repeatedly emphasizes this fact; thus, on p. 195, he insists that an infinitesimal is not a constant, but is a *variable*. Clearly, the word "finite" was unintentional, but it is an oversight that may easily lead the beginner into error.

† On p. 400, near the middle: "We will now show that the series obtained by differentiating the power series is uniformly convergent when x is *within* the interval of convergence * * *." This is wrong as it stands; the series is *not* in general uniformly convergent for this range of values for x. It may, however, be corrected by replacing "x" by the words "the interval for x: $a \leqq x \leqq b$."