NOTE ON MR. GEORGE PEIRCE'S APPROXIMATE CONSTRUCTION FOR π.

BY M. EMILE LEMOINE.

(Read before the American Mathematical Society, August 20, 1901.)

THERE appeared in the BULLETIN for July, 1901, a construction by Mr. George Peirce for obtaining the approximate length of π in a circle of radius 1. There are numerous constructions of this kind, and it may be of interest to indicate the method which permits a comparison of these constructions with one another as to their graphical simplicity. Their relative theoretic exactness is determined by calculating the true value of the length which in each case approximately represents π.

As examples of these comparisons, I take the construction of Mr. Peirce and three others, and employ the geometrographic method (see Mathematical papers of the Chicago congress, 1893, p. 143, or in more complete form, La géométrografie, Paris, Naud, 1901) which is applicable with rigor and facility. I will designate by :

A. The construction of Mr. Peirce.

B A very old construction, attributed to Heinrich Kühn, in the *Novi Commentarii Acad. Petropol.*, Vol. III (1753).

C. A construction given by myself for $\pi/2$.

D. A construction due to Professor Pleskot of the Czech Realschule of Prague (*Journal de mathématiques élémentaires* de M. de Longchamps, 1895, p. 125); this also gives $\pi/2$.

The geometrographic notation is so simple that I indicate it at once, so that any geometer not acquainted with it may have no difficulty in comprehending this note.

1. Placing *one* point of the compass on a given point is designated as "operation C_1" or op (C_1); hence, speculatively, including a given length between the points is op. ($2C_1$).

2. Placing a point of the compass on an *undetermined* point of a straight line is op. (C_2).

3. Drawing a circle is op. (C_3).

4. Making the edge of the ruler pass through *one* point is op. (R_1) ; hence, speculatively, making it pass through two points is op. ($2R_1$).

5. Drawing a straight line is op. (R_2).

This is all for the *canonical* geometrography, *i. e.*, where the only instruments used are ruler and compass.

If the square is also admitted, the same notation is retained as for the operations with the ruler, but the R is accented : op. (R_1'), op. $(2R_1')$. I also accent any operation with the ruler that serves *directly* for a construction requiring the square. This is done merely for the purpose of making the symbols themselves indicate the extent to which the square is used.

Placing the edge of the square (or ruler) in coincidence with a given straight line, an operation which never occurs in canonical geometrography, is regarded as passing this edge through two points and denoted by op. $(2R_1')$. Finally, sliding one side of the square along the ruler until the other side passes through a given point is op. (E). This is the only symbol peculiar to the square.

Any canonical construction is thus represented by a symbol of the form op. $(l_1R_1 + l_2R_2 + m_1C_1 + m_2C_2 + m_3C_3)$. The sum $l_1 + l_2 + m_1 + m_2 + m_3$ is called the coefficient of simplicity or simply the *simplicity*. The sum $l_1 + m_1 + m_2$ of the coefficients of the *preparatory* operations is called the coefficient of exactness or simply the *exactness* ; l_2 and m_3 are the number of straight lines and circles drawn.

When the square is admitted, the symbol of a construction will be similarly

$$\text{op. } (l_1R_1 + l_1'R_1' + l_2R_2 + kE + m_1C_1 + m_2C_2 + m_3C_3);$$

$l_1 + l_1' + l_2 + k + m_1 + m_2 + m_3$ is the simplicity, $l_1 + l_1' + k + m_1 + m_2$ the exactness ; l_2 and m_3 are the number of straight lines and circles drawn. For brevity, a circle with center A and radius R or MN may be denoted by $A(R)$ or $A(MN)$.

NOTE.—Geometrography is essentially speculative ; it assumes, for instance, like geometry, that the lines and circles are completely drawn, that the paper is a plane of infinite extent, that the drawing instruments are as large or as small as may be necessary, etc. It therefore guides the draughtsman as rational mechanics guides the engineer, but *far* more exactly.

EXAMINATION OF THE CONSTRUCTIONS.

A. (*Figure* 1.)—At the outset, the figure in the plane consists only of a circle of radius 1 with marked center O.

1°. *Canonical.*—Draw any diameter $AC(R_1 + R_2)$. By means of the intersection of the two circles $A(\rho)$, $C(\rho)(2C_1 + 2C_3)$, ρ being any sufficient length, draw the diameter $BD(2R_1 + R_2)$ perpendicular to AC. Place one point of the compass on D, the other on O, and draw $D(DO)(2C_1 + C_3)$. Draw $AD(2R_1 + R_2)$ which cuts $D(DO)$ in E, between A and D. Draw $CE(2R_1 + R_2)$ which locates G on BD and F on the given circle. Draw $C(GB)(3C_1 + C_3)$ which cuts FC produced in I. FI is the desired length. Op. $(7R_1 + 4R_2 + 7C_1 + 4C_3)$, simplicity 22, exactness 14, 4 lines, 4 circles.

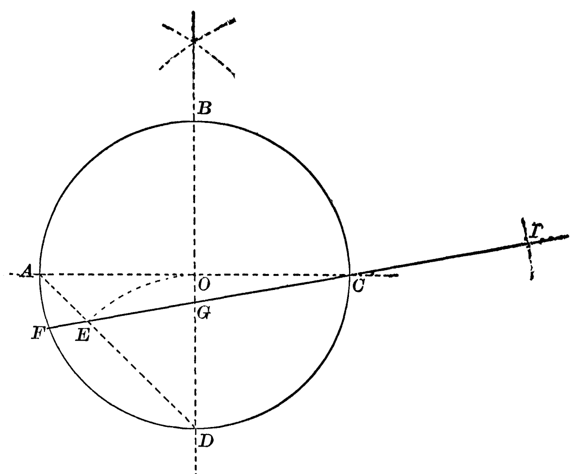

FIG. 1.

2°. *With the square.*—The instrument can only be used as follows: Draw AC with one side of the right angle serving as ruler $(R_1' + R_2)$. Without moving the square, place the ruler against the hypotenuse and slide the square along the ruler until the other side passes through $O(E)$. Draw $BD(R_2)$ and proceed as in the canonical construction above. Op. $(4R_1 + R_1' + 4R_2 + E + 5C_1 + 2C_3)$, simplicity 17, exactness 11, 4 lines, 2 circles.

B. (*Figure* 2.)—This is based on the relation $\sqrt{2} + \sqrt{3} = 3.1462 \cdots$.

1°. *Canonical.*—With any point of the given circle as center draw $B(BO)(C_1 + C_2 + C_3)$ cutting the given circle in C and D. Draw $C(BO)(C_1 + C_3)$ which cuts $B(BO)$ in P. Draw $PO(2R_1 + R_2)$ which cuts the given circle in K. We have $KD = \sqrt{2}$, $CD = \sqrt{3}$. Draw $KD(2R_1 + R_2)$, then $D(DC)(2C_1 + C_3)$ which cuts KD produced in I and I'. KI is the desired approximate length of the semicircumference. Op. $(4R_1 + 2R_2 + 4C_1 + C_2 + 3C_3)$, simplicity 14, exactness 9, 2 lines and 3 circles.

The construction can be performed otherwise and a symbol obtained of the same simplicity,

$$\text{op.}\ (2R_1 + R_2 + 6C_1 + C_2 + 4C_3)$$

(see *Bulletin de la Société mathématique de France*, volume 23, 1895).

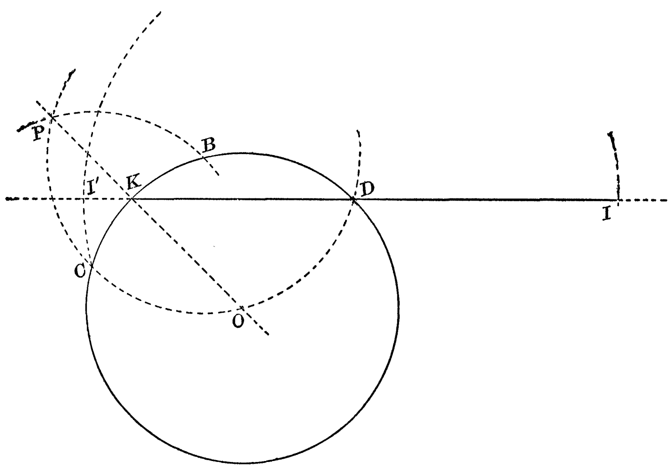

FIG. 2.

Remark.—Since $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2}) = 1$, we have approximately $KI' = 1/\pi$, since $\sqrt{3} + \sqrt{2}$ is π.

2°. *With the Square.*—No simplification can be obtained in this case.

C. (*Figure* 3.)—This rests on the fact that the cosine of the smallest positive angle whose sine is the side of the inscribed regular decagon in the circle of radius 1 is

$$\sqrt{\tfrac{1}{2}(\sqrt{5} - 1)} = 0.7861,$$

while

$$\tfrac{1}{4}\pi = 0.78539 \cdots .$$

1°. *Canonical.*—Draw a diameter $AB(R_1 + R_2)$. Draw $A(AB)(2C_1 + C_3)$, which cuts AB in B'. Draw

$$B'(AB)(C_1 + C_3),$$

which cuts $A(AB)$ in C and C'. Draw $C'(AB)(C_1 + C_3)$, which cuts $B'(AB)$ in D. Draw $AD(2R_1 + R_2)$, which cuts

$A(AB)$ in E between A and D. Draw $C(CE)(2C_1 + C_3)$, which cuts AB in G between A and B and in G'. Draw $A(AG)(2C_1 + C_3)$, which cuts the given circle in H. The distance BH, which need not be drawn, is one-quarter of the circumference.

Op. $(3R_1 + 2R_2 + 8C_1 + 5C_3)$, simplicity 18, exactness 11, 2 lines, 5 circles.

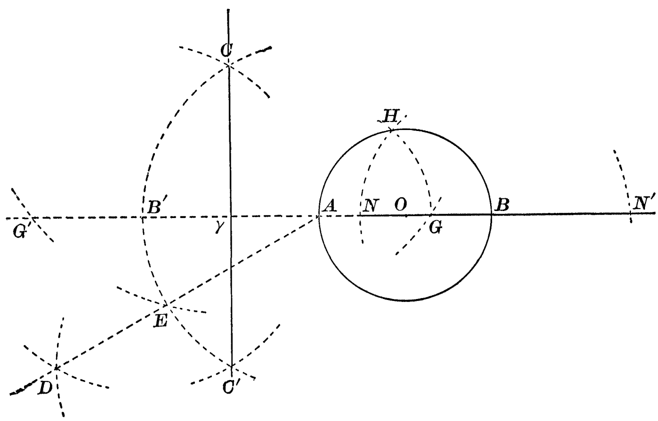

FIG. 3.

In fact it suffices to show that $AG = AH = AB\dfrac{\sqrt{5}-1}{2}$.

For $\frac{1}{2}(\sqrt{5}-1)$ will then be the sine of the angle ABH, and hence the cosine of the same angle will be $\sqrt{\frac{1}{2}(\sqrt{5}-1.)}$ But the isosceles triangle CAE is right-angled at A, hence $CE = AB\sqrt{2}$. If we denote by γ the point where CC' cuts BA, we have, in the equilateral triangle $CB'A$, $C\gamma = \frac{1}{2}AB\sqrt{3}$, whence

$$\overline{CG}^2 = \overline{CE}^2 = \overline{C\gamma}^2 + \overline{\gamma G}^2,$$

or

$$2\overline{AB}^2 = \tfrac{3}{4}\overline{AB}^2 + \overline{\gamma G}^2,$$

or

$$\overline{\gamma G}^2 = \tfrac{5}{4}\overline{AB}^2,$$

$$\therefore AG = \gamma G - \gamma A = \tfrac{1}{2}AB(\sqrt{5}-1).$$

Remark.—We should find $AG' = \frac{1}{2}AB(\sqrt{5}+1)$. G and G' therefore divide AB in extreme and mean ratio (sectio aurea). Now in the construction above G and G' are located by the symbol op. $(3R_1 + 2R_2 + 6C_1 + 4C_3)$, but if we merely wish to divide a line AB in extreme and mean ratio, the symbol should be diminished by $(R_1 + R_2)$, since AB need not be drawn. It appears then that to divide a line in extreme and mean ratio, the symbol

$$\text{op. } (2R_1 + R_2 + 6C_1 + 4C_3)$$

suffices; simplicity 13, exactness 8, 1 line, 4 circles.

The symbol of the old classical construction for the same problem, as ordinarily given, is

$$\text{op. } (6R_1 + 3R_2 + 11C_1 + 9C_3).$$

With some geometrographic precautions it can be reduced to op. $(6R_1 + 3R_2 + 10C_1 + 8C_3)$. By employing the square it becomes op. $(4R_1 + 2R_1' + E + 3R_2 + 8C_1 + 6C_3)$. The construction given above is in fact one of the numerous geometrographic constructions* which are known for the problem of the sectio aurea.

2°. *With the square.*—The square can be used to determine the point E, by drawing a perpendicular from A to $B'C'$. But this reduces the symbol only by a unit,

$$\text{op. } (R_1 + 2R_1' + E + 2R_2 + 7C_1 + 4C_3),$$

simplicity 17, exactness 11, 2 lines, 4 circles.

D. (*Figure* 4.)—1°. *Canonical.*—Draw any diameter AB $(R_1 + R_2)$; draw $A(AO)(2C_1 + C_3)$ cutting the given circumference in C and D; draw $CD(2R_1 + R_2)$ cutting AB in E. On AB produced lay off $EF = 2CD(4C_1 + 2C_3)$; draw FD $(2R_1 + R_2)$ and on this line take H, between F and D, so that $FH = AB(3C_1 + C_3)$. HD is a quarter circumference.

In fact, we find

$$HD = \tfrac{1}{2}\sqrt{51} - 2 = 1.570714 \cdots$$

while the quarter circumference is $1.570796 \cdots$.

$$\text{Op. } (5R_1 + 3R_2 + 9C_1 + 4C_3),$$

simplicity 21, exactness 14, 3 lines, 4 circles.

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* A construction for a given problem is called its geometrographic construction if it has the minimum coefficient of simplicity of all the known constructions. If there are several constructions of the minimum simplicity they are all called geometrographic.

2°. *With the square.*—The square is not useful in this construction.

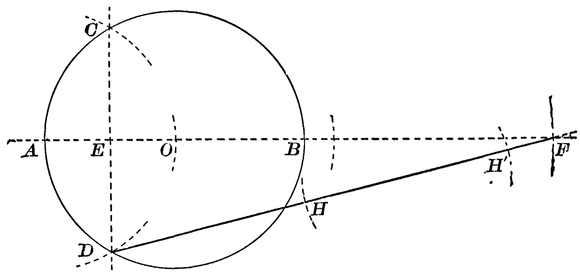

FIG. 4.

SUMMARY.

Let $\triangle$ be the difference between π and its approximation.

A. The symbol for the canonical construction of π is

op. $(7R_1 + 4R_2 + 7C_1 + 4C_3)$; S. 22, E. 14 ; 4 lines, 4 circles.

With the square

op. $(4R_1 + R_1' + 4R_2 + E + 5C_1 + 2C_3)$; S. 17, E. 11 ; 4 lines, 2 circles. $\triangle = + 0.0012 \cdots$.

B. The symbol for the canonical construction of π is

op. $(4R_1 + 2R_2 + 4C_1 + C_2 + 3C_3)$; S. 14, E. 19 ; 2 lines, 3 circles.

There is no advantage in using the square.

$$\triangle = + 0.0047 \cdots.$$

C. The construction gives $\frac{1}{2}\pi$ with the canonical symbol

op. $(3R_1 + 2R_2 + 8C_1 + 5C_3)$; S. 18, E. 11 ; 2 lines, 5 circles.

With the square

op. $(R_1 + 2R_1' + E + 2R_2 + 7C_1 + 4C_3)$; S. 17, E. 11 ; 2 lines, 4 circles.

To obtain π it is necessary to draw $B(BH)$ cutting AB in two points N and N' whose distance represents π ; this adds $(2C_1 + C_3)$ to the preceding symbols.

The symbols then become, for the canonical construction, op. $(3R_1 + 2R_2 + 10C_1 + 6C_3)$; S. 21, E. 13 ; 2 lines, 6 circles.

With the square

op. $(R_1 + 2R_1' + E + 2R_2 + 9C_1 + 5C_3)$; S. 20, E. 13 ; 2 lines, 5 circles. $\triangle = + 0.0030 \cdots$.

D. The construction gives $\frac{1}{2}\pi$ with the canonical symbol

op. $(5R_1 + 3R_2 + 9C_1 + 4C_3)$; S. 21, E. 14 ; 3 lines, 4 circles.

The square cannot be used with advantage.

To obtain π it is still necessary to draw $H(HD)(2C_1 + C_3)$ cutting FD in H'. HH' represents π.

Op. $(5R_1 + 3R_2 + 11C_1 + 5C_3)$; S. 24, E. 16 ; 3 lines, 5 circles. $\triangle = -0.0002$.

Remark.—If the required length is $\frac{1}{2}\pi$, it is necessary in the construction A to bisect FI, in the construction B to bisect KI. This requires op. $(2R_1 + R_2 + 2C_1 + 2C_3)$. The simplicities of the canonical constructions A, B, C, D for $\frac{1}{2}\pi$ are therefore 29, 21, 18, 21, respectively.*

The geometrographic symbol thus affords the greatest facility for the comparison of constructions and the choice of the simplest or the one best adapted for the purpose.

Trisection of an Angle.

I take advantage of this opportunity to mention an approximate trisection of a given angle, which is very little known. This construction was communicated to me by Carl Störmer, a young Norwegian mathematician, who got it from a sea-captain evidently not well versed in mathematics, since he claimed to have discovered an exact trisection by means of the ruler and compass.

In Fig. 5, let $AOB = \alpha < \frac{1}{2}\pi$ be the given angle. "Draw any circle with center at O, cutting OA in A, OB in B. Join the middle point C of OB to the end A' of the diameter OA. Draw CA' cutting the diameter perpendicular to OA in D. Through D draw a parallel to AA' cutting the given circle in E on the side of A'. Draw EB, and parallels to EB through A' and O. These two parallels cut the circumfer-

* Mr. E. B. Escott calls my attention to two other approximate constructions for π, one given by R. A. Proctor in his Light science for leisure hours, Vol. I, the other by A. A. Kochanski, in the *Acta eruditorum* for 1685 (comp. Cantor's Geschichte, Vol. III, p. 21). For the former, $\triangle = 0.0002$; for the latter, $\triangle = 0.00006$.

ence in F and G between A and B so that arc $FG = \frac{1}{3}$ arc BA."

This is the enunciation as communicated to me. I will now treat the resulting construction geometrographically.

1°. *Geometrographic canonical construction.*—Fig. 5, $\alpha < 90°$. Observe first that in the preceding construction, if only the value of $\frac{1}{3}\alpha$ is required, we may stop after drawing DE and draw OE. The angle $A'OE$ will be $\frac{1}{3}\alpha$ to the degree of approximation obtainable by the construction. In fact OG, parallel to $A'F$, divides the angle AOF rigorously into two equal parts, and since, from the above, arc $GF = \frac{1}{3}\alpha$, it follows that the arcs GA and its equal $A'E$ are also each $= \frac{1}{3}\alpha$.

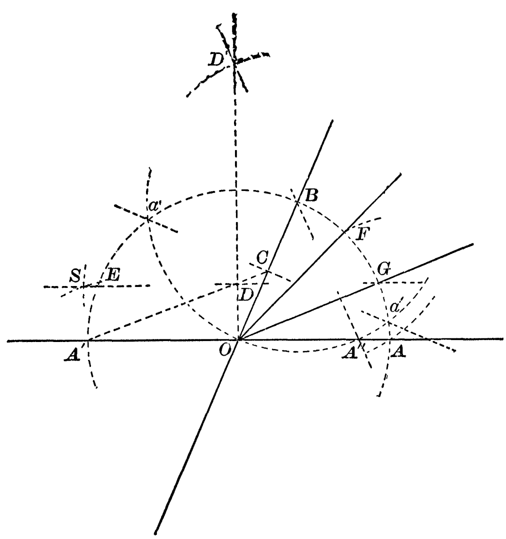

FIG. 5.

With any radius ρ, draw $O(\rho)(C_1 + C_3)$ cutting OA in A, OB in B. Draw $B(\rho)(C_1 + C_3)$ intersecting $O(\rho)$ in a, a'; connect a and $a'(2R_1 + R_2)$, thus locating C on OB. Let A'' be the point where $B(\rho)$ cuts OA. Draw $BA''(2R_1 + R_2)$ cutting $B(\rho)$ in D'. Draw $D'O(2R_1 + R_2)$, which is the perpendicular to OA at O. Draw $CA'(2R_1 + R_2)$ locating D on OD'. To pass a parallel to OA through D, draw $D(\rho)(C_1 + C_3)$ and, having taken DO on the compass while the one point was at D in drawing $D(\rho)$, draw $A'(DO)(2C_1 + C_3)$ cutting $D(\rho)$ in S outside of $O(\rho)$. Draw DS $(2R_1 + R_2)$ cutting $O(\rho)$ in E on the side of A' and in G on the side of A (for arc $A'E = \frac{1}{3}\alpha = AG$). Draw $OG(2R_1 + R_2)$; then $AOG = \frac{1}{3} AOB$ will be obtained by

$$\text{op. } (12R_1 + 6R_2 + 5C_1 + 4C_3),$$

simplicity 27, exactness 17 ; 6 lines, 4 circles. To trisect completely the angle AOB, we must draw also $G(GA)$ $(2C_1 + C_3)$ cutting $O(\rho)$ in F, and finally $OF(2R_1 + R_2)$. In all,

$$\text{op. } (14R_1 + 7R_2 + 7C_1 + 5C_3),$$

simplicity 33, exactness 21 ; 7 lines, 5 circles.

2°. *With the square.*—Draw $O(\rho)$, $B(\rho)$, and their intersections, $(2R_1 + R_2 + 2C_1 + 2C_3)$. Draw $CA'(2R_1 + R_2)$ and the perpendicular OD' to $OA(2R_1' + E + R_2)$. Draw the $(2R_1 + R_2)$ parallel through D to $OA(2R_1' + E + R_2)$, and GO

$$\text{op. } (6R_1 + 4R_1' + 2E + 5R_2 + 2C_1 + 2C_3),$$

S. 21, E. 14 ; 5 lines, 2 circles. To trisect AOB completely,

$$\text{op. } (8R_1 + 4R_1' + 2E + 6R_2 + 4C_1 + 3C_3),$$

S. 27, E. 18; 6 lines, 3 circles. M. Störmer informs me that on following out the construction he finds

$$\measuredangle FOG = \tfrac{1}{3}\alpha = \sin^{-1}\frac{\sin\alpha}{2 + \cos\alpha}$$

and that the maximum error is less than 20′.

For $\alpha > 90°$ the *graphical* operation is somewhat more complicated since it is necessary to operate on $\alpha - 90°$.

1°. *Canonical*, Fig. 6.—Erect at O a perpendicular OB' to OB by drawing any circle $\omega(\rho_1)$ passing through $O(C_1 + C_3)$. If this circle cuts OB in B_1, draw $B_1\omega(2R_1 + R_2)$ cutting $\omega(\rho_1)$ in δ; then $O\delta(2R_1 + R_2)$ is the required perpendicular to OB. Taking B' in the proper sense, we operate on the angle AOB' as formerly on AOB, at least with very slight modification. Draw $O(\rho)(C_1 + C_3)$ locating A and B', then $B'(\rho)(C_1 + C_3)$ locating a, a', and A'', then $aa'(2R_1 + R_2)$ locating C in the middle of OB'. Draw also $a(\rho)(C_1 + C_3)$, which will be of use presently. $a(\rho)$ cuts aa' in β on the same side of OB' as A. Draw $B'A''(2R_1 + R_2)$, which locates D'. Draw $OD'(2R_1 + R_2)$ and $CA'(2R_1 + R_2)$, which locates D. Fix the point $S(3C_1 + 2C_3)$ as in the case $\alpha < 90°$ and draw $SD(2R_1 + R_2)$ locating G' on $O(\rho)$. We have then arc $AG' = \frac{1}{3}(\alpha - \frac{1}{2}\pi)$. Draw $G'(O\beta)(3C_1 + C_3)$ locating G on $O(\rho)$ so that arc $AG'G = \frac{1}{3}\alpha$, since

$$\text{arc } GG' = \text{arc } O\beta = \text{arc } 30^\circ = \text{arc } \tfrac{1}{6}\pi.$$

Draw $OG(2R_1 + R_2)$, and we have $AOG = \frac{1}{3}a$.

$$\text{op. } (16R_1 + 8R_2 + 10C_1 + 7C_3),$$

S. 41, E. 26 ; 8 lines, 7 circles.

To trisect AOB completely we must still draw $G(GA)$ $(2C_1 + C_3)$ locating F on $O(\rho)$ and $OF(2R_1 + R_2)$,

$$\text{op. } (18R_1 + 9R_2 + 12C_1 + 8C_3),$$

S. 47, E. 30 ; 9 lines, 8 circles.

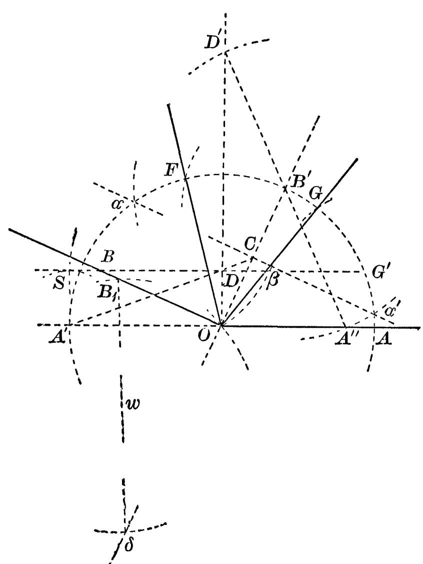

FIG. 6.

2°. *With the square.*—We find for constructing $GOA = \frac{1}{3}a$, op. $(6R_1 + 6R_1' + 3E + 6R_2 + 6C_1 + 4C_3)$, S. 31, E. 21 ; 6 lines, 4 circles. For trisecting AOB completely,

$$\text{op. } (8R_1 + 6R_1' + 3E + 7R_2 + 8C_1 + 5C_3),$$

S. 37, E. 25 ; 7 lines, 5 circles.

NOTE.—Since geometrography is a very new science and therefore little known to many mathematicians, I have developed the constructions in great detail in order that they may be followed readily by all whom this paper may interest.

Geometrography is treated didactically in the Traité de géométrie of Rouché et de Comberousse (7th edition, volume 1, Gauthier-Villars, Paris, 1900), in the *Archiv der Mathematik und Physik*, April and May, 1901, and more fully in my La géométrografie, Paris, Naud, in press, 8vo. 100 pp.

CONCERNING THE ELLIPTIC $\wp(g_2, g_3, z)$-FUNCTIONS AS COÖRDINATES IN A LINE COMPLEX, AND CERTAIN RELATED THEOREMS.

BY DR. H. F. STECKER.

(Read before the American Mathematical Society, October 26, 1901.)

Introduction.

SYSTEMS, that have appeared from time to time, of coördinates for the Kummer surface, each more or less related to the elliptic functions, suggest that the existence of such systems of coördinates may be but the partial manifestation of a more general truth; that is to say, since the Kummer surface is definitely related to a line complex of the second order, *i. e.*, is its surface of singularities, any system of coördinates on such a surface ought to arrange itself under a more general system relating at least to the complex of second order, and presumably to the general complex.

The following paper concerns itself with this general question and its application to the Kummer surface and certain other configurations.

§ I.

If we write the general quartic which enters into the discussion of the elliptic functions in the form

$$F(z) \equiv z^4 + \alpha z^3 + \beta z^2 + \gamma z + \delta \equiv \prod_{1,2,3,4.} (z_2^{(\kappa)} z_1 - z_1^{(\kappa)} z_2) = 0,$$

and if

$$(i, \kappa) \equiv \begin{vmatrix} Z_1^{(i)} & Z_2^{(i)} \\ Z_1^{(\kappa)} & Z_2^{(\kappa)} \end{vmatrix},$$

then $F(z)$ has the following irrational invariants: