SOME REMARKS ON TETRAHEDRAL GEOMETRY.

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1. Any two quadric surfaces have in general a common self-conjugate tetrahedron. If we refer to it a system of homogeneous point coördinates x_1, x_2, x_3, x_4, the equations of the two quadrics take the forms :

(1)
$$g_1 x_1^2 + g_2 x_2^2 + g_3 x_3^2 + g_4 x_4^2 = 0,$$
$$h_1 x_1^2 + h_2 x_2^2 + h_3 x_3^2 + h_4 x_4^2 = 0.$$

Now the polar planes of a point with respect to these two quadrics cut each other in a straight line, which we may also find as the polar line of the given point in the following way, starting from the quartic curve of intersection of the two quadrics : We draw the two chords of the quartic curve which pass through the point, and determine on each of them the fourth harmonic to its two curve points and the given point. Joining these two fourth harmonic points we have at once the required polar line.

We fix a straight line in space by Plückerian line coördinates, viz., if x_i, y_i be the coördinates of any two points upon the line, by the six expressions

(2)
$$p_{ik} = x_i y_k - x_k y_i.$$

Between them the identical relation exists

(3)
$$p_{23} p_{14} + p_{31} p_{24} + p_{12} p_{34} = 0.$$

The polar lines of all points in space, with respect to a quartic space curve of the first species, form a complex of lines. This complex has been called by Reye a tetrahedral complex. If z_i be the coördinates of the point, we immediately find for the coördinates of its polar line

(4)
$$p_{ik} = g_{ik} z_i z_k,$$

by making

(5)
$$g_{ik} = g_i h_k - g_k h_i.$$

These magnitudes g_{ik}, which also satisfy the condition

(6)
$$g_{23} g_{14} + g_{31} g_{24} + g_{12} g_{34} = 0$$

may be regarded as the coördinates of the quartic curve referred to its fundamental tetrahedron, whose vertices are the vertices of the quadric cones passing through the curve.

The equations (4) give a representation of the lines of the tetrahedral complex by means of the coördinates of their "poles." From them we find the other representation

$$(7) \qquad \frac{p_{23}p_{14}}{g_{23}g_{14}} = \frac{p_{31}p_{24}}{g_{31}g_{24}} = \frac{p_{12}p_{34}}{g_{12}g_{34}},$$

whence the necessary condition (3) is, by (6), at once derived.

The fundamental tetrahedron has the singular property that every straight line lying in one of its faces, and every line passing through one of its vertices belongs to the tetrahedral complex. In fact, one coördinate of each of the three pairs p_{23}, p_{14} ; p_{31}, p_{24} ; p_{12}, p_{34} then vanishes, and the double equation (7) is satisfied.

The tetrahedral complex is of the second degree. Those of its lines which pass through any point constitute a quadric cone, which is circumscribed about the fundamental tetrahedron ; and the lines lying in any plane envelop a conic section of this plane, which is inscribed in the fundamental tetrahedron, as will be shown hereafter.

Since the tetrahedral complex is given by the ratios of the three binary products

$$p_{23}\; p_{14}, \quad p_{31}\; p_{24}, \quad p_{12}\; p_{34},$$

and their sum is always 0, any equation

$$(8) \qquad \lambda_1 p_{23}p_{14} + \lambda_2 p_{31}p_{24} + \lambda_3 p_{12}p_{34} = 0$$

may be regarded as defining a tetrahedral complex.

We may enquire for the quartic space curves, by aid of which this complex is to be obtained. Their fundamental tetrahedron must, of course, be that of the tetrahedral complex, and the condition their coördinates g_{ik} must satisfy is at once found, by supposing

$$(6) \qquad g_{23}g_{14} + g_{31}g_{24} + g_{12}g_{34} = 0,$$

to be

$$(9) \qquad \lambda_1 g_{23}g_{14} + \lambda_2 g_{31}g_{24} + \lambda_3 g_{12}g_{34} = 0.$$

This form is the same as that of the proposed equation of the complex of lines, and we can say that this triply infinite set of quartic curves is also a tetrahedral complex.

2. Instead of taking the polar planes of points with respect to two quadric surfaces, we may join the poles of any plane with respect to the same surfaces, and we shall thus obtain the same tetrahedral complex of lines. For, if u_i, v_i be the coördinates of any two planes passing through a given straight line, we may also denote the magnitudes

$$(10) \qquad q_{ik} = u_i v_k - u_k v_i$$

as coördinates of the line, and we then have

$$(11) \qquad \begin{aligned} q_{23} &= p_{14}, & q_{31} &= p_{24}, & q_{12} &= p_{34}, \\ q_{14} &= p_{23}, & q_{24} &= p_{31}, & q_{34} &= p_{12}. \end{aligned}$$

But the line, which joins the poles, with respect to the surfaces (1), of a plane with coördinates w_i has itself in this sense the following coördinates :

$$(12) \qquad q_{ik} = g_{ik} \frac{w_i}{g_i h_i} \frac{w_k}{g_k h_k},$$

whence, by virtue of (11), the relation (7) is again derived.

Now let us consider, instead of only two quadrics, all quadrics having the same common curve. Then for any point we obtain a pencil of polar planes with respect to these surfaces, and the pencils thus found for all points of space are in a projective relation. This relation is such that the planes passing through the same vertex of the fundamental tetrahedron are always corresponding planes; and the planes joining the lines of the tetrahedral complex, as axes of these pencils of polar planes, to the vertices of the fundamental tetrahedron have all therefore the same anharmonic ratio. Hence reciprocally : *If we determine the straight lines such that the planes joining them with four given points have the same anharmonic ratio, then these lines form a tetrahedral complex.*

If we take the quadrics inscribed in the same quartic developable, then the poles of any plane with respect to these quadrics lie upon a straight line, and the straight lines thus obtained are in a projective relation. This relation is such, that the points where the straight lines meet the planes of the fundamental tetrahedron are corresponding points of the lines. Hence, as above : *If we determine the straight lines whose points of intersection with four given planes have the same anharmonic ratio, these lines form a tetrahedral complex.*

As a corollary we find that, the fundamental tetrahedron being known, a tetrahedral complex is wholly given by *one* of its lines. With regard to every tetrahedron the straight lines in space are divided, in an entirely determinate manner, into a set of tetrahedral complexes.

If we draw in two faces of the fundamental tetrahedron, through their points of intersection with a line of the tetrahedral complex, the straight lines to those vertices of the tetrahedron which are not common to both planes, then we get, varying the line of the complex, in the two planes two projective pencils of rays, and every line meeting two corresponding rays of these pencils belongs to the tetrahedral complex. Hence it is evident how all lines of the complex may be found, if the fundamental tetrahedron and one line are given.

The analytical proof is very easy. For, the points of intersection of the line whose coördinates are given by (4) with the first and second plane of the fundamental tetrahedron have coördinates x_i and y_i, determined thus :

$$x_1 = 0, \quad x_2 : x_3 : x_4 = p_{12} : p_{13} : p_{14} = g_{12}z_2 : g_{13}z_3 : g_{14}z_4,$$
$$y_2 = 0, \quad y_1 : y_3 : y_4 = p_{21} : p_{23} : p_{24} = g_{21}z_1 : g_{23}z_3 : g_{24}z_4.$$

Hence

$$\mu = \frac{x_3}{x_4} = \frac{g_{13}}{g_{14}} \cdot \frac{z_3}{z_4}, \quad \nu = \frac{y_3}{y_4} = \frac{g_{23}}{g_{24}} \cdot \frac{z_3}{z_4},$$

$$\frac{\mu}{\nu} = - \frac{g_{31} g_{24}}{g_{23} g_{14}} = const.,$$

which proves the theorem.

3. From the poles of any planes, with respect to one quadric, we obtain the poles of the same planes, with respect to another quadric, by a collinear transformation. Hence : *If we transform the points of space by a collinear transformation and join every point with its corresponding point, these lines will form a tetrahedral complex, whose fundamental tetrahedron is determined by the self-conjugate points.* Reciprocally : *If we transform the planes of space by a collinear transformation and cut every plane by its corresponding plane, the lines of intersection will form a tetrahedral complex, whose fundamental tetrahedron is determined by the self-conjugate planes.*

For an analytical proof suppose

$$h_1 = h_2 = h_3 = h_4 ;$$

then [compare (4)] a line of the complex has coördinates of the following form :

(13) $$p_{ik} = (g_i - g_k)z_iz_k$$
or

(13') $$p_{ik} = y_iz_k - y_kz_i,$$
by making

(14) $$y_i = g_iz_i \qquad (i = 1, 2, 3, 4).$$

But these equations represent a general collinear transformation, whose fundamental tetrahedron is that of the tetrahedral complex.

Instead of the one transformation just obtained, we may employ any transformation of the form

(15) $$y_i = (g_i - \lambda)z_i,$$

where λ is an arbitrary parameter, and arrive at the same tetrahedral complex. All these transformations are said to form a *pencil*. They convert every line of the complex into itself, and we obtain by them successively all points of the line from any one point upon it.

Starting from planes, we get our tetrahedral complex by any transformation of the form

(16) $$v_i = (\gamma_i - \lambda)w_i, \qquad (i = 1, 2, 3, 4),$$

where v_i and w_i are plane coördinates and

(17) $$\gamma_i = \frac{1}{g_i}.$$

In fact, these equations give, for the line of intersection of two corresponding planes, the following coördinates:

(18) $$g_{ik} = v_iw_k - v_kw_i = (\gamma_i - \gamma_k)w_iw_k,$$

and this line belongs to the same tetrahedral complex as that given by (13).

The transformation (16) is expressed, in point coördinates, as follows:

(19) $$y_i = \frac{1}{\gamma_i - \lambda}z_i, \qquad (i = 1, 2, 3, 4).$$

All these transformations, which we obtain by varying λ, are said to constitute a *series*. By them the planes through any line of the complex pass one into another, and are all to be found from any one among them.

4. If, in the expressions

(13) $$p_{ik} = (g_i - g_k)z_iz_k$$

we put

(20) $z_i = f_i u_i$,   whence $u_i = \dfrac{z_i}{f_i}$,

and, regarding the u_i as plane coördinates, write q_{ik} for p_{ik}, we do not change the tetrahedral complex.   But

(21) $\dfrac{z_1}{f_1}\, x_1 + \dfrac{z_2}{f_2}\, x_2 + \dfrac{z_3}{f_3}\, x_3 + \dfrac{z_4}{f_4}\, x_4 = 0$

represents the polar plane of the point with coördinates z_i with respect to the surface

(22) $\dfrac{x_1^2}{f_1} + \dfrac{x_2^2}{f_2} + \dfrac{x_3^2}{f_3} + \dfrac{x_4^2}{f_4} = 0$,

and this is a general quadric, for which the fundamental tetrahedron is a self-conjugate one.   Thus we see, that *a tetrahedral complex remains unaltered, if in place of each of its lines we take the reciprocal polar of this line with respect to any quadric surface having the fundamental tetrahedron for a self-conjugate tetrahedron.*
     On the other hand, let

$$g_{ik} = y_i z_k - y_k z_i$$

be the coördinates of any line ; then its reciprocal polar for the surface (22) has the coördinates

$$p_{ik} = \dfrac{g_{ik}}{f_i f_k},$$

hence, whatever be the values of the f_i, we have

$$\dfrac{p_{23} p_{14}}{g_{23} g_{14}} = \dfrac{p_{31} p_{24}}{g_{31} g_{24}} = \dfrac{p_{12} p_{34}}{g_{12} g_{34}}.$$

If we take the reciprocal polars of a straight line for all quadric surfaces having a certain common self-conjugate tetrahedron, these polars will form a tetrahedral complex which contains the given line and whose fundamental tetrahedon is the common self-conjugate tetrahedron.
     5. By regarding, in the equations (19), z_i as fixed and λ as variable, we see that every point is transformed by the transformations of the series into the points of a cubic space curve.   We must suppose the right members of the equations to be multiplied by the common factor

$$(\gamma_1 - \lambda)\,(\gamma_2 - \lambda)\,(\gamma_3 - \lambda)\,(\gamma_4 - \lambda)\ ;$$

then they all are cubic functions of λ and vanish for three of the values γ_1, γ_2, γ_3, γ_4, whence for $\lambda = \gamma_i$ three of the coördinates y are $= 0$, and only y_i is $\neq 0$. Thus the cubic curve is circumscribed about the fundamental tetrahedron. I call this curve a pole curve of the tetrahedral complex, and the point with coördinates z_i its pole.

This curve has the singular property, that *every one of its chords is a line of the tetrahedral complex*, as it is easily shown. For, let λ, λ' be the parameters of the points where the pole curve is met by a chord; then the coördinates of this chord are

$$p_{ik} = \frac{z_i}{\gamma_i - \lambda} \cdot \frac{z_k}{\gamma_k - \lambda'} - \frac{z_k}{\gamma_k - \lambda} \cdot \frac{z_i}{\gamma_i - \lambda'}.$$

Hence, if we put

$$w_i = \frac{z_i}{(\gamma_i - \lambda)\,(\gamma_i - \lambda')},$$

write g_{ik} for p_{ik}, and suppress the common factor $\lambda - \lambda'$, we find

$$g_{ik} = (\gamma_i - \gamma_k)\,w_i w_k \qquad\text{[see (18)]}.$$

Every cubic space curve is contained upon a doubly infinite set of quadric surfaces. They all are ruled surfaces, and one of the two sets of straight lines upon them is formed by chords of the cubic curve. In our case the surfaces passing through the pole curve are covered by an infinity of complex lines, and may be generated by a line moving in the tetrahedral complex. We shall call them simply quadrics of the complex. Any two pole curves lie upon *one* quadric, and this is always a quadric of the complex.

6. Before inquiring into the general equation of these quadrics, let us find first the equations of the four cones projecting a pole curve from the vertices of the fundamental tetrahedron. This is immediately done if we omit one of the four equations (19), for instance the last, and eliminate λ from the proportion

$$y_1 : y_2 : y_3 = \frac{z_1}{\gamma_1 - \lambda} : \frac{z_2}{\gamma_2 - \lambda} : \frac{z_3}{\gamma_3 - \lambda}$$

or $\qquad \dfrac{z_1}{y_1} : \dfrac{z_2}{y_2} : \dfrac{z_3}{y_3} = (\gamma_1 - \lambda) : (\gamma_2 - \lambda) : (\gamma_3 - \lambda)$.

Multiplying the several terms of this proportion by $\gamma_2 - \gamma_3$, $\gamma_3 - \gamma_1$, $\gamma_1 - \gamma_2$ and adding, we get

$$(\gamma_2 - \gamma_3)\frac{z_1}{y_1} + (\gamma_3 - \gamma_1)\frac{z_2}{y_2} + (\gamma_1 - \gamma_2)\frac{z_3}{y_3} = 0.$$

Finally we make

(23) $$r_{ik} = \frac{\gamma_i - \gamma_k}{z_i z_k},$$

and then obtain the required equations of the four cones in question

(24) $$\begin{cases} r_{34}y_3y_4 + r_{42}y_4y_2 + r_{23}y_2y_3 = 0, \\ r_{43}y_4y_3 + r_{14}y_1y_4 + r_{31}y_3y_1 = 0, \\ r_{24}y_2y_4 + r_{41}y_4y_1 + r_{12}y_1y_2 = 0, \\ r_{32}y_3y_2 + r_{13}y_1y_3 + r_{21}y_2y_1 = 0. \end{cases}$$

The magnitudes r_{ik}, again subject to the condition

(25) $$r_{23}r_{14} + r_{31}r_{24} + r_{12}r_{34} = 0,$$

fix the pole curve among the cubic curves circumscribed about the fundamental tetrahedron, and may be regarded as its coördinates in the system of these curves. Then the equations (23) show that one of these curves is a pole curve of the tetrahedral complex of lines given by the equation

$$\lambda_1 p_{23}p_{14} + \lambda_2 p_{31}p_{24} + \lambda_3 p_{12}p_{34} = 0$$

if between its coördinates the relation exists

$$\lambda_1 r_{23}r_{14} + \lambda_2 r_{31}r_{24} + \lambda_3 r_{12}r_{34} = 0.$$

The pole curves of a tetrahedral complex of lines form again a tetrahedral complex among the cubic space curves circumscribed about the fundamental tetrahedron.

7. If we multiply the equations (24) respectively by η_1, η_2, η_3, η_4 and add them all, we immediately find the following expression for a quadric surface containing the pole curve :

(26) $$\begin{aligned} &(\eta_1 - \eta_4)r_{23}y_2y_3 + (\eta_2 - \eta_3)r_{14}y_1y_4 \\ &+ (\eta_2 - \eta_4)r_{31}y_3y_1 + (\eta_3 - \eta_1)r_{24}y_2y_4 \\ &+ (\eta_3 - \eta_4)r_{12}y_1y_2 + (\eta_1 - \eta_2)r_{34}y_3y_4 = 0. \end{aligned}$$

These surfaces constitute a double infinity, since the equation of one of them is not altered, if we add any arbitrary constant to the η's, or multiply them by a common factor.

If we put the foregoing equation identical with the following

(27) $$a_{23}y_2y_3 + a_{14}y_1y_4 + a_{31}y_3y_1 + a_{24}y_2y_4 + a_{12}y_1y_2 + a_{34}y_3y_4 = 0,$$

supposing always

(28) $$a_{ik} = a_{ki}$$

and denoting generally the a_{ik} as coördinates of a quadric circumscribed about the fundamental tetrahedron, we find for these magnitudes such expressions as

$$a_{23} = (\eta_1 - \eta_4)r_{23}, \quad a_{14} = (\eta_2 - \eta_3)r_{14}, \quad \text{etc.,}$$

or if we substitute the values of r_{ik} from (23), we get

(29) $$a_{23} = \frac{(\eta_1 - \eta_4)(\gamma_2 - \gamma_3)}{z_2 z_3}, \quad a_{14} = \frac{(\eta_2 - \eta_3)(\gamma_1 - \gamma_4)}{z_1 z_4}, \quad \text{etc.,}$$

and hence

(30) $$\frac{a_{23}a_{14}}{G_1} + \frac{a_{31}a_{24}}{G_2} + \frac{a_{12}a_{34}}{G_3} = 0,$$

where

$$G_1 : G_2 : G_3$$
$$= (\gamma_2 - \gamma_3)(\gamma_1 - \gamma_4) : (\gamma_3 - \gamma_1)(\gamma_2 - \gamma_4) : (\gamma_1 - \gamma_2)(\gamma_3 - \gamma_4)$$
(31)
$$= (g_2 - g_3)(g_1 - g_4) : (g_3 - g_1)(g_2 - g_4) : (g_1 - g_2)(g_3 - g_4)$$

so that

(32) $$G_1 + G_2 + G_3 = 0 ;$$

and

(33) $$\frac{p_{23}p_{14}}{G_1} = \frac{p_{31}p_{24}}{G_2} = \frac{p_{12}p_{34}}{G_3}$$

represents the tetrahedral complex of lines. The equation (30), which defines a set of *four* dimensions among the *five*-dimensional multitude of quadrics circumscribed about the fundamental tetrahedron, gives the necessary and sufficient condition that one of these quadrics is a quadric of the tetrahedral complex.

Especially we may consider the cones among these quadrics of the complex. Every point in space is the vertex of one of them, and the cone itself is generated by the complex lines passing through this point. The equation of such a complex cone is at once derived from (26), if we

make $$(\eta_1, \eta_2, \eta_3, \eta_4) = (\gamma_1, \gamma_2, \gamma_3, \gamma_4),$$

and substitute for the r_{ik} their values (23). We then get

(34) $$\begin{cases} a_{23} = \dfrac{G_1}{z_2 z_3}, \quad a_{31} = \dfrac{G_2}{z_3 z_1}, \quad a_{12} = \dfrac{G_3}{z_1 z_4}, \\[2mm] a_{14} = \dfrac{G_1}{z_1 z_4}, \quad a_{24} = \dfrac{G_2}{z_2 z_4}, \quad a_{34} = \dfrac{G_3}{z_3 z_4}, \end{cases}$$

whence

(35) $$\sqrt{a_{23}a_{14}} + \sqrt{a_{31}a_{24}} + \sqrt{a_{12}a_{34}} = 0.$$

Rationalizing, we may write this same relation

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{vmatrix} = 0,$$

and this is in fact the condition, that the quadric (27) should become a cone. Thus, the complex cone belonging to a point with coördinates z_i has the equation

$$(36) \; G_1 \left(\frac{y_2 y_3}{z_2 z_3} + \frac{y_1 y_4}{z_1 z_4} \right) + G_2 \left(\frac{y_3 y_1}{z_3 z_1} + \frac{y_2 y_4}{z_2 z_4} \right) + G_3 \left(\frac{y_1 y_2}{z_1 z_2} + \frac{y_3 y_4}{z_3 z_4} \right) = 0.$$

8. The foregoing considerations may be repeated in the reciprocal form, without undergoing any other change or restriction, except that in place of the γ_i we must take their reciprocal values g_i. Instead of the pole curves *circumscribed* about the fundamental tetrahedron we then get cubic space curves *inscribed* in it, whose osculating planes intersect in pairs in lines of the complex. The coördinates of the points of these curves are represented by a variable parameter as follows :

$$(37) \qquad\qquad x_i = m_i (g_i - \lambda)^3, \qquad (i = 1, 2, 3, 4),$$

where the m_i are certain magnitudes. Indeed, it is at once clear that the points in which this curve intersects any plane of the fundamental tetrahedron coincide. These four planes are, therefore, osculating planes of the curve and their points of contact correspond respectively to the values g_1, g_2, g_3, g_4 of the parameter λ.

There exists not only a system of quadrics *circumscribed* about the fundamental tetrahedron, which are covered by complex lines, but also a system of *inscribed* quadrics enjoying the same property. Any quadric inscribed to the fundamental tetrahedron has an equation of the form

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} & x_1 \\ a_{21} & 0 & a_{23} & a_{24} & x_2 \\ a_{31} & a_{32} & 0 & a_{34} & x_3 \\ a_{41} & a_{42} & a_{43} & 0 & x_4 \\ x_1 & x_2 & x_3 & x_4 & 0 \end{vmatrix} = 0,$$

and the same equation (30) expresses the condition that the surface may be generated by lines of the complex.

The lines of the tetrahedral complex which lie in a given plane envelop a conic section of this plane touching the four planes of the fundamental tetrahedron. We call it a a complex curve. Then from (36) we see at once that, if

$$w_1 x_1 + w_2 x_2 + w_3 x_3 + w_4 x_4 = 0$$

be the equation of the plane, the equation of its complex curve in plane coördinates v_i is

$$(38) \quad G_1\left(\frac{v_2 v_3}{w_2 w_3} + \frac{v_1 v_4}{w_1 w_4}\right) + G_2\left(\frac{v_3 v_1}{w_3 w_1} + \frac{v_2 v_4}{w_2 w_4}\right) + G_3\left(\frac{v_1 v_2}{w_1 w_2} + \frac{v_3 v_4}{w_3 w_4}\right) = 0.$$

Hence the four cones projecting the conic from the vertices of the fundamental tetrahedron have, in point coördinates, the following equations:

$$(39) \quad \begin{cases} \qquad\quad s_{12}\sqrt{x_2} + s_{13}\sqrt{x_3} + s_{14}\sqrt{x_4} = 0, \\ s_{21}\sqrt{x_1} \qquad\quad + s_{23}\sqrt{x_3} + s_{24}\sqrt{x_4} = 0, \\ s_{31}\sqrt{x_1} + s_{32}\sqrt{x_2} \qquad\quad + s_{34}\sqrt{x_4} = 0, \\ s_{41}\sqrt{x_1} + s_{42}\sqrt{x_2} + s_{43}\sqrt{x_3} \qquad\quad = 0, \end{cases}$$

if we make

$$(40) \qquad s_{23} = \sqrt{G_1 w_2 w_3}, \quad s_{14} = \sqrt{G_1 w_1 w_4}, \quad \text{etc.}$$

The s_{ik}, bound to the identical relation

$$(41) \qquad s_{23}s_{14} + s_{31}s_{24} + s_{12}s_{34} = 0,$$

are the coördinates of the complex-curve among the totality of conics inscribed to the fundamental tetrahedron. From (40) we derive

$$(42) \qquad \frac{s_{23}s_{14}}{G_1} = \frac{s_{31}s_{24}}{G_2} = \frac{s_{12}s_{34}}{G_3}.$$

The complex-curves of a tetrahedral complex of lines form again a tetrahedral complex.

9. We add, in conclusion, a few remarks on the pole curves of our complex, the coördinates of whose points are represented by a variable parameter, as follows:

$$(19) \qquad y_i = \frac{z_i}{\gamma_i - \lambda} \qquad\qquad (i = 1, 2, 3, 4).$$

The plane joining three points of the curve corresponding to parameters λ, λ', λ'' has the equation :

$$(43) \quad \begin{aligned} &\frac{(\gamma_1 - \lambda)(\gamma_1 - \lambda')(\gamma_1 - \lambda'') y_1}{(\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3)(\gamma_1 - \gamma_4) z_1} \\ &+ \frac{(\gamma_2 - \lambda)(\gamma_2 - \lambda')(\gamma_2 - \lambda'') y_2}{(\gamma_2 - \gamma_1)(\gamma_2 - \gamma_3)(\gamma_2 - \gamma_4) z_2} \\ &+ \frac{(\gamma_3 - \lambda)(\gamma_3 - \lambda')(\gamma_3 - \lambda'') y_3}{(\gamma_3 - \gamma_1)(\gamma_3 - \gamma_2)(\gamma_3 - \gamma_4) z_3} \\ &+ \frac{(\gamma_4 - \lambda)(\gamma_4 - \lambda')(\gamma_4 - \lambda'') y_4}{(\gamma_4 - \gamma_1)(\gamma_4 - \gamma_2)(\gamma_4 - \gamma_3) z_4} = 0. \end{aligned}$$

Hence we find the equation of the osculating plane at the point whose parameter is λ

$$(44) \quad \frac{(\gamma_1 - \lambda)^3 y_1}{f'(\gamma_1) z_1} + \frac{(\gamma_2 - \lambda)^3 y_2}{f'(\gamma_2) z_2} + \frac{(\gamma_3 - \lambda)^3 y_3}{f'(\gamma_3) z_3} + \frac{(\gamma_4 - \lambda)^3 y_4}{f'(\gamma_4) z_4} = 0.$$

where $f'(\lambda)$ denotes the derivate function of

$$f(\lambda) = (\gamma_1 - \lambda)(\gamma_2 - \lambda)(\gamma_3 - \lambda)(\gamma_4 - \lambda).$$

Now the osculating planes of a skew cubic in the three points where a plane meets the curve intersect in a point of this plane, and thus by means of the cubic curve there is associated with every plane a point lying in it, and reciprocally to every point a plane passing through it. This correspondence between the planes and points in space has been called by *Moebius* a null system (Nullsystem);[*] every point is the null point of its corresponding plane, and every plane the null plane of its corresponding point. Denoting by u_i plane coördinates and by x_i the coördinates of the conjugate point, we have, for any such transformation, a representation of the following form :

$$(45) \quad \begin{aligned} u_1 &= \qquad\quad p_{12}x_2 + p_{13}x_3 + p_{14}x_4, \\ u_2 &= p_{21}x_1 \qquad\quad + p_{23}x_3 + p_{24}x_4, \\ u_3 &= p_{31}x_1 + p_{32}x_2 \qquad\quad + p_{34}x_4, \\ u_4 &= p_{41}x_1 + p_{42}x_2 + p_{43}x_3 \qquad\quad , \end{aligned}$$

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[*] This name has been chosen, because for the lines in a plane passing through its null point, or for the lines through a point lying in its null plane, the moment of a certain system of forces is zero.

where always
$$p_{ik} + p_{ki} = 0.$$

If we ask what the values of these magnitudes p_{ik} are for the pole curve (19) just considered, we find the following expressions :

$$(46) \quad \begin{aligned} p_{23} &= \frac{(\gamma_1 - \gamma_4)(\gamma_2 - \gamma_3)^2}{z_2 z_3}, & p_{14} &= \frac{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_4)^2}{z_1 z_4}, \\ p_{31} &= \frac{(\gamma_2 - \gamma_4)(\gamma_3 - \gamma_1)^2}{z_3 z_1}, & p_{24} &= \frac{(\gamma_3 - \gamma_1)(\gamma_2 - \gamma_4)^2}{z_2 z_4}, \\ p_{12} &= \frac{(\gamma_3 - \gamma_4)(\gamma_1 - \gamma_2)^2}{z_1 z_2}, & p_{34} &= \frac{(\gamma_1 - \gamma_2)(\gamma_3 - \gamma_4)^2}{z_3 z_4}, \end{aligned}$$

or, if we like,

$$(46') \quad \begin{aligned} p_{23} &= G_1 r_{23}, & p_{31} &= G_2 r_{31}, & p_{12} &= G_3 r_{12}, \\ p_{14} &= G_1 r_{14}, & p_{24} &= G_2 r_{24}, & p_{34} &= G_3 r_{34}, \end{aligned}$$

[compare (23) and (31)].

To verify this result, it suffices to show that the plane corresponding to any point of the pole curve is the osculating plane at this point. Let λ be the parameter of the point and
$$u_1 x_1 + u_2 x_2 + u_3 x_3 + u_4 x_4 = 0$$

the equation of the corresponding plane. Then, considering first the coördinate u_4, we find for it

$$(\gamma_1 - \lambda)(\gamma_2 - \lambda)(\gamma_3 - \lambda) u_4 z_4$$

$$= (\gamma_2 - \gamma_3)(\gamma_1 - \gamma_4)^2 (\gamma_2 - \lambda)(\gamma_3 - \lambda)$$
$$+ (\gamma_3 - \gamma_1)(\gamma_2 - \gamma_4)^2 (\gamma_3 - \lambda)(\gamma_1 - \lambda)$$
$$+ (\gamma_1 - \gamma_2)(\gamma_3 - \gamma_4)^2 (\gamma_1 - \lambda)(\gamma_2 - \lambda).$$

But the right side of this equation is easily shown to be

$$(\gamma_2 - \gamma_3)(\gamma_3 - \gamma_1)(\gamma_1 - \gamma_2)(\gamma_4 - \lambda)^2.$$

This equation is to be multiplied by $(\gamma_4 - \lambda)$ on both sides ; and if we add the corresponding equations for u_1, u_2, u_3, which are at once obtained by a cyclic permutation, and omit the common factor $(\gamma_1 - \lambda)(\gamma_2 - \lambda)(\gamma_3 - \lambda)(\gamma_4 - \lambda)$ at the left, we finally get

$$u_1 = (\gamma_3 - \gamma_4)(\gamma_4 - \gamma_2)(\gamma_2 - \gamma_3)\frac{(\gamma_1 - \lambda)^3}{z_1},$$

$$u_2 = (\gamma_4 - \gamma_1)(\gamma_1 - \gamma_3)(\gamma_3 - \gamma_4)\frac{(\gamma_2 - \lambda)^3}{z_2},$$

$$u_3 = (\gamma_1 - \gamma_2)(\gamma_2 - \gamma_4)(\gamma_4 - \gamma_1)\frac{(\gamma_3 - \lambda)^3}{z_3},$$

$$u_4 = (\gamma_2 - \gamma_3)(\gamma_3 - \gamma_1)(\gamma_1 - \gamma_2)\frac{(\gamma_4 - \lambda)^3}{z_4},$$

as it should be.

From (46) we still derive the following relation :

$$(47) \qquad \sqrt[3]{p_{23}p_{14}} + \sqrt[3]{p_{31}p_{24}} + \sqrt[3]{p_{12}p_{34}} = 0.$$

This is the necessary condition for the determining magnitudes of a zero system, if it is to be found from a skew cubic *circumscribed* about the fundamental tetrahedron. But we should have found the same equation by starting from a skew cubic *inscribed* in the fundamental tetrahedron. In fact, by applying the polar transformation, which is simply represented as follows :

$$(y_1, \ y_2, \ y_3, \ y_4) = (u_1, \ u_2, \ u_3, \ u_4),$$

we get instead of the circumscribed pole curve an inscribed cubic space curve, which furnishes the same zero system, and hence also the equation (47) remains unaltered.