This theorem can also be immediately applied to Bessel's functions whose order is not zero. Let $F_n(x)$ be any real solution of Bessel's equation

$$(4)\qquad \frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0.$$

Using polar coördinates r, ϑ we have as a solution of (3) when n is real

$$u=\cos n\vartheta\cdot F_n(r),$$

when n is pure imaginary

$$u=e^{in\vartheta}\cdot F_n(r).$$

Applying the theorem just quoted to these solutions we get the theorems:

If $n^2\leqq 1$, $F_n(x)$ *vanishes at least once in any interval of length* $2c=4.810\cdots$ *which does not include the origin.*

If $n>1$, $F_n(x)$ *vanishes at least once in any interval of length* $2c$ *throughout which* $|x|>c\left[\csc\frac{\pi}{2n}-1\right]$.

As a special application I note that we thus get an upper limit for the value of the smallest root of $F_n(x)$ and thus in particular of $J_n(x)$.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.

A GENERALIZATION OF APPELL'S FACTORIAL FUNCTIONS.

BY DR. E. J. WILCZYNSKI.

(Read before the American Mathematical Society at the Annual Meeting, December 28, 1898.)

LET
$$F(s, z)=0$$

be an algebraic equation defining s as function of z. Let R, the corresponding Riemann's surface, be of class p. By a system of crosscuts $a_1,\cdots,a_p$; $b_1,\cdots,b_p$; $c_1,\cdots,c_p$ the $(2p+1)$-ply connected surface R is changed into a simply connected surface R_{abc}.

We can solve the following problem : Construct a function $\varphi(z)$ of a point on this surface, a given branch of which is multiplied by a given uniform function of z and s whenever the point crosses one of the crosscuts. It is not in general possible to construct such a function $\varphi(z)$ which is uniform on R_{abc}. Except in special cases there will arise new branch points for $\varphi(z)$ upon this surface, but it is possible to describe the behavior of the function at those points, as well as to give the conditions for their non-occurrence.*

Call generally λ a point on the left, and ρ a point on the right side of any crosscut. Then $\dfrac{\varphi(\lambda)}{\varphi(\rho)}$ is to be a uniform function of s and z for every crosscut. A slight modification of Appell's proof, which applies to the case of constant factors, shows that these multipliers must equal unity for the crosscuts c_k, which we therefore can suppose to be removed. Also the factor belonging to b_k is the same function for both parts of the crosscut.

We proceed then to construct a function $\varphi(z)$ which has the properties that

$$\begin{array}{lll} \text{along } a_k & \varphi(\lambda) = m_k\varphi(\rho) & \\ & & (k = 1, 2, \cdots, p) \\ \text{along } b_k & \varphi(\lambda) = n_k\varphi(\rho), & \end{array}$$

where m_k and n_k are arbitrarily given uniform functions of s and z.

Let $w_1(z), w_2(z), \cdots, w_p(z)$ be the p normal integrals of the first kind. Their moduli of periodicity are exhibited in the well known table

Crosscut.	a_1	a_2	$\cdots$	a_p	b_1	b_2	$\cdots$	b_p
w_1	πi	0	$\cdots$	0	b_{11}	b_{12}	$\cdots$	b_{1p}
w_2	0	πi	$\cdots$	0	b_{21}	b_{22}	$\cdots$	b_{2p}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
w_p	0	0	$\cdots$	πi	b_{p1}	b_{p2}	$\cdots$	b_{pp}

where $$b_{ik} = b_{ki}.$$
It is clear then that

$$(1) \qquad \eta = e^{\frac{1}{\pi i}[w_1(z)\log m_1 + w_2(z)\log m_2 + \cdots + w_p(z)\log m_p]}$$

has the required factors for the crosscuts a_k, so that

* A similar generalization of the functions defined by linear differential equations appears in the April number of the *Amer. Jour. of Math.*

$$\xi = \frac{\varphi}{\eta} \tag{2}$$

has the factors unity for a_k. For crosscut b_k we have

$$\varphi(\lambda) = n_k \varphi(\rho)$$

$$\eta(\lambda) = \eta(\rho) e^{\frac{1}{\pi i}[b_{1k} \log m_1 + b_{2k} \log m_2 + \cdots + b_{pk} \log m_p]}.$$

Hence $\quad \xi(\lambda) = \xi(\rho) \qquad$ along crosscut a_k,

$\qquad \xi(\lambda) = \mu_k \xi(\rho) \qquad$ along crosscut b_k,

where

$$\log \mu_k = \log n_k - \frac{1}{\pi i}[b_{1k} \log m_1 + b_{2k} \log m_2 + \cdots + b_{pk} \log m_p]. \tag{3}$$

Now we can determine p quantities $v_1, v_2, \cdots, v_p$ in such a way that the following function, which we can then call $\xi(z)$, has these properties, viz. :

$$\xi(z) = e^{v_1 \pi_{\alpha_1 \beta_1}(z) + v_2 \pi_{\alpha_2 \beta_2}(z) + \cdots + v_p \pi_{\alpha_p \beta_p}(z)}, \tag{4}$$

$\pi_{\alpha_i \beta_i}(z)$ denoting a normal integral of the third kind, which becomes infinite for α_i and β_i as the expression

$$\log(z - \beta_i) - \log(z - \alpha_i).$$

For, along a_k, $\pi_{\alpha_i \beta_i}(\lambda) = \pi_{\alpha_i \beta_i}(\rho)$, so that $\xi(\lambda) = \xi(\rho)$; and, along b_k,

$$\pi_{\alpha_i \beta_i}(\lambda) = \pi_{\alpha_i \beta_i}(\rho) + 2[w_k(\beta_i) - w_k(\alpha_i)].$$

So $\xi(z)$ will have the required factors along b_k if

$$\log \mu_k = 2 \sum_{i=1}^{p} v_i [w_k(\beta_i) - w_k(\alpha_i)] \qquad (k = 1, 2, \cdots, p). \tag{5}$$

As α_i and β_i are perfectly arbitrary, we may choose them so that the determinant

$$\Delta = | w_k(\beta_i) - w_k(\alpha_i) | \quad (i, k = 1, 2, \cdots, p)$$

does not vanish. We can then determine $v_1, \cdots, v_p$ so as to verify (5); v_i will be a homogeneous linear function of $\log \mu_1, \cdots, \log \mu_p$, with constant coefficients, say

$$(6) \qquad v_i = \sum_{k=1}^{p} a_{ik} \log \mu_k \qquad (i = 1, 2, \cdots, p);$$

v_i being thus determined,

$$(7) \qquad \varphi(z) = e^{\frac{1}{\pi i}\sum\limits_{k=1}^{p} w_k(z) \log m_k + \sum\limits_{k=1}^{p} v_k \pi_{\alpha_k \beta_k}(z)}$$

is a function with the required properties.

The most general function with these properties is

$$\varphi(z) f(z)$$

where $f(z)$ is a function whose factors at the crosscuts are all equal to unity.

Let us proceed to examine the behavior of $\varphi(z)$ for all of the points of the Riemann's surface. This function aside from the crosscut factors is obviously uniform in the vicinity of all points except α_k, β_k, and the zeros and poles of m_k and n_k.

In the vicinity of α_k we have

$$\pi_{\alpha_k \beta_k}(z) = -\log(z - \alpha_k) + P(z - \alpha_k),$$

where $P(z - \alpha_k)$ represents an ordinary power series, and the other integrals, as well as $\log m_k$ and $\log n_k$, are all uniform in the vicinity of α_k if we assume that the poles of m_k and n_k are different from α_k and β_k. Then if z describes a positive circuit around α_k, $\varphi(z)$ changes into

$$A_k \varphi(z)$$

where

$$(8) \quad A_k = e^{-2\pi i v k} = \prod_{j=1}^{p} \mu_j^{-2\pi i a_{kj}} = \prod_{j=1}^{p} \left(\frac{m_1^{b_{1j}} m_2^{b_{2j}} \cdots m_p^{b_{pj}}}{n_j^{\pi i}} \right)^{2a_{kj}}$$

$$(k = 1, 2, \cdots, p).$$

A circuit around β_k multiplies $\varphi(z)$ by $\dfrac{1}{A_k}$, so that a circuit around α_k and β_k leaves $\varphi(z)$ unaltered, provided it is so taken as not to enclose any of the other critical points.

We have just seen that in the vicinity of α_k and β_k,

$$\varphi(z) = e^{-v_k \log(z - \alpha_k)} P_k(z - \alpha_k),$$

$$\varphi(z) = e^{+v_k \log(z - \beta_k)} Q_k(z - \beta_k),$$

respectively. We have assumed that the poles and zeros of

m_k and n_k do not coincide with any of the points α_k and β_k. Hence for $z = \alpha_k$, $v_k = v_k'$ a finite quantity, and also the value $v_k = v_k''$ for $z = \beta_k$. We can therefore multiply $\varphi(z)$ by finite powers of $z - \alpha_k$ and $z - \beta_k$ respectively, so that the product does not become infinite for these values of z. We will then say that $\varphi(z)$ is regular in the vicinity of these points.

In addition to these singular points α_k and β_k, which we can call points of the first kind, we must consider the points of the second kind, viz.: the zeros and poles of m_k and n_k. They give rise to a double kind of multiformity in $\varphi(z)$.

A circuit around a zero ε_i of m_k multiplies $\varphi(z)$ by an expression of the form

$$(9) \qquad B_{ki} = e^{2w_k(z) + e_1\pi_{\alpha_1\beta_1}(z) + e_2\pi_{\alpha_2\beta_2}(z) + \cdots + e_p\pi_{\alpha_p\beta_p}(z)}$$

where $e_1, e_2, \cdots, e_p$ are constants. It is seen at once that these factors B_{ki} have the same value for all simple zeros of m_k, say B_k, and that for a zero of multiplicity λ the factor is B_k^{λ}. For simple poles the factor is $\frac{1}{B_k}$.

Similarly a circuit around a simple zero of n_k multiplies $\varphi(z)$ by a function C_k which is just like B_k except that it contains no integral of the first kind.

But the zeros and poles of m_k and n_k give rise to multiformity in still another way. For the expression A_k by which $\varphi(z)$ is multiplied on making a circuit around α_k is itself multiplied by constants for circuits around the zeros and poles of m_k and n_k. Thus a circuit around a simple zero of m_k multiplies A_k by a constant D_k, and a circuit around a simple zero of n_k by a constant E_k. Obviously if m_k and n_k are rational functions of s and z a circuit enclosing all of the poles and zeros of one of these functions and no other critical points leaves A_k unaltered.

The points α_k and β_k also give rise to a secondary multiformity other than that already mentioned. For a circuit around α_i multiplies B_k and C_k by constant factors b_{ki}, c_{ki} for $i = 1, 2, \ldots, p$.

Finally B_k and C_k are themselves factorial functions, with constant crosscut factors equal to unity for the crosscuts a_l, and to F_{kl}, G_{kl} for B_k and C_k respectively at the crosscuts b_l. Only in case $F_{kl} = G_{kl} = 1$, will *all branches* of $\varphi(z)$ have the factors m_l and n_l at the crosscuts.

If $\varphi(z)$ denote any particular branch of our multiform function, every other branch is therefore found from it by multiplying by all possible combinations of the different factors mentioned.

There are altogether

$$7p + 4p^2$$

factors of which $2p + 4p^2$ are constants.

If m_k and n_k are rational functions of s and z whose poles and zeros are distinct from the points α_k and β_k, $\varphi(z)$ is everywhere regular. Every other function of the same kind is contained in the form

$$\varphi(z)R(s,z),$$

$R(s, z)$ being a rational function of s and z.

In general it is impossible to avoid the introduction of α_k and β_k as secondary branch points, however they may be chosen. But if there are relations between m_k and n_k so that μ_k is a constant, all of the quantities v_k may be taken equal to unity provided that β_i and α_i are chosen in accordance with the relation (5) putting $v_i = 1$, or more generally

$$\log \mu_k = 2 \sum_{i=1}^{q} [w_k(\beta_i) - w_k(\alpha_i)], \qquad (q \geqq p)$$

$$(k = 1, 2, \cdots, p),$$

which can in general be done. A special case hereof is that of constant crosscut factors. The zeros and poles of m_k and n_k will still be branch points. β_k and α_k will be zeros and poles of $\varphi(z)$.

If $\varphi(z)$ is to be uniform on the surface R_{ab}, m_k and n_k must have the form

$$m_k = e^{\rho_k(s,z)} \qquad n_k = e^{\sigma_k(s,z)}$$

where ρ_k and σ_k are uniform functions of s and z. This still leaves $\varphi(z)$ multiform in the vicinity of α_k and β_k. In order that $\varphi(z)$ may be uniform there also, all of the quantities v_k must be integers. But then according to (6) μ_k must be constants, so that we are led back to the case just treated.

If ρ_k and σ_k are rational functions of s and z they can be represented as sums of integrals of the second kind and their derivatives. Suppose the simplest case, that all of the poles of ρ_k and σ_k are distinct; then ρ_k and σ_k are linear functions of integrals of the second kind only. Our function $\varphi(z)$ then reduces to an exponential into which enter products of integrals of the three kinds. It can then be essentially considered as a product of divers θ functions.

All of these functions can appear as integrals of differential equations, a fact which we hope to discuss on some future occasion.

UNIVERSITY OF CALIFORNIA, BERKELEY,
November 7, 1898.

ON THE ARITHMETIZATION OF MATHEMATICS.

BY PROFESSOR JAMES PIERPONT.

(Read before the American Mathematical Society at the Meeting of February 25, 1899.)

*Introduction.**—The following lines are an attempt to show why arithmetical methods form the only sure foundation in analysis at present known. Certain general reasons are indicated in a very suggestive paper by Klein.† I have striven to carry these ideas further and indicate exactly why arguments based on intuition cannot be considered final in analysis. To do this I have grouped certain well known facts so as to support the conclusion which is formulated at the end of this paper. Doubtless a similar train of thought has occurred to others who have dwelt on this fascinating subject, lying on the border line between mathematics and metaphysics ; but I have seen nothing of the kind in print. The argument falls under two heads. The first deals with magnitudes or quantities (Grössen). It is very easy to point out the gross lack of rigor in this respect and to show how its correction leads inevitably to the modern theory of irrational numbers as developed by Weierstrass, Dedekind, or G. Cantor. The matter is so obvious that I have devoted, only a few lines to it. The second heading treats of our intuition. This requires more detail, and I have not hesitated to make the argument appeal to all by citing numerous examples.

---

* These prefatory remarks have been added to the paper since its presentation.

† "Ueber Arithmetisirung der Mathematik." *Göttinger Nachrichten* (Geschäftliche Mittheilungen) 1895, p. 82. See also Miss Maddison's translation in the BULLETIN, 2d series, vol. 2, p. 241.