It may be noticed that W. K. Clifford arrived at these results in precisely the same way (*Proceedings of the London Mathematical Society*, vol. 7, p. 29), and when he found that he had been anticipated by Darboux, expressed his opinion that this, viz: the work of Darboux that we are now considering, is a book which it is almost inexcusable in a geometer not to have read, marked, learned and inwardly digested.

There are many other interesting investigations in the book, especially a study at considerable length of the intersection of a sphere with a cyclide; an extension also of Ivory's theorem to confocal cyclides is worthy of notice. If $A$, $B$, $C$ are three points on the cyclide and $A'$, $B'$, $C'$ are three corresponding points on a confocal cyclide, M. Darboux shows that the relation

$$AB'.BC'.CB'=BA'.CB'.BC'$$

connects the distances between the points.

R. A. ROBERTS.

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## BESSEL FUNCTIONS.

*A Treatise on Bessel Functions and their Applications to Physics.* By ANDREW GRAY and G. B. MATHEWS. Macmillan & Co. 1895. 8vo, x+292 pp.

The transcendental functions to which Bessel's name has been attached are not only of the highest importance in mathematical physics, second perhaps only to the trigonometric and exponential functions, they are also of great interest to the student of pure mathematics both from the formal side and from the point of view of the theory of functions. There has, however, up to this time been no connected treatment of these functions in the English language, with the exception of the utterly inadequate treatment contained in the last sixty-five pages of Todhunter's book, *The Functions of Laplace, Lamé and Bessel*, published twenty years ago. The German monographs by C. Neumann and Lommel make no attempt to cover more than small portions of the subject, and the same is true to an even greater extent of the sections devoted to Bessel's functions in Heine's *Kugelfunctionen*, Basset's *Hydrodynamics*, Rayleigh's *Sound* and elsewhere. Messrs. Gray and Mathews have therefore filled a real gap in mathematical literature.

The authors make it clear in their preface that their own interest is chiefly for the applications to mathematical physics. They have, however, none of the intolerance for and dislike of pure mathematics which is unfortunately too often manifested. The authors' position may, perhaps, be best explained in their own words: "Some readers may be inclined to think that the earlier chapters contain a needless amount of tedious analysis. . . . . . As a matter of fact it will be found that little, if any, of the analytical theory included in the present work has failed to be of some use or other in the later chapters; and we are so far from thinking that anything superfluous has been inserted that we could almost wish that space would have allowed of a more extended treatment, especially in the chapters on the complex theory and on definite integrals."

The book consists of fifteen chapters, followed by a few pages of miscellaneous matter and tables. Of the fifteen chapters, eight (87 pages) are devoted to the theory, while the remaining seven (135 pages) treat of the applications. This separation of theory and application is by no means complete, much analysis which might easily have been included in the earlier chapters being left until needed, while a few physical problems are introduced for the sake of illustration into the chapters on the theory of Bessel's functions. We think there can be no doubt that the authors have been wise in carrying this separation at least as far as they have done, as it conduces greatly to the clearness of the book, and makes it, what is very important, a convenient book of reference. The book might have been made even more useful in this respect than it is now by a somewhat extended collection of formulæ, including in particular a table of definite integrals involving Bessel's functions, which could have been placed with the numerical tables at the end of the book.

CHAPTER I: *Introductory* shows how Bessel's functions naturally present themselves in two simple physical and in one astronomical problem. The problems chosen (vibration of a heavy chain hanging from one end, cooling of a cylinder, Kepler's equation) are not only interesting in themselves; they have also a decided historic interest as being among the first problems in which Bessel's functions were used.*

* There is a slight historical slip on p. 2, in the statement that after the problem of the heavy chain "the next appearance of a Bessel function is in Fourier's *Théorie analytique de la chaleur.*" The problem of the vibration of a circular membrane had been treated by Euler in 1764 (Cf. *Bulletin*, February, 1893, p. 108). A reference to Bessel's paper of 1818 might also well have been made.

CHAPTER II: *Solution of the Differential Equation.* The title of this chapter indicates the starting point chosen by the authors for the systematic study of Bessel's functions. The differential equation is solved by means of power series and a few simple relations between the Bessel's functions thus obtained are established. On p. 8 it is pointed out that the series for $J_{-n}(x)$ ceases to exist (" becomes unintelligible " is the unfortunate expression used) when $n$ is a positive integer, and on p. 12 the function is defined in this case by the formula : $J_{-n}(x)=(-1)^n J_n(x)$, even the mere statement of the reason for this definition (viz.: to preserve the continuity with regard to $n$) being relegated to a footnote. The treatment of such a fundamental point as this, which would not have covered half a page, ought surely to have been given. We are glad to find the Bessel's function of the second kind * $Y_n(x)$ introduced here by means of power series ; the interesting, but for the beginner artificial, form in terms of Bessel's functions of the first kind is given later (pp. 21–23).

CHAPTER III: *Functions of Integral Order. Expansions in series of Bessel Functions.*—This chapter opens with Schlömilch's elegant definition of Bessel's functions of the first kind of integral orders as the coefficients in the expansion of $e^{\frac{x}{2}(t-1/t)}$ according to ascending and descending powers of $t$. Here as elsewhere in the book the authors have adopted a half-way course with regard to mathematical rigor which can perhaps be justified when we consider the class of readers for which the book is written. Thus they usually establish the convergency of the series or integrals with which they deal, but leave the question as to whether the series or integrals converge to the right value, and also questions as to rearranging the terms of a series, differentiating a series, etc., almost, if not quite, unconsidered. It should, however, be added that the authors sometimes regard this method of obtaining results as merely heuristic and establish their results when once found by other methods.

CHAPTER IV: *Semiconvergent Expansions* might well have been postponed until later, as at this point only Bessel's functions of the first kind can be treated at all, and that in a very imperfect way. The term *semiconvergent* is here used in its proper sense, a series of this sort being divergent, but such that for certain values of the argument and of $n$

* The authors have seen fit to attach Neumann's name to this function, although in the case $n=0$ the function had been used by Euler.

the sum of its first $n$ terms gives an approximation to the function we wish to represent. The use of the word in this sense is more than forty years old, and it is greatly to be regretted that of late years the term has been used in English in a totally different sense, viz.: non-absolutely or conditionally convergent. After the general solution $u_r$ of Bessel's equation has been developed into a semiconvergent series the statement is made (p. 37): "It is to be observed ..... that it is not to be inferred that the expression above given for $u_r$ reflects in any adequate way the *functional* properties of an exact solution ..... These considerations, however, do not debar us from employing the expressions $u_r$ up to a certain point for approximate numerical calculation ....." The distinction here made between the numerical values of a function and its "functional properties" is one which would, we think, be found in the minds of a very large number of people. There exists, however, no such distinction in reality, as a function is nothing more nor less than what might be called an ideal table of numerical values, and any properties of a function are merely propositions concerning these numerical values. The peculiarity of a semiconvergent series is that it does not permit us, like the ordinary representations of functions, to compute these numerical values to any required degree of accuracy, but merely to a certain limited degree of accuracy. The subject of semiconvergent series together with the allied subject of the asymptotic values of functions is one with which most readers of this *Treatise* are sure to be so unfamiliar, and it is at the same time a subject of such importance both practically and theoretically, that we think a more extended and more accurate treatment might well have been given of it.

CHAPTER V: *The Zeros of the Bessel Functions* gives far too brief a treatment (a little over six pages) of one of the most important and fascinating questions connected with Bessel's functions. The chapter opens with Bessel's original proof that $J_0(x)$ has an infinite number of real roots, this proof being reproduced "on account not only of its historical interest, but of its directness and simplicity." The fact that the considerations here given not only establish the existence of an infinite number of real roots, but also serve to completely separate them, is not, however, touched upon. Stokes's method of computing the large roots of $J_0(x)$ by means of semiconvergent series is also treated in this chapter, but no discussion is given of the degree of accuracy which it enables us to attain. Another matter which might well have found a place, is the application of the method developed by Sturm,* or of some suitable modification of it † to Bessel's equation. At least a reference should have been made to Hurwitz's paper in Math. Ann., vol. 33.

CHAPTER VI: *Fourier-Bessel Expansions.* Certain definite integral formulæ are first deduced by means of Green's theorem. Then follows a simple problem in the flow of heat which leads up to the development of an arbitrary function in the form: $\sum_{s=1}^{s=\infty} A_s J_0(\lambda_s r)$ where the quantities $\lambda_s$ are the roots of the equation $\lambda k J_0'(\lambda) + h J_0(\lambda) = 0$. Other developments of a similar nature follow. The work in this chapter is merely formal, no questions of convergency being discussed.

CHAPTER VII: *Complex Theory.* We agree with the authors in regretting that they did not feel able to devote more space to this interesting subject. Bessel's functions are treated in this chapter by means of their definition as definite integrals taken along paths in the complex plane. The functions with pure imaginary argument are here introduced, and the chapter closes with Lipschitz's strict treatment of the semiconvergent expansions already given.

CHAPTER VIII: *Definite Integrals Involving Bessel Functions.* The first half of this chapter is devoted to a clear and well-arranged exposition of the most fundamental parts of this side of the subject which seems to have appealed strongly to the authors. The last half of the chapter deals with the expression of an arbitrary function by means of a remarkable double integral involving Bessel's functions. This integral, which was in substance discovered by C. Neumann, belongs to an extended class of integrals of which Fourier's integral may serve as a type. In fact this integral bears precisely the same relation to the expansion in terms of Bessel's functions considered in Chapter VI. that Fourier's integral bears to Fourier's series.‡ This fact is unfortunately not brought out by the authors although the similarity to Fourier's integral is mentioned.

CHAPTER IX: *The Relation of the Bessel Functions to*

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* Liouville's Journal, vol. I.

† See, for instance, Riemann-Hattendorf's *Partielle Differentialgleichungen*, p. 266-269, where, however, only the simplest case is considered and the presentation is somewhat faulty.

‡ This analogy seems the most natural, at least for a treatise on Bessel's functions. It is, however, also true, as has been pointed out by C. Neumann and Mehler, that one may regard this integral as having the same relation to the ordinary development in spherical harmonics, that Fourier's integral bears to Fourier's series.

*Spherical Harmonics.* In view of the physical tendencies of the book it might perhaps have been well, instead of giving a purely analytical treatment of this subject, to develop it in connection with the fact that cylinder coördinates may be regarded as a limiting case of polar coördinates in space. The treatment given, however, brings out clearly the analytic relations.

Before going on to the portion of the book in which the applications of Bessel's functions are taken up let us look back at the theoretical chapters we have just been considering in a somewhat more comprehensive way than we have yet done. While the treatment is too incomplete and even at times inaccurate to satisfy the pure mathematician, it is well adapted to the object the authors had in view of giving to the physicist a working knowledge of the more important properties of Bessel's functions. Even to the student of pure mathematics these chapters give, on account of the numerous sides of the subject on which they touch, an introduction to the theory of Bessel's functions which may fairly be considered satisfactory when supplemented by a certain amount of collateral reading. The presentation is clear and interesting and the references to the original memoirs useful.

The writers have avoided giving undue prominence to one side of the theory at the expense of other sides. There is, however, one matter of no little importance as it seems to us which the authors have hardly touched upon, namely, the differential equation which Bessel's functions satisfy. This equation to be sure is the starting point in Chapter II, but, when once its solutions have been expanded into series, these series, or the definite integrals which are proved equal to them, are made use of whenever any property of the functions is to be established.* Many of the properties of Bessel's functions may, however, be derived with ease and elegance directly from the differential equation. For this purpose one of the binomial forms (line 4, Chapter II, and (81) p. 35) is usually most convenient. The formulæ on pages 16 and 53 might well have been obtained in this way, while perhaps the most important matter which could best be so treated is the subject of the roots of Bessel's functions and allied questions. It should be borne in mind in introducing a student to the study of Bessel's functions that these are only an example of functions which occur frequently in mathematical physics, and which satisfy linear

* Slight exceptions to this statement will be found in Chapters IV, and VI.

differential equations of the second order. With the exception of a few simple cases, which like Bessel's functions can be represented by definite integrals and simple series, no other means is usually available for discussing the properties of these functions, than the differential equation which they satisfy. It seems therefore advisable that in the case of Bessel's functions, although other methods are more or less applicable, the differential equation should not be altogether neglected.

Another matter, a brief treatment of which might well have been included in this treatise, is the expression for the ratio of Bessel's functions of orders $n$ and $n+1$ as a continued fraction.

Passing now to the chapters in which the applications to physical problems are considered, we are struck throughout by the very substantial knowledge of mathematical physics assumed on the part of the reader. This contrasts with the small amount of pure mathematics assumed in the earlier chapters. The simpler problems involving Bessel's functions, such for instance as are taken up in Byerly's text book on *Fourier's Series*, are usually omitted on the ground that the method of treatment is obvious. Here, as in the case of the earlier part of the book, the great diversity of subjects treated, borrowed from German as well as English writers, is an excellent feature.

CHAPTER X: *Vibrations of Membranes.* The free vibrations of circular membranes are here very briefly treated. Many interesting questions connected more or less closely with this subject and involving the use of Bessel's functions are not even mentioned, owing doubtless to the excellent treatment in Lord Rayleigh's *Theory of Sound*. It may be mentioned that formula (21) at the close of the chapter reduces at once by means of the relation

$$J_n(x)\,Y'_n(x) - Y_n(x)\,J'_n(x) = \frac{1}{x},$$

to the form:

$$L = \frac{\pi}{2\varkappa^2[J_n(\varkappa a)]^2}\left(\frac{1}{[Y_n(\varkappa b)]^2} - \frac{1}{[Y_n(\varkappa a)]^2}\right).$$

CHAPTER XI: *Hydrodynamics*, opens with a tripos problem concerning vortex motion in which the lines of flow lie in planes passing through an axis. It does not look as though the authors had noticed that the transcendental equation (10) for determining $n$ has no real roots. In order to avoid imaginaries, the functions $J_1$, $Y_1$, and cosh should have been used instead of $I_1$, $K_1$ and cos. There follows a sketch of an investigation by Lord Kelvin on the small oscillations of a cylindrical vortex about a state of steady motion, and on waves in a cylindrical tank. The chapter closes with a treatment of the two dimensional motion of a viscous liquid which is of peculiar interest as introducing for the first time in the book Bessel functions whose arguments are neither real nor pure imaginary, but of the form $a\sqrt{i}$ where $a$ is real.

CHAPTER XII: *Steady Flow of Electricity or of Heat in Uniform Isotropic Media.* While the problems considered in the preceding chapter led to developments in series of the sort considered in chapter VI, those considered in the present chapter lead to developments not in series, but in definite integrals. The first problem taken up, while simple, is typical of the kind of question treated in this chapter, and it will be worth while for us to consider it in some detail. The problem is, to find the electrostatic potential due to a circular conducting disk charged with electricity. The origin is taken at the centre of the disk and cylinder coördinates $z$, $r$, $\varphi$ are used, the axis of $z$ being perpendicular to the plane of the disk. As the potential is symmetrical with regard to the plane of the disk we need consider only that half of space in which $z$ is positive. The method pursued by the authors is to write down the expression:

$$V=\frac{2c}{\pi}\int_0^\infty e^{-\lambda z}\sin(\lambda r_1)J_0(\lambda r)\frac{d\lambda}{\lambda},$$

$c$ being the potential on the disk and $r_1$ the radius of the disk, and then to verify that this is the desired potential.* This method, while perfectly conclusive, is not satisfactory to the reader as it does not put him in a position to solve a similar problem for himself. We may, however, proceed as follows: From the known values of the surface density at points of the plate we have when $z=+0$ and $r<r_1$:

$$\frac{\partial V}{\partial z}=-\frac{2c}{\pi}\frac{1}{\sqrt{r_1^2-r^2}}.$$

When $z=0$ and $r>r_1$, we have from symmetry: $\frac{\partial V}{\partial z}=0$. We need then merely to find a solution of Laplace's equation which satisfies these conditions and vanishes in the proper way at infinity. Since we have circular symmetry about the axis of $z$ we shall try to build up our potential from so-

* This verification is unnecessarily long; formula (13) *p.* 126 and the corresponding formula when $r>r_1$ being all that is necessary.

lutions of the form: $e^{-\lambda z} J_0(\lambda r)$. Since there is no reason to restrict the value of $\lambda$ here we shall naturally form from these particular solutions not a series but an integral:

$$V = \int_0^\infty a(\lambda)\, e^{-\lambda z} J_0(\lambda r)\, d\lambda.$$

From this we get:

$$\frac{\partial V}{\partial z} = -\int_0^\infty \lambda\, a(\lambda)\, e^{-\lambda z} J_0(\lambda r)\, d\lambda.$$

We have then merely to determine the coefficient $a(\lambda)$ in such a way that:

$$\int_0^\infty \lambda\, a(\lambda)\, J_0(\lambda r)\, d\lambda = \begin{cases} \dfrac{2c}{\pi} \dfrac{1}{\sqrt{r_1^2 - r^2}} & \text{when } r < r_1, \\ 0 & \text{when } r > r_1. \end{cases}$$

This can be done at once by means of formula (166) $p$. 80, which gives:

$$a(\lambda) = \frac{2c}{\pi} \int_0^{r_1} \frac{\rho}{\sqrt{r_1^2 - \rho^2}} J_0(\lambda\rho)\, d\rho.$$

This definite integral can be evaluated (cf. Sonine, Math. Ann. Bd. 16 $p$. 36 bottom) and gives:

$$a(\lambda) = \frac{2c}{\pi} \frac{\sin(\lambda r_1)}{\lambda}.$$

Thus we are led to the value of $V$ given above.

Further problems of a similar, though more complicated nature are treated in the remainder of this chapter which follows closely a paper by H. Weber. Many readers will doubtless wish that the authors had seen fit to devote a little more space to Riemann's treatment of Nobili's rings, as the original is far from easy reading.

CHAPTER XIII: *Propagation of Electromagnetic Waves along Wires.*—As a starting point for this recently developed and already very important subject the equations of the electromagnetic field are first given in general form, and then so specialized as to fit the case in which the waves are guided by a straight wire. $x$ being the distance measured along the wire, $\rho$ the distance from the wire, and $t$ the time, the cases considered are those in which the forces are the real parts of expressions of the form:

$$f(\rho)\, e^{(mx - nt)i}$$

$n$ being real, but $m$, in general, complex. The Bessel's functions which are thus introduced have complex arguments. The methods of work are in the main, those of J. J. Thomson and Hertz. No questions involving developments in series are taken up.

CHAPTER XIV : *Diffraction of Light* contains rather elaborate investigations, chiefly due to Lommel, of the diffraction produced by a small circular opening in a screen, in the course of which interesting numerical results accompanied by diagrams are given. Bessel's functions are here introduced in a way different from anything that has gone before (except the treatment of Kepler's equation), occurring directly as definite integrals and not as elsewhere as the solution of a differential equation. Some extensive analytical investigations find place in this chapter which might better, we think, have been placed in the earlier theoretical part of the book.

CHAPTER XV : *Miscellaneous Applications*, though short, is of decided interest. We choose for special mention from among the half dozen questions here considered the following beautiful problem due to Greenhill: a circular elastic cylinder of small cross section, (*e. g.*, a knitting needle) is held in a vertical position with its lower end clamped and upper end free ; to find the greatest length consistent with stability.

There follow two notes, one on Bessel's functions of the second kind, and one on the formula due to J. McMahon for the calculation of the roots of Bessel's functions. The next fifteen pages are devoted to examples to be worked out by the reader. Among these pure mathematics predominates but physical problems also occur in considerable number.

Last but not least come the numerical tables. Of these the first three are due in their present form to Meissel, while the other three are taken from the reports of the British Association. Table I is a twelve place table of the values of $J_0(x)$ and $J_1(x)$ from $x=0$ to $x=15.5$ at intervals of 0.01. This is merely an enlargement of Bessel's original table. Table II which is here published for the first time gives for positive integral values of $n$ and $x$ the values of $J_n(x)$ from $x=1$ to $x=24$ to eighteen decimal places. Table III gives to sixteen decimal places the first fifty roots of $J_1(x)=0$ with the corresponding values of $J_0(x)$. Table IV gives the complex values of $J_0(x\sqrt{i})$ for real values of $x$ from 0 to 6 at intervals of 0.2. The values are given to nine decimal places. Table V gives the values of $I_1(x)=-i\,J_1(ix)$ from $x=0$ to $x=5.1$, at intervals of 0.01 to nine decimal places. Table VI gives, chiefly to twelve significant figures, the values of $I_n(x)$ for integral values of $n$ from 0 to 11 and for values of $x$ at intervals of 0.2 from 0 to 6. This collection of tables will be highly appreciated by all who have to use Bessel's functions in numerical work. A table similar to table III but giving the roots of the equation $J_0(x)=0$ and the corresponding values of $J_1(x)$ would be a welcome addition in a future edition. Such a table would be useful in the numerous problems involving the development of unity in the interval from zero to $a$ in a series of the form $\sum_{s=0}^{s=\infty} A_s J_0(\lambda_s r)$ where $\lambda_s$ is the $s^{\text{th}}$ root of the equation $J_0(\lambda a)=0$.

A short bibliography which though confessedly incomplete will be found useful, and a drawing of the curves $y=J_0(x)$ and $y=J_1(x)$ close the volume.

Serious misprints seem to be rare. On page 14, however, there is one which deserves mention as it occurs in an important formula. The last term in formula (31) reads $-\sum_1^n \frac{1}{2s}$. The minus sign should be changed to plus.

MAXIME BÔCHER.

HARVARD UNIVERSITY,
March, 1896.

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## MODERN METHODS OF ANALYTICAL GEOMETRY.

*An Introductory Account of Certain Modern Ideas and Methods in Plane Analytical Geometry.* By CHARLOTTE ANGAS SCOTT, D. Sc., Girton College, Cambridge; Professor of Mathematics in Bryn Mawr College, Pennsylvania. XII+288 pp., octavo. London and New York, Macmillan and Co. $2.50.

A minor excellence of this book, for which many readers will feel truly grateful, is the fact that it is written in the English of English speaking and writing people. Private abbreviations, cabalistic marks necessitating constant reference to an elusive "list of signs," Teutonisms, and Greek logomachy in the way of "tetrastigms," etc., are agreeably absent. The parvenu "join" is flattered with recognition, but this term is now in such general use that to protest further against it will be of little avail. It is in a measure a consolation that no one is as yet permitted to "enthuse" over this acquisition to the language. "Joining line" is a