GAUSS'S THIRD PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA.

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IT is hoped that the following note may be of interest to some readers of the BULLETIN as indicating the connection between Gauss's third proof that every algebraic equation has a root (*Ges. Werke,* vol. III, p. 59 and p. 107), and those branches of mathematics which have since been developed under the names of the Theory of Functions and the Theory of the Potential. The considerations which follow have doubtless suggested themselves to other readers of Gauss's proof, and it even seems extremely probable that Gauss was led to the discovery of his proof by some method not very different from that here indicated.

Each of the three paragraphs of the present note which are marked with a roman numeral contains a complete proof of the theorem that every algebraic equation has a root. These proofs are arranged in the order of increasing complexity of detail, but of decreasing number of theorems assumed known. The last is essentially Gauss's proof.

Let $f(z) = 0$ be the equation (of the nth degree) for which we wish to prove the existence of a root, and suppose that in the polynomial $f(z)$ the coefficient of z^n is 1. The idea which underlies the proof we shall give is, that if we can prove that $\phi(z)/f(z)$, where $\phi(z)$ is a polynomial, does not remain finite for all values of z, $f(z) = 0$ must have a root. In what follows I let $\phi(z) = zf'(z) = zdf(z)/dz$. We will write $F(z) = \dfrac{zf'(z)}{f(z)} = u(x, y) + iv(x, y)$, where $z = x + yi$. Before taking up the various forms of proof we will note two points: 1st, $u(o, o) = 0$; 2d, if we describe a circle of radius a about the origin, $u(x, y)$ can be made positive at all points on the circumference of this circle by taking a sufficiently large, since $F(\infty) = n$.

Proof I.—$F(z)$ is obviously a monogenic function so that its real part $u(x, y)$ satisfies Laplace's equation. Moreover, if $f(z) = 0$ had no root, u would be finite, continuous, and single valued together with its first derivatives throughout the entire z-plane. We could therefore apply to it the proposition (familiar from the theory of the potential) that the average value of u upon the circumference of any circle is equal to the value of u at the centre. The value of u at the origin, however is zero while the average value of u upon the circumference of a sufficiently large circle with centre at the origin is positive. We are thus led to a contradiction if we assume that $f(z) = 0$ has no root.

This proof, while of extreme simplicity, presupposes a certain knowledge of the theory of Laplace's equation. The proof ordinarily given of the average value theorem we have just used is based upon Green's theorem. Only that special case of Green's theorem is necessary, however, which is known as Gauss's theorem, viz.:

$$\int \frac{\partial u}{\partial n} ds = 0,$$

the integral being taken around the boundary of a region in which u satisfies Laplace's equation, and is, together with its first derivatives, finite, continuous, and single valued, and in which n denotes the (say external) normal. To deduce our average value theorem for the circle of radius a from this we have merely to take as our path of integration a circle of radius $r \leq a$ concentric with the given circle of radius a. Using polar coordinates, Gauss's theorem may be written

$$\int_0^{2\pi} \frac{\partial u}{\partial r} d\phi = 0,$$

from which follows

$$\int_0^a \int_0^{2\pi} \frac{\partial u}{\partial r} d\phi dr = 0.$$

If we now reverse the order of integration and indicate the value of u at the centre by u_0, and the values of u on the circle of radius a by u_a, we get

$$\int_0^{2\pi} [u_a - u_0] d\phi = 0,$$

from which the desired mean value theorem follows at once.

It remains then to prove Gauss's theorem. Instead of deducing this theorem from Laplace's equation we may deduce it more simply for our purposes by the use of conjugate functions. The whole proof of the fundamental theorem of algebra may then be stated as follows:

Proof II.—If $f(z) = 0$ had no root, both u and v would be everywhere finite, continuous, and single valued, together with their first derivatives. Moreover, being conjugate functions they satisfy the relation

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial s},$$

n and s representing any two directions such that the angle from n to s is 90°.* We have then, the integrals being taken around a circle of radius r and with centre at the origin,

$$\int_0^{2\pi} \frac{\partial u}{\partial r} d\phi = \frac{1}{r}\int_0^{2\pi} \frac{\partial v}{\partial \phi} . d\phi = 0,$$

from which we deduce

$$\int_0^a \int_0^{2\pi} \frac{\partial u}{\partial r} d\phi dr = 0.$$

If we here reverse the order of integration and remember that u vanishes at the origin, we get

$$\int_0^{2\pi} u d\phi = 0,$$

the integral being taken around a circle with centre at the origin and with any radius a. But this is impossible since by taking a sufficiently large we can make u everywhere positive.

Finally, if we do not wish to make use of even the fundamental formulæ for monogenic functions, we may proceed as follows:

Proof III.—Let $f(z) = z^n + (a_1 + b_1 i)z^{n-1} + \ldots + (a_{n-1} + b_{n-1} i)z + a_n + b_n i = \sigma + \tau i$, and let $zf'(z) = \sigma' + \tau' i$. Then, letting $z = r(\cos\phi + i \sin\phi)$, we have:

$$\begin{aligned}
\sigma &= r^n \cos n\phi + a_1 r^{n-1} \cos(n-1)\phi + \ldots \\
&\qquad - b_1 r^{n-1} \sin(n-1)\phi - \ldots, \\
\tau &= r^n \sin n\phi + a_1 r^{n-1} \sin(n-1)\phi + \ldots \\
&\qquad + b_1 r^{n-1} \cos(n-1)\phi + \ldots, \\
\sigma' &= nr^n \cos n\phi + (n-1)a_1 r^{n-1} \cos(n-1)\phi + \ldots \\
&\qquad - (n-1)b_1 r^{n-1} \sin(n-1)\phi - \ldots,
\end{aligned}$$

* The familiar formulæ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ merely express this fact for two special pairs of directions, and conversely the more general statement of the text can readily be deduced from these two formulæ.

$$\tau' = nr^n \sin n\phi + (n-1)a_1 r^{n-1} \sin (n-1)\phi + \ldots$$
$$+ (n-1)b_1 r^{n-1} \cos (n-1)\phi + \ldots,$$

$$F(z) = \frac{\sigma' + \tau' i}{\sigma + \tau i} = \frac{\sigma\sigma' + \tau\tau'}{\sigma^2 + \tau^2} + \frac{\sigma\tau' - \tau\sigma'}{\sigma^2 + \tau^2} i = u + vi.$$

We wish now to find the derivatives of u and v with regard to r and ϕ. For this purpose we note the following relations:

$$\frac{\partial \sigma}{\partial r} = \frac{\sigma'}{r}, \quad \frac{\partial \sigma}{\partial \phi} = -\tau',$$

$$\frac{\partial \tau}{\partial r} = \frac{\tau'}{r}, \quad \frac{\partial \tau}{\partial \phi} = \sigma'.$$

We also have formulæ of precisely the same sort for expressing the derivatives of σ' and τ' with regard to r and ϕ in terms of σ'' and τ'' where

$$\sigma'' = n^2 r^n \cos n\phi + (n-1)^2 a_1 r^{n-1} \cos (n-1)\phi + \ldots$$
$$- (n-1)^2 b_1 r^{n-1} \sin (n-1)\phi - \ldots,$$

$$\tau'' = n^2 r^n \sin n\phi + (n-1)^2 a_1 r^{n-1} \sin (n-1)\phi + \ldots$$
$$+ (n-1)^2 b_1 r^{n-1} \cos (n-1)\phi + \ldots$$

We get then by direct differentiation

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \phi}$$
$$= \frac{(\sigma^2+\tau^2)(\sigma\sigma''+\tau\tau'') + (\sigma\tau'-\tau\sigma')^2 - (\sigma\sigma'+\tau\tau')^2}{r(\sigma^2+\tau^2)^2} = T.$$

Now form the double integral

$$\Omega = \int_0^a \int_0^{2\pi} T d\phi dr.$$

If here we integrate first with regard to ϕ and then with regard to r, we obviously get $\Omega = 0$. If, however, we integrate first with regard to r and then with regard to ϕ, we get, remembering that u vanishes at the origin,

$$\Omega = \int_0^{2\pi} u d\phi,$$

the integral being taken around the circumference of a circle with radius a and centre at the origin, so that Ω will be positive if a is sufficiently large. The fact that we get different values for Ω according to the order of integration shows that T cannot be everywhere finite, continuous, and single valued, and this can be explained only by the vanishing of $\sigma^2 + \tau^2$ (since r, which also occurs in the denominator of T is a factor of each term of the numerator). A point where $\sigma^2 + \tau^2$ vanishes is a root of $f(z) = 0$.

In the proofs above given I have started with Gauss from the function $\dfrac{zf'(z)}{f(z)}$. There are, however, other functions which might have been used in almost exactly the same way, as for instance $\dfrac{z^n}{f(z)}$ and $\dfrac{1}{f(z)}$. In fact Gauss's proof would be somewhat simplified by the use of this last function.

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NOTES.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York, Saturday afternoon, April 27, at three o'clock. There were fourteen members present. In the absence of the president and vice-president, Professor Mansfield Merriman occupied the chair. On the recommendation of the council the following persons, nominated at the preceding meeting, were elected to membership: Professor Sara Antoinette Acer, Wells College, Aurora, N. Y.; Dr. Harris Hancock, University of Chicago, Chicago; Professor Munroe Benjamin Snyder, Central High School, Philadelphia. One nomination for membership was received. The following papers were presented:

(1) "On the derivation of the equations of rotation of bodies of variable form," by Professor R. S. WOODWARD.

(2) "A theory of mathematical methods," by Dr. E. M. BLAKE.

(3) "Kinetic stability of central orbits," by Professor W. WOOLSEY JOHNSON.

Professor Johnson's paper appears in the present number of the BULLETIN on page 193.

B. G. TEUBNER, of Leipsic, announces as in press the third volume of Dr. ERNST SCHRÖDER'S *Algebra der Logik;* it is devoted to the algebra and logic of relatives. The same publisher has in preparation an edition, in two 8vo volumes, of JULIUS PLÜCKER'S collected mathematical and physical papers. This publication is due to the initiative of the Göttingen Academy of Sciences. The first volume, which will be