It follows that, while with any law of central attraction a circular orbit is possible with any radius r_0, it will be $\left\{\begin{matrix}\text{stable}\\ \text{unstable}\end{matrix}\right\}$ according as $\left.\dfrac{u_0}{P_0}\dfrac{dP}{du}\right]_0$ is $\left\{\begin{matrix}\text{less}\\ \text{greater}\end{matrix}\right\}$ than 3.*

8. The case in which $P = \mu u^3$ is peculiar, since the criterion is then identically equal 3. The special case occurs when $C = 0$, the orbit being an equiangular spiral unless $h^2 = \mu$, which makes $\gamma = 90°$, when it becomes a circle, and the circle must be regarded as described with kinetic instability.

LAGRANGE'S PLACE IN THE THEORY OF SUBSTITUTIONS. †

BY DR. JAMES PIERPONT.

IN the present brief note I cannot vindicate Lagrange's right to the title of creator of the theory of substitutions; but I hope, by presenting a few examples of his methods, to show the importance of considering him from this point of view. Lagrange was led to the study of this theory by his attempts to solve equations of degree higher than the fourth. Speaking of the inherent difficulties which this thorny subject offered to the investigator, he remarks: ‡

"The theory of equations is of all parts of analysis the one, we would think, which ought to have acquired the greatest degree of perfection, by reason both of its importance and of the rapidity of the progress that its first inventors made; but although the mathematicians of later days have not ceased to apply themselves, there remains much in order that their efforts may meet with the success that one could desire. In regard to the resolution of literal equations one has hardly advanced further than one was in Cardan's time, who was the first to publish the resolution of equations of the third and fourth degree. The first successes of the Italian analysts in this branch seem to have marked the limit of possible discoveries: at least it is certain that all attempts that have been made up to the present to push back the limits of this branch of algebra have hardly served for other purposes than

* An equivalent criterion is otherwise derived in Thomson and Tait's Natural Philosophy, § 350.

† Read before the Yale Mathematical Club.

‡ Lagrange: Nouveaux Mémoires, Acad. Sciences Berlin, years 1770–71. Also, Œuvres, vol. III, pp. 205–421, Réflexions sur la résolution algébrique des équations.

to find new methods to solve the equations of third and fourth degree, none of which seem applicable to equations of higher degrees."

In his great paper published in the Mémoires of the Academy of Sciences at Berlin in the years 1770–71, under the title "Réflexions sur la résolution algébrique des equations," Lagrange proposed to examine the different methods which one had found up to then to solve algebraic equations, to reduce them to general principles, and to show *à priori* why these methods succeeded in case of the cubic and biquadratic, but failed for equations of higher degree. To do this Lagrange took up successively the various methods proposed by Cardan, Ferrari, Descartes, Tschirnhaus, Euler, and Bézout, and showed that the roots of the various resolvents upon whose solution the solution of the given equation depended were *rational* functions of the roots of the given equation. Here, then, was a great and fundamental step in advance. The problem of the solution of equations was shown to depend upon the properties of *rational* functions of the roots. To study the properties of these functions, Lagrange invented a "*calcul des combinaisons,*" as he styled it, which was nothing else than the first rudiments of the theory of substitutions.

By means of this new *calcul* Lagrange was placed in a position to tell in advance the result and character of certain investigations, in much the same way that algebra serves for numerical problems. Lagrange himself characterizes his method in the following words:

"These, then, if I mistake not, are the true principles for the resolution of equations. The analysis is reduced, as is seen, to a species of calculus of combinations by means of which one finds *à priori* the results one should expect." Lagrange gives the new calculus a broad and solid basis. Among the various theorems he established for rational functions of the roots of a general equation of nth degree, one is of fundamental importance: a function V which takes on $n!$ values for the substitutions of the symmetric group is root of an irreducible equation of degree $n!$ whose coefficients are rationally known. The roots of this equation are rational functions of one another, and possess the remarkable property that every rational function of the roots of the given equation can be expressed rationally in one of them, and hence the roots themselves. Similar functions he defines as those having the same group. Two similar functions are rationally expressible by each other. A rational function of the roots which takes on ρ values for the symmetric group, is root of an equation of degree ρ, whose coefficients are rational in the coefficients of the given equation. Further, ρ is a divisor of $n!$

If ϕ and ψ take on respectively $m\rho$ and ρ values for the symmetric group, then ϕ is a root of an equation of mth degree whose coefficients are rational in ψ. The function ψ can be expressed rationally in ϕ.

Let us see how such theorems as these enable Lagrange to assign *à priori* the reason for the success of the various methods proposed to solve the cubic and biquadratic, and their failure when applied to equations of higher degrees. I have already remarked that Lagrange found that the resolving functions employed by his predecessors were rational. In final analysis, he found that they all belonged to the type

$$t = x_1 + \alpha x_2 + \alpha^2 x_3 + \dots + \alpha^{n-1} x_n,$$

where $x_1, x_2, \dots x_{n-1}$ are the roots of the given equation $f(x) = 0$ and α is an imaginary nth root of unity. Let us see to what equations these functions lead. Two cases present themselves according as n is prime or composite. Let first n be prime.

Then the function

$$\theta_1 = t_1^n = (x_1 + \alpha x_2 + \dots + \alpha^{n-1} x_n)^n$$

remains unchanged for the cyclic substitutions

$$| z, z + a | \quad (\text{mod } n) \quad a = 0, \quad 1 \dots n - 1.$$

For the substitutions

$$| z, bz | \quad (\text{mod } n) \quad b = 1, 2 \dots n - 1,$$

θ_1 takes on $n - 1$ values

$$\theta_1 \theta_2 \dots \theta_{n-1},$$

obtained by replacing α by respectively α^2, α^3, ...

Consider the equation

$$\theta = (\theta - \theta_1) \dots (\theta - \theta_{n-1}) = 0.$$

Its coefficients are symmetrical functions of $\theta_1, \theta_2, \dots$. Let ψ be such a function. It is root of an equation of degree $\rho = (n - 2)\,!$ If ψ can be found in any way, the coefficients of $\theta = 0$ being rational in ψ, are rationally known, and the solution of $f(x) = 0$ depends now upon an equation of degree $n - 1$.

When $n = 3$, $\rho = 1$; $n = 5$, $\rho = 6$.

Thus for the cubic we see that the coefficients of $\theta = 0$ are rational, and the solution depends therefore upon an equation of the second degree only. As soon, however, as the prime

$n > 3$, the coefficients depend upon an equation of degree higher than the given equation. For $n = 5$ it is already of sixth degree. In passing I note that Lagrange by the considerations of his new *calcul* made the solution of the quintic depend upon a sextic. The methods of Tschirnhaus, Euler, and Bézout lead to equations of twenty-fourth degree. Lagrange sought in vain to find a resolving function which should satisfy an equation of degree less than five.

It is not uninteresting also to remark, that whenever an equation of prime degree is algebraically soluble, Lagrange's method leads us directly to the solution. When $n = 5$, this was already noticed by Malfatti, a contemporary of Lagrange.

When n is a composite number, the foregoing considerations do not hold.

Let $n = p\nu$ where p is a prime factor.

Lagrange writes the roots in the array

$$\begin{array}{lcl} x_1 x_2 & \cdots\cdots & x_\nu \\ x_{\nu+1} & \cdots\cdots & x_{2\nu} \\ x_{2\nu+1} & \cdots\cdots & x_{3\nu} \\ \cdots & \cdots\cdots & \cdots \\ x_{(p-1)\nu+1} & \cdots & x_{p\nu} \end{array}$$

Let the equation whose roots form the ith row be $\phi_i = 0$. The coefficients of this equation are symmetric functions of the elements of the corresponding row.

If $X_i = x_{(i-1)\nu+1} + x_{(i-1)\nu+2} + \ldots x_{i\nu}$, $i = 1, 2 \ldots p$, the coefficients of $\phi_i = 0$ are rational in X_i, and when X_1, $X_2 \ldots$ are known, the coefficients of $\phi_1 = 0$, $\phi_2 = 0 \ldots$ are rationally known, and the solution of an equation of degree $p\nu$ is reduced to the solution of p equations of degree ν. If now ν be composite we may break ν into two factors, $\nu = p_1\nu_1$, p_1 being prime, and proceed as before.

Let us now return to the determination of the quantities X_1, $X_2 \ldots$ which are roots of an equation of prime degree, namely,

$$X = (X - X_1)(X - X_2) \ldots (X - X_p) = 0,$$

whose coefficients are roots of a rational equation of degree $\rho = \frac{n\,!}{(\nu\,!)^p p\,!}$.

The equation $X = 0$ being prime, may be solved by the foregoing method.

When

$$n = 4, \ \rho = 3; \quad n = 6, \ \rho = 10 \text{ or } 15.$$

Here again Lagrange's methods gave him a clear insight into the reason for the success and failure of his predecessors' methods, according as $n \leqq 4$ or > 4.

Particularly instructive and important is the application Lagrange made of his methods to the equations upon which the division of the circumference of the circle into n equal parts depends,* for it would have required a far less attentive reader than Lagrange's illustrious disciple, Abel, not to have perceived what slight modifications were necessary in order to apply Lagrange's methods to the corresponding equations in the theory of elliptic functions.

The equations in question have the form

$$x^{n-1} + x^{n-2} + \ldots + x + 1 = 0$$

where we suppose n prime. Lagrange proceeds as follows: Let $n - 1 = pq$, p prime. If r be an imaginary nth root of unity and a a primitive congruence root for n, we can arrange the pq roots thus:

$$\begin{array}{lllll} r & r^{a^p} & r^{a^{2p}} & \ldots & r^{a^{(q-1)p}} \\ r^a & r^{a^{p+1}} & r^{a^{2p+1}} & \ldots & r^{a^{(q-1)p+1}} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ r^{a^{p-1}} & r^{a^{2p-1}} & r^{a^{3p-1}} & \ldots & r^{a^{pq-1}} \end{array}$$

If $X_1, X_2, \ldots$ denote, as in the general case just treated, the sums of the elements of the various rows, Lagrange showed that the equation

$$X = (X - X_1) \ldots (X - X_p) = 0$$

is rational and can be algebraically solved, so that the solution of the original equation depends upon the solution of p equation of degree q, and so forth. To solve the equation $X = 0$, he employs as in the general case the resolving function

$$\theta_1 = t_1^p = (X_1 + \alpha X_2 + \ldots + \alpha^{p-1} X_p)^p.$$

But this quantity is here rationally known (α being supposed known), since, for any substitution which changes r into r^a, X_1 goes over into X_2, X_2 into X_3, etc.; thus θ_1 is unaltered. Developing and arranging according to powers of α, we have

$$\theta_1 = \xi_0 + \alpha \xi_1 + \ldots + \alpha^{p-1} \xi_{p-1},$$

* LAGRANGE: Traité de la résolution des équations numériques de tous les degrés. Paris, 1808. pp. 275–311.

where the ξ's are unchanged for r, r^a. But the ξ's being rational functions of r, r^a, r^{a^2} . . . , we have, e.g.,

$$\xi_0 = A + Br + Cr^a + \ldots + Nr^{a^{n-2}}.$$

As this is unchanged for r, r^a, we have by comparison

$$B = C = \ldots = N, \quad \text{or}$$

$$\xi_0 = A + B(r + r^a + \ldots) = A - B;$$

that is, ξ_0 is known. As θ_1 is thus known, we get

$$X_1 + \alpha X_2 + \ldots + \alpha^{p-1}X_p = \sqrt[p]{\theta_1};$$

similarly $$X_1 + \beta X_2 + \ldots + \beta^{p-1}X_p = \sqrt[p]{\theta_2};$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$X_1 + \omega X_2 + \ldots + \omega^{p-1}X_p = \sqrt[p]{\theta_{p-1}};$$

also $$X_1 + \quad X_2 + \ldots + X_p = -1,$$

where $\alpha, \beta \ldots \omega$ are the $p - 1$ imaginary pth roots of unity.

This system of linear equations gives us X_i; for example,

$$X_1 = \frac{-1 + \sqrt[p]{\theta_1} + \sqrt[p]{\theta_2} + \ldots + \sqrt[p]{\theta_{p-1}}}{p}.$$

The roots of the equation

$$x^{n-1} + x^{n-2} + \ldots + x + 1 = 0$$

are rational functions of one of them, x_0:

$$(1) \qquad x_\iota = \theta_\iota(x_0), \qquad \iota = 1, 2 \ldots n - 1.$$

They enjoy further the property that

$$(2) \qquad \theta_\iota \theta_\kappa x_0 = \theta_\kappa \theta_\iota x_0.$$

But just these properties (1), (2) are enjoyed by the n^2 roots of the equation $F(x) = 0$ for dividing the argument of sn$z$, namely,

$$x_{p,q} = \operatorname{sn}\left(\frac{z}{n} + \frac{4pK + 4qiK'}{n}\right),$$

when we consider the quantities k^2, sn$z$, cn$z$, dn$z$, and

$$y_{p,q} = \text{sn}\left(\frac{4pK + 4qiK'}{n}\right),$$

as known.

How closely Abel follows Lagrange in his solution of equations of the type $F(x) = 0$ is shown by the following sketch of his method :*

Let $f(x) = 0$ be any (irreducible) equation whose roots enjoy the properties (1), (2). We may represent them by the array

$$\begin{array}{lll} x_0 & \theta x_0 \ldots & \theta^{n-1}x_0; \\ x_1 & \theta x_1 \ldots & \theta^{n-1}x_1; \\ \cdot & \cdot \quad \cdot & \cdot \quad \cdot \\ x_{m-1} & \theta x_{m-1} \ldots & \theta^{n-1}x_{m-1}. \end{array}$$

Consider the equation $\phi(x) = 0$, whose roots are the elements of the first row. If we suppose the coefficients of this equation to be known, it is soluble. In fact setting with Lagrange

$$\psi_0 = (x_0 + \alpha\theta x_0 + \ldots + \alpha^{n-1}\theta^{n-1}x_0)^n,$$

ψ_0 remains unchanged for $| \, x_0, \theta x_0 \, |$, as in the case of the cyclotomic equations just considered.

But ψ_0 is rational, for if we denote by ψ_m what ψ_0 becomes after the substitution $| \, x_0, \theta^m x_0 \, |$, since $\psi_m = \psi_0$, we have $\psi_0 = \psi_1 = \ldots \psi_{n-1}$, whence

$$\psi_0 = \frac{1}{n}(\psi_0 + \psi_1 + \ldots + \psi_{n-1}),$$

a symmetric function of the roots of $\phi(x) = 0$.

Thus, precisely as before, we have a system of linear equations which gives, for example,

$$x_0 = \frac{-A + \sqrt[n]{\psi_0} + \sqrt[n]{\psi_1} + \ldots + \sqrt[n]{\psi_{n-1}}}{n},$$

where A is the coefficient of x^{n-1} in $\phi(x) = 0$.

We return to the equation upon which the coefficients of $\phi(x) = 0$ depend.

* ABEL: Mémoire sur une classe particulière d'équations résolubles algébriquements. *Crelle*, vol. 4. Also, Œuvres, 2d ed., vol. I, pp. 478–507.

Let X_1 be the oft-considered symmetric function of the elements of the first row,

$$X_1 = x_0 + \theta x_0 + \ldots + \theta^{n-1} x_0, \text{ etc.}$$

The equation

$$X = (X - X_1)(X - X_2) \ldots (X - X_m) = 0$$

is rational. In fact

$$X_1 = F(x_0) = F(\theta x_0) = \ldots = F(\theta^{n-1} x_0).$$

Similarly $X_2 = F(x_1) = F(\theta x_1) = \ldots = F(\theta^{n-1} x_1).$

. . . .

Hence $X_1^\kappa = \frac{1}{n}\{(F(x_0))^\kappa + \ldots + (F(\theta^{n-1} x_0))^\kappa\};$

$$X_2^\kappa = \frac{1}{n}\{(Fx_1)^\kappa + \ldots + (F(\theta^{n-1} x_1))^\kappa\},$$

. . . .

and thus $X_1^\kappa + X_2^\kappa + \ldots + X_m^\kappa$ is rational.

Now the equation $X = 0$, having the same properties as the original equation $f(x) = 0$, this last is algebraically soluble, and we have the theorem that the equation upon which the division of the argument of the elliptic function $\mathrm{sn}(z)$ depends is (under the previous assumptions) algebraically soluble.

Leaving Abel now, I must pass on to a last and even more striking example of the wonderful powers of Lagrange's "*calcul*" to announce *à priori* the results which one should expect. Lagrange, as I have remarked, had vainly endeavored to find a rational resolving function for the quintic which would satisfy an equation of degree less than five, and so place one in a position to effect the solution of this celebrated equation. Ruffini, an Italian contemporary of Lagrange, and his ardent disciple, succeeded by Lagrange's own methods in proving that no such function existed; in fact he demonstrated quite generally that no rational function of n elements existed which took on three or four values for the symmetric group, n being > 4. But Ruffini was too convinced of the latent power of his great countryman's methods to stop here: he boldly undertook* by their means to prove that the alge-

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* P. RUFFINI: Reflessioni intorno alla soluzione delle equazioni algebraiche generali. Modena, 1813.

braical solution of the general equation of degree > 4 was impossible. Although not altogether successful in his attempt, I wish to show with what simple means he did prove that if the expression for a root can be given such a form that the radicals in it are rational functions of the roots of the given equation, then the algebraical solution is impossible when the degree of the equation surpasses four.

In fact such an expression for the root could always be arrived at as follows: Let A_1 be a rational quantity, and let n_1 be a prime; then $P_1^{n_1} = A_1$ defines a first irrationality. Let A_2 be any rational function of quantities originally rational and P_1; then $P_2^{n_2} = A_2$ defines a second irrationality. Continuing in this way, any root of the given equation has the form $x_1 = A$, where A is a rational function of $P_1, P_2 \ldots$

Let now s be the cyclic substitution $s = (1\,2\,3\,4\,5)$, and let $P_s, P_{s^2}, P_{s^3}, P_{s^4}$ be the values of P_1 for s, s^2, s^3, and s^4, respectively. Then $P_s = \beta P_1$ where $\beta^{n_1} = 1$. Operating with s, $s^2 \ldots$ this gives

$$P_{s^2} = \beta^2 P_1, \quad P_{s^3} = \beta^3 P_1, \quad P_{s^4} = \beta^4 P_1; \quad \therefore \ \beta^5 = 1.$$

Similarly let P_σ be the value of P_1 for $\sigma = (1\,2\,3)$; then $P_\sigma = \gamma P_1$, and thus $\gamma^3 = 1$. But $P_{s\sigma} = \beta\gamma P_1$; hence, since $(s\sigma)^5 = 1$, $\beta^5\gamma^5 = 1$; $\therefore \gamma = 1$. Similarly, if $\rho = (3\,4\,5)$, then $P_\rho = P_{\rho^2} = P_1$. Now $\rho\sigma = s$; hence $P_{\rho\sigma} = P_s = P_1$. But $P_s = \beta P_1$; thus $\beta = 1$. Hence P_1 remains unaltered for s, thus also P_2, etc.; hence finally A. Thus the right hand of $x_1 = A$ is unchanged for s while the left-hand side is changed. Thus it is impossible to solve algebraically the general equation whose degree surpasses four.

The limited time at my disposal has not permitted me to discuss Lagrange's claims in detail; but the few examples I have chosen from Lagrange himself and from his immediate disciples will show, I think, how incomparably superior his methods were to those of his predecessors, Hudde, Saunderson, Le Sœur, and Waring; and it would be no difficult or ungrateful task to show how easily the ideas of Galois spring from the same source that inspired Ruffini, Cauchy, and Abel

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