

GERGONNE'S PILE PROBLEM.

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IN Ball's *Mathematical Recreations*, pp. 101-6, is described the familiar three-pile trick with twenty-seven cards, mentioned by Bachet; also Gergonne's generalization for a pack of m^m cards. Suppose the pack is dealt into m piles of m^{m-1} cards each, and after the first deal the pile indicated as containing the selected card is taken up a th; after the second deal is taken up b th, and so on; finally, after the m th deal is taken up k th. Then the card selected will be the n th from the top where,

$$\begin{aligned} \text{if } m \text{ is even, } & n = km^{m-1} - jm^{m-2} + \dots + bm - a + 1; \\ \text{if } m \text{ is odd, } & n = km^{m-1} - jm^{m-2} + \dots - bm + a. \end{aligned}$$

If in the latter case we put the pile indicated always in the middle of the pack, then $a = b = \dots = j = k = \frac{m+1}{2}$, and we find $n = \frac{m^m + 1}{2}$, or the card will appear in the middle of the pack.

Dr. C. T. Hudson* has discussed the general problem: To deal a pack of ab cards into a piles of b cards each and so stack the piles after each deal that after the n th deal any selected card may be r th in the whole pack. Let the card selected be the s th card from the bottom of the pile containing it after the first deal. Let p_1, p_2, \dots, p_n be the places the pile of the selected card is to hold after the first, second, \dots n th stacking of the piles. After the first stacking the number of the selected card, counting from the bottom of the pack, will be the $b(p_1 - 1) + s$; after the second, $b(p_2 - 1) + \frac{b(p_1 - 1) + s + m_1}{a}$; after the n th,

$$\begin{aligned} \frac{1}{a^{n-1}} [& b \{ a^{n-1}(p_n - 1) + a^{n-2}(p_{n-1} - 1) + \dots + a(p_2 - 1) \\ & + (p_1 - 1) \} + s + m_1 + am_2 + \dots + a^{n-2}m_{n-1}], \end{aligned}$$

where m_1, m_2, \dots, m_{n-1} are integers $\leq a - 1$, including 0. Since $m_1 + am_2 + \dots + a^{n-2}m_{n-1}$ lies between 0 and $a^{n-1} - 1$, the selected card's place r lies between

$$\frac{1}{a^{n-1}} [b \{ a^{n-1}(p_n - 1) + \dots + a(p_2 - 1) + (p_1 - 1) \} + s]$$

* *Educational Times Reprints*, 1868, vol. 9, pp. 89-91.

and

$$\frac{1}{a^{n-1}}[b\{a^{n-1}(p_n-1)+\dots+a(p_2-1)+(p_1-1)\}+(s-1)]+1.$$

$\therefore p_1 + ap_2 + \dots + a^{n-1}p_n$ lies between $\frac{a^{n-1}r-s}{b} + \frac{a^n-1}{a-1}$ and $\frac{a^{n-1}(r-1)-(s-1)}{b} + \frac{a^n-1}{a-1}$, or since $b \geq s \geq 1$, between $\frac{a^{n-1}r-b}{b} + \frac{a^n-1}{a-1}$ and $\frac{a^{n-1}(r-1)}{b} + \frac{a^n-1}{a-1}$. Thus p_1, p_2, \dots, p_n are the successive remainders on dividing any integer between these limits n times by a .

If we make $p_1 = p_2 = \dots = p_n = \frac{a+1}{2}$ and $r = \frac{ab+1}{2}$,

we find that $a^{n-1} \geq b$, which is the condition giving the least value of n such that the selected card shall be brought to the middle of the pack in n deals, the pile containing it being placed in the middle of the pack after each deal. This condition was given by Dr. Hudson in the form $a^{n-1} + 2 \geq b$, the inaccuracy of which is more clearly shown by the following theorems, derived last October while unaware of Hudson's and Gergonne's results.

Theorem I. For a pack of P^m cards, m operations (of dealing into P piles of P^{m-1} cards each and placing the pile containing the selected card into the middle of the pack) are necessary and sufficient to shuffle the bottom card of a pile into the middle of the pack.

Let $I(x/y)$ denote x/y when it is an integer, but the integer just greater than x/y when it is a fraction. Write $p = \frac{P-1}{2}$.

After one deal our card is the bottom or first card in its pile, beneath which p piles of P^{m-1} cards are then placed. Hence after the first operation the number of our card in the pack is $(p \cdot P^{m-1} + 1)$. After the second deal its number in its pile is $I\left(\frac{p \cdot P^{m-1} + 1}{P}\right) = (p \cdot P^{m-2} + 1)$. Then p piles of P^{m-1} cards each are placed beneath this pile. Hence after the second operation the number of our card in the pack is $(p \cdot P^{m-1} + p \cdot P^{m-2} + 1)$. After the third operation it is $(p \cdot P^{m-1} + p \cdot P^{m-2} + p \cdot P^{m-3} + 1)$; after the m th it is $(p \cdot P^{m-1} + p \cdot P^{m-2} + \dots + p \cdot P + p + 1) = p\left(\frac{P^m-1}{P-1}\right) + 1 = \left(\frac{P^m+1}{2}\right)$ or that of the *middle* card in the pack.

Theorem II. For a pack of $P(P^{m-1} + 2)$ cards, $m + 1$ operations are necessary and sufficient to shuffle the bottom card of one of the P piles into the middle of the pack.

After the first deal it is the first card in its pile; after the second, its number in its pile is $I\left(\frac{p(P^{m-1} + 2) + 1}{P}\right) = (p \cdot P^{m-2} + 1)$; after the third, $I\left(\frac{p(P^{m-1} + 2) + p \cdot P^{m-2} + 1}{P}\right) = (p \cdot P^{m-2} + p \cdot P^{m-3} + 1)$; after the m th, $(p \cdot P^{m-2} + p \cdot P^{m-3} + \dots + p \cdot P + p + 1)$; after the $(m + 1)$ st deal it is $I\left(\frac{p(P^{m-1} + 2) + p \cdot P^{m-2} + \dots + p + 1}{P}\right) = (p \cdot P^{m-2} + p \cdot P^{m-3} + \dots + p + 2) = p\left(\frac{P^{m-1} - 1}{P - 1}\right) + 2 = \frac{(P^{m-1} + 2) + 1}{2}$

or it is the *middle* card in its pile.

The number of operations necessary to shuffle the card selected into the middle of the pack is evidently greatest when this card is the bottom [or top] card in its pile. Further, this number will increase (not continuously, however), if the number of cards in each pile be increased, the number of piles being constant. Hence from Theorems I and II it follows: For a pack of n cards, n being an odd multiple of the odd number P such that $P^{m-1} < n < P^m$, m operations are sufficient (and, for the extreme cases, necessary) to shuffle a card chosen arbitrarily into the middle of the pack.

If in the condition $a^{n-1} = b$ we make $b = a^{m-1}$, we get m as the least value of n ; if we make $b = a^{m-1} + 2$, we get $m + 1$ as its least value. But from Dr. Hudson's condition m would be the least value of n in the latter case, contrary to Theorem II.

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At an international meeting held in Paris in 1889, under the auspices of the French Mathematical Society, it was resolved to prepare a complete bibliography of the literature of mathematics for the period 1800-1889 and of the history of mathematics since 1600. An international committee was