figure'; and to his absolute use of the term *double point*, speaking of a double point in a correspondence as if it were a double point on the circle considered, and then defining consecutive points as points that "fall together in a double point."

If Professor Smith could be induced to translate his work into ordinary mathematical English, we feel sure that he would greatly increase its usefulness not only in aiding "the first upward steps of the climber," but also in preparing him for the "much higher ascent."

CHARLOTTE ANGAS SCOTT.

BRYN MAWR COLLEGE, PA., *March*, 1893.

WRONSKI'S EXPANSION.

BY PROF. W. H. ECHOLS.

IN 1810 Höené Wronski presented to the French Academy of Sciences the following formula, without demonstration,

$$fx = a_0 + a_1\omega_1 + a_2\omega_2 + \ldots \text{ ad. inf.}, \quad . \quad . \quad . \quad (1)$$

in which fx, ω_1, ω_2, . . . are arbitrary functions of x, and a_0, a_1, . . . are independent of x. This formula, or rather the law for the formation of the coefficients, he called *la loi suprême.*

Lagrange and Lacroix were appointed as a committee to examine Wronski's memoir and to report on it to the Academy. This report is an admirable production and in every way worthy of the distinguished names attached to it. It is especially noticeable for its conservative tone and yet its acknowledged recognition of the importance and possible future of the formula. The commissioners must have been very much impressed, to have repeated section III. in section IV., "Ce qui a frappé vos commissaires dans le mémoire de M. Wronski, c'est qu'il tire de sa formule toutes celles que l'on connaît pour le développement des fonctions, et qu'elles n'en sont que des cas très-particuliers." It would seem remarkable in view of this that nothing has been done toward developing his work and placing it on a sound scientific basis. The whole of Wronski's work and method of work appears to be purely qualitative ; it is truly algorithmic, inasmuch as he uses all symbols of quantity with as little regard for their quantitative relations as though they were purely operational symbols without quantitative properties. When one reads Wronski's published writings, one should read them with this understanding, not for the positive results, but for the suggestions which the analytical forms hold out. When we remember that the work of Cauchy and Abel had yet to be done in clearing up this matter of series, it was well that the clear head of Lagrange was called into service in this matter.

The report, while in every way complimentary to Wronski, was misconstrued by him and he did not avail himself of the excellent suggestions made by the commissioners. He took offence, which showed itself in his "Mémoire sur la réfutation de la théorie des fonctions analytiques de Lagrange."

In the following year he presented a second memoir to the Academy, submitting his *problème-universel des mathématiques,* a particular case of *la loi suprême,* upon which was based subsequently his so called refutation of the theory of functions of Lagrange. This was also presented without demonstration. It was referred to a committee for examination which was composed of Arago and Legendre. This report does not deal so kindly with Wronski, and throughout his life he never recovered from the bitterness which it engendered in him against the authors and even the Academy itself. These two reports are given in full in the "Supplément à la Réforme de la Philosophie," p. lxxxvi. I prefer to consider the more generous and liberal opinion of Lagrange and Lacroix, rather than the severe one of Arago and Legendre, as indicating the existence of truth and promise of future utility in the unfortunate Wronski's undemonstrated formula.

In 1811 Wronski published his "Introduction à la Philosophie des Mathématiques," quarto, pp. 270. This is chiefly occupied with the metaphysical foundation of mathematics and the classification of its orders and branches. In 1812, after the report on his first memoir, he published the "Réfutation de la Théorie des Fonctions, etc.," and in 1815 was published the "Philosophie de la Technie Algorithmique," quarto, pp. 286. This last is the most important of his writings. The work opens with the words, "En quoi consistent les mathématiques ?—N'y aurait-il pas moyen d'embrasser, par un seul problème, tous les problèmes de ces sciences, et de résoudre généralement ce problème universel ? Telles sont les questions qui doivent couronner la philosophie du géomètre; et telles sont aussi les questions qu'à la fin de notre Philosophie des Mathématiques, sous la marque (XXXII.), nous avons embrassées dans notre loi algorithmique absolue, comme nous allons le montrer." The formula marked (XXXII.) is (1) at the head of this paper.

Wronski unhesitatingly lays down such general laws of form as

$$Fx = A_1\phi_1x + A_2\phi_2x + \ldots,$$
$$Fx = (A_1 + \phi_1x)(A_2 + \phi_2x) \ldots,$$
$$Fx = A_0 + \cfrac{\phi_1x}{A_1 + \cfrac{\phi_2x}{A_2 + \ldots}}$$

for arbitrary functions ϕ.

The last ninety pages of the Philosophie de la Technie are devoted to the so-called demonstration of la loi suprême, which consists in determining a form for the coefficients. The major portion of the ninety pages are taken up with the development of his "shinn" functions (Wronskians).

The problème-universel is expressed in the formula

$$0 = fx + x_1f_1x + x_2f_2x + \ldots,$$

and upon it Wronski founded his "Résolution générale des equations algébriques de tous les degrés." In 1817 the Academy of Sciences of Lisbon proposed as the subject of a prize the demonstration of Wronski's formula. The prize was in the following year awarded to M. Torriani for the refutation of it.

In 1847 were published the "Messianisme" and "Réforme du Savoir Humain," which contain the "Réforme des Mathématiques," "Résolution des equations algébriques," "Réforme de la Philosophie," etc.

The literature on Wronski and his methods consists of: a paper by Professor Cayley in the *Quarterly Journal* (1873) "On Wronski's Theorem," which brought forth two others in the *Nouvelles Annales*, for April and July 1874, by M. Abel Transon. In the *Comptes Rendus* (April 1881), by M. Yvon Villarceau under the section Mécanique Céleste,—" Note sur les Methodes de Wronski. "Exposé des Méthodes Générales en Mathématiques, d'après Höené Wronski," by M. Emile West (Gauthier-Villars, 1886). A small pamphlet published in 1890 by the same firm, "Loi Téléologique (12 pp.) précédée d'un Autobiographie et Inventaire de l'Œuvre (62 pp.)." The autobiography is taken from the "Réforme de la Philosophie." In *Bibliotheca Mathematica*, 1892, No. 2, a short note of five pages, "Sur les découvertes Mathématiques de Wronski," by S. Dickstein, gives a list of the titles of Wronski's works and of the papers by other writers which refer to them. The author of this note has in preparation a complete bibliography of the published works and manuscripts of Wronski, which will be published by the Academy of Sciences of Cracow and will form the basis for the edition of Wronski's mathematical works which the Academy proposes to publish. Wronski's manuscripts are in the library of Kornik (near Posen), and are the property of Count Zamoyski. In the preface to his "Exposé," M. West says: "Wronski's works are very rare, and it is difficult to read them at first. In this work we propose to make known new mathematical processes which are applicable to the most difficult questions in mathematics. The methods in question, while put forth in the beginning of the century by a mathematician whose name was disgraced under circumstances useless to recall, remain even to-day unknown. Höené Wronski has been dead for thirty years, and the causes which turned attention away from his work no longer exist; our generation is not interested in the old quarrels over the truly remarkable works of a learned and profound mathematician."

At the close of his article on Wronski's methods in celestial mechanics, M. Villarceau says: "Those who wish to undertake a similar work we advise to consult, if possible, the manuscripts left by Wronski. Judging them by the works he has published, one will be sure to find a precious mine of analytical developments, very correctly executed."

Höené Wronski, born in Posen, August 24, 1778, died in Neuilly, August 9, 1853. At sixteen years of age he was an officer of artillery under Kosciusko. He was made prisoner by the Russians at the battle of Maciecowicz, October 10, 1794, and at the close of the Polish revolution he entered the Russian army and reached therein the grade of Lieutenant-Colonel. He resigned in 1797 to devote himself exclusively to the study of philosophy and the mathematical sciences, studied two years in Germany, and then in France, where he remained until his death.

Those who have written upon Wronski and his methods are enthusiastic upon the subject, but none of them seems to have gone down into his work or to be able to do more than write with high praise of its character and seek to draw the attention of others to his writings. Now that the Academy of Cracow has taken the matter in hand we may hope that the question as to the real merit of Wronski's work will be settled. When this has been done, if the writer may offer his impression acquired through a somewhat cursory examination, it is quite likely that it will be found that Wronski's work is mainly formal, or, as I have said, qualitative. None the less is the honor due him for it, for it is the law of mathematical evolution that qualitative analysis shall precede the quantitative, and it is rare that the same mind possesses equally the power of imagination in mathematical theory and realization of its application. Wronski in the highest degree possessed that mind which, "having seen a leaf and a drop of water, can construct the forests, the rivers, and the seas."

LA LOI SUPRÊME.

It has been frequently remarked that Wronski gives a demonstration of the *loi suprême* in the "Philosophie de la Technie." This is wrong, for Wronski never shows the possibility of the equality sign in (1). What he has done amounts to this: he has assumed the infinite series to be true; upon this assumption he proceeds to determine a form for the constant coefficients. This he does in the last ninety pages of the "Philosophie de la Technie." It is in the course of this analysis that he develops the "combinatory sums" (determinant functions) now known (after Muir) by his name. In the supplement to M. West's "Exposé" one finds the deduction of the coefficient forms after Wronski's own method, which is much abbreviated by the use of modern determinant forms. The most modern form of this determination can be found in Laurent's "Traité d'Analyse," vol. III., p. 352, where the general coefficient is given as follows:

$$a_n = \frac{\Omega_n(fa)}{\Omega_n(\omega_n)} - \sum_{\mu=n+1}^{\mu=\infty} a_\mu \frac{\Omega_n(\omega_\mu)}{\Omega_n(\omega_n)}, \tag{2}$$

wherein some arbitrary value a has replaced the variable x, and

$$\Omega_n(F) = \begin{vmatrix} \omega_1' & \ldots & \omega_{n-1}', & F' \\ \cdot & \cdot & \cdot & \cdot \\ \omega_1^n & \ldots & \omega_{n-1}^n, & F^n \end{vmatrix};$$

the symbol ω^r meaning that after differentiating ω r times x is changed to a. It is to be observed, however, that Wronski in his work does not confine himself to the operation of differentiation, but works out his results by use of an arbitrary symbol of repetitive operation, of which Δ and D are to be regarded as merely particular cases.

The value of any coefficient in the series (1) is thus given by (2) in terms of an infinite series involving all the coefficients which follow it. We may, however, by successive substitution, express a_n in an infinite series, distant terms of which involve coefficients of as high an order as we choose. From the nature of this expression the value of a_n is thus indeterminate inasmuch as we can have no means of testing the arithmetical equivalence of such a series. It would seem that form (2), or the modified form of it obtained through substitution, leads to no solution of the problem in view.

If we make the upper limit of μ finite, say $m > n$, it is easy to show for small values of m, and doubtless it can be readily shown for large values of m, that (2) may be converted into the proper form for the coefficient, which is

$$a_n = (-1)^n \frac{\begin{vmatrix} \omega' & \dots & \omega'_{n-1} & \omega'_{n+1} & \dots & \omega'_{m-1} & f'a \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \omega_1^{m-1} & \dots & \omega_{n-1}^{m-1} & \omega_{n+1}^{m-1} & \dots & \omega_{m-1}^{m-1} & f^{m-1}a \end{vmatrix}}{\begin{vmatrix} \omega'_1 & \dots & \omega'_{m-1} \\ \cdot & \cdot & \cdot \\ \omega_1^{m-1} & \dots & \omega_{m-1}^{m-1} \end{vmatrix}}. \quad (3)$$

The only reason that can be assigned for Wronski's not giving this general form to his coefficient is that he could not see at that time how to evaluate this ratio when m became infinite, this being dependent (when the determinants become difference-products) on Jacobi's theorem for writing out the minor of a simple alternant in terms of the sums of the products of the elements taken with repetition, etc. (see *Annals of Mathematics*, vol. 7, Nos. 4, 5); yet the only particular case of (2) which leads to correct results is evident from (3), and does not require Jacobi's theorem, so that it is strange that with all his ingenuity and fertility of resource he did not make use of it.

If the ω functions in (1) be such that ω_q^r vanishes for $r < q$, either through the operation performed on the functions or through the substitution of a specific value for the variable, then the last column in $O_n(\omega_\mu)$, $(\mu = n + 1 \dots \infty)$ vanishes, as does also the second term in the right of (2), leaving for the value of a_n,

$$a_n = \begin{vmatrix} \omega'_1, & 0 \dots 0, & f' \\ \omega''_1, & \omega''_2 \dots 0, & f'' \\ \cdot & \cdot & \cdot \\ \omega^n_1, & \omega^n_2 \dots \omega^n_{n-1}, & f^n \end{vmatrix} \div \begin{vmatrix} \omega'_1, & 0 \dots 0 \\ \omega''_1, & \omega'_2 \dots 0 \\ \cdot & \cdot & \cdot \\ \omega^n_1, & \omega^n_2 \dots \omega^n_n \end{vmatrix}, \quad (4)$$

which is the form of the coefficient in (1) giving the theorem known as Wronski's expansion. If here we put $\omega_n = (\phi x)^n$ and let $\phi(a) = 0$, we deduce Burmann's series, which including Lagrange's, Laplace's, and therefore Taylor's series was doubtless the cause of section III. in the report of Lagrange and Lacroix upon Wronski's memoir.

If in (1) the ω functions be such that ω_q^r vanishes when $r > q$, then we have from (2)

$$a_n = \frac{f_a^n}{\omega_n^n} - \sum_{\mu = n+1}^{\mu = \infty} a_\mu \frac{\omega_\mu^n}{\omega_n^n},$$

$$= \frac{f_a^n}{\omega_n^n} - \sum_{\mu = 1}^{\mu = \infty} \frac{f^{n+\mu} a}{\prod\limits_{r=1}^{\mu} \omega_{n+r}^{n+r}}. \qquad (5)$$

This is wrong, for the correct value of a_n in this case must be $f^n a/\omega_n^n$. In particular, if $\omega^q = x_q$, then $a_0 = f0$, and

$$a_n = \frac{f^n x}{n!} - \frac{f^{n+1}x}{n!(n+1)!} - \ldots,$$

which should be the coefficient $f^n x/n!$ in Bernouilli's series.

The value (4) under the conditions imposed is correct, and gives that particular case mentioned in the BULLETIN, vol. 2, p. 140. It seems that throughout Wronski's whole work he has aimed at the generalization given in the BULLETIN as quoted above, which I have called a *composite*.

If inferences may be drawn while the investigations are yet incomplete, this composite may prove useful; and if so, it is my sincere hope that it may be the means of lifting in some measure the weight of opprobrium from the memory of the unfortunate Wronski, whose pathetic story appeals so strongly to the sympathy of all of his co-workers.

CHARLOTTESVILLE, *February*, 1893.

NOTE ON THE SUBSTITUTION GROUPS OF SIX, SEVEN, AND EIGHT LETTERS.

BY F. N. COLE, PH.D.

A LIST of the groups of six, seven, and eight letters is given by Mr. Askwith in vol. 24 of the *Quarterly Journal of Mathematics*, and Professor Cayley has revised and tabulated Mr. Askwith's results in vol. 25 of the same journal.* Noticing

* A list of groups as far as ten letters was given by Kirkman in the *Proceedings of the Literary and Philosophical Society of Manchester*, vol. 3 (1864), p. 144.