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A Lorentzian quantum geometry

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Abstract

We propose a formulation of a Lorentzian quantum geometry based on the framework of causal fermion systems. After giving the general definition of causal fermion systems, we deduce space-time as a topological space with an underlying causal structure. Restricting attention to systems of spin dimension two, we derive the objects of our quantum geometry: the spin space, the tangent space endowed with a Lorentzian metric, connection and curvature. In order to get the correspondence to differential geometry, we construct examples of causal fermion systems by regularizing Dirac sea configurations in Minkowski space and on a globally hyperbolic Lorentzian manifold. When removing the regularization, the objects of our quantum geometry reduce precisely to the common objects of Lorentzian spin geometry, up to higher-order curvature corrections.

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Contents

1 **Introduction** 1199

2 **Causal fermion systems of spin dimension two** 1201

2.1 **The general framework of causal fermion systems** 1201

2.2 **The spin space and the Euclidean operator** 1202

2.3 **The connection to Dirac spinors, preparatory considerations** 1204

3 **Construction of a Lorentzian quantum geometry** 1206

3.1 **Clifford extensions and the tangent space** 1206

3.2 **Synchronizing generically separated sign operators** 1215

3.3 **The spin connection** 1217

3.4 **The induced metric connection, parity-preserving systems** 1224

3.5 **A distinguished direction of time** 1226

3.6 **Reduction of the spatial dimension** 1228

3.7 **Curvature and the splice maps** 1229

3.8 **Causal sets and causal neighborhoods** 1232

4 **Example: the regularized Dirac sea vacuum** 1233

4.1 **Construction of the causal fermion system** 1235

4.2 **The geometry without regularization** 1240

4.3 **The geometry with regularization** 1245

4.4 **Parallel transport along timelike curves** 1247

5 **Example: the fermionic operator in a globally hyperbolic space-time** 1249

5.1 The regularized fermionic operator 1249

5.2 The Hadamard expansion of the fermionic operator 1258

5.3 The fermionic operator along timelike curves 1261

5.4 The unspliced versus the spliced spin connection 1263

5.5 Parallel transport along timelike curves 1268

6 Outlook 1270

Acknowledgment 1272

Appendix A The expansion of the Hadamard coefficients 1272

References 1288

1 Introduction

General relativity is formulated in the language of Lorentzian geometry. Likewise, quantum field theory is commonly set up in Minkowski space or on a Lorentzian manifold. However, the ultraviolet divergences of quantum field theory and the problems in quantizing gravity indicate that on the microscopic scale, a smooth manifold structure might no longer be the appropriate model of space-time. Instead, a "classical" Lorentzian manifold should be replaced by a "quantum space-time". On the macroscopic scale, this quantum space-time should go over to a Lorentzian manifold, whereas on the microscopic scale it should allow for a more general structure. Consequently, the notions of Lorentzian geometry (like metric, connection and curvature) should be extended to a corresponding "quantum geometry".

Although different proposals have been made, there is no consensus on what the mathematical framework of quantum geometry should be. Maybe the mathematically most advanced approach is Connes' non-commutative geometry [8], where the geometry is encoded in the spectral triple $(\mathcal{A}, \mathcal{D}, \mathcal{H})$ consisting of an algebra $\mathcal{A}$ of operators on the Hilbert space $\mathcal{H}$ and a generalized Dirac operator $\mathcal{D}$. The correspondence to differential geometry is obtained by choosing the algebra as the commutative algebra of functions on a manifold, and $\mathcal{D}$ as the classical Dirac operator, giving back the setting of spin geometry. By choosing $\mathcal{A}$ as a non-commutative algebra, one can extend the notions of differential geometry to a much broader setting. One disadvantage of non-commutative geometry is that it is mostly worked out in the Euclidean setting (however, for the connection to the Lorentzian case see [34, 32]). Moreover, it is not clear whether the spectral triple really gives a proper description of quantum effects on the microscopic scale. Other prominent approaches are canonical quantum gravity (see [28]), string theory (see [4]) and loop quantum gravity (see [35]); for other interesting ideas see [7, 21].

In this paper, we present a framework for a quantum geometry which is naturally adapted to the Lorentzian setting. The physical motivation is coming from the fermionic projector approach [12]. We here begin with the more general formulation in the framework of *causal fermion systems*. We give general definitions of geometric objects like the tangent space, spinors, connection and curvature. It is shown that in a suitable limit, these objects reduce to the corresponding objects of differential geometry on a globally hyperbolic Lorentzian manifold. But our framework is more general, as it also allows us to describe space-times with a non-trivial microstructure (like discrete space-times, space-time lattices or regularized space-times). In this way, the notions of Lorentzian geometry are extended to a much broader context, potentially including an appropriate model of the physical quantum space-time.

More specifically, in Section 2 we introduce the general framework of causal fermion systems and define notions of *spinors* as well as a *causal structure*. In Section 3, we proceed by constructing the objects of our Lorentzian quantum geometry: we first define the *tangent space* endowed with a Minkowski metric. Then we construct a *spin connection* relating spin spaces at different space-time points. Similarly, a corresponding *metric connection* relates tangent spaces at different space-time points. These connections give rise to corresponding notions of *curvature*. We also find a distinguished *time direction* and discuss the connection to causal sets.

In the following Sections 4 and 5, we explain how the objects of our quantum geometry correspond to the common objects of differential geometry in Minkowski space or on a Lorentzian manifold: In Section 4 we construct a class of causal fermion systems by considering a Dirac sea configuration and introducing an ultraviolet regularization. We show that if the ultraviolet regularization is removed, we get back the topological, causal and metric structure of Minkowski space, whereas the connections and curvature become trivial. In Section 5 we consider causal fermion systems constructed from a globally hyperbolic space-time. Removing the regularization, we recover the topological, causal and metric structure of the Lorentzian manifold. The spin connection and the metric connection go over to the spin and Levi–Civita connections on the manifold, respectively, up to higher order curvature corrections.

2 Causal fermion systems of spin dimension two

2.1 The general framework of causal fermion systems

We begin with the general definition of causal fermion systems (see [16, 18] for the physical motivation and [20, Section 1] for more details on the abstract framework).

Definition 2.1. Given a complex Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ (the *particle space*) and a parameter $n \in \mathbb{N}$ (the *spin dimension*), we let $\mathcal{F} \subset \mathrm{L}(\mathcal{H})$ be the set of all self-adjoint operators on $\mathcal{H}$ of finite rank, which (counting with multiplicities) have at most n positive and at most n negative eigenvalues. On $\mathcal{F}$ we are given a positive measure ρ (defined on a σ-algebra of subsets of $\mathcal{F}$), the so-called *universal measure*. We refer to $(\mathcal{H}, \mathcal{F}, \rho)$ as a *causal fermion system in the particle representation.*

On $\mathcal{F}$ we consider the topology induced by the operator norm

$$\|A\| := \sup\{\|Au\|_{\mathcal{H}} \text{ with } \|u\|_{\mathcal{H}} = 1\}. \tag{2.1}$$

A vector $\psi \in \mathcal{H}$ has the interpretation as an occupied fermionic state of our system. The name "universal measure" is motivated by the fact that ρ describes a space-time "universe". More precisely, we define *space-time M* as the support of the universal measure, $M := \operatorname{supp} \rho$; it is a closed subset of $\mathcal{F}$. The induced measure $\mu := \rho|_M$ on M allows us compute the volume of regions of space-time. The interesting point in the above definition is that by considering the spectral properties of the operator products xy, we get relations between the space-time points $x, y \in M$. The goal of this article is to analyze these relations in detail. The first relation is a notion of causality, which was also the motivation for the name "causal" fermion system.

Definition 2.2 (causal structure). For any $x, y \in \mathcal{F}$, the product xy is an operator of rank at most $2n$. We denote its non-trivial eigenvalues (counting with algebraic multiplicities) by $\lambda^{xy}_1, \ldots, \lambda^{xy}_{2n}$. The points x and y are called *timelike* separated if the λ^{xy}_j are all real. They are said to be *spacelike* separated if the λ^{xy}_j are complex and all have the same absolute value. In all other cases, the points x and y are said to be *lightlike* separated.

Restricting the causal structure of $\mathcal{F}$ to M, we get causal relations in space-time.

In order to put the above definition into the context of previous work, it is useful to introduce the inclusion map $F : M \hookrightarrow \mathcal{F}$. Slightly changing our point of view, we can now take the space-time (M, μ) and the mapping $F : M \to \mathcal{F}$ as the starting point. Identifying M with $F(M) \subset \mathcal{F}$ and constructing the measure ρ on $\mathcal{F}$ as the push-forward,

$$\rho = F_* \mu \;:\; \Omega \mapsto \rho(\Omega) := \mu(F^{-1}(\Omega)), \tag{2.2}$$

we get back to the setting of Definition 2.1. If we assume that $\mathcal{H}$ is finite dimensional and that the total volume $\mu(M)$ is finite, we thus recover the framework used in [17, Section 2] for the formulation of so-called *causal variational principles*. Interpreting $F(x)$ as local correlation matrices, one can construct the corresponding fermion system formulated on an indefinite inner product space (see [17, Sections 3.2 and 3.3]). In this setting, the dimension f of $\mathcal{H}$ is interpreted as the number of particles, whereas $\mu(M)$ is the total volume of space-time. If we assume furthermore that ρ is a finite counting measure, we get into the framework of *fermion systems in discrete space-time* as considered in [14, 13]. Thus Definition 2.1 is compatible with previous papers, but it is slightly more general in that we allow for an infinite number of particles and an infinite space-time volume. These generalizations are useful for describing the infinite volume limit of the systems analyzed in [17, Section 2].

2.2 The spin space and the Euclidean operator

For every $x \in \mathcal{F}$, we define the *spin space* S_x by

$$S_x = x(\mathcal{H}); \tag{2.3}$$

it is a subspace of $\mathcal{H}$ of dimension at most $2n$. On S_x we introduce the *spin scalar product* $\prec .|. \succ_x$ by

$$\prec u|v \succ_x = -\langle u | xv \rangle_{\mathcal{H}} \quad \text{(for all } u, v \in S_x); \tag{2.4}$$

it is an indefinite inner product of signature (p, q) with $p, q \leq n$. A *wave function* ψ is defined as a ρ-measurable function which to every $x \in M$ associates a vector of the corresponding spin space,

$$\psi \;:\; M \to \mathcal{H} \quad \text{with } \psi(x) \in S_x \quad \text{for all } x \in M. \tag{2.5}$$

Thus, the number of components of the wave functions at the space-time point x is given by $p + q$. Having four-component Dirac spinors in mind, we are led to the case of *spin dimension two*. Moreover, we impose that S_x has maximal rank.

Definition 2.3. Let $(\mathcal{H}, \mathcal{F}, \rho)$ be a fermion system of spin dimension two. A space-time point $x \in M$ is called *regular* if S_x has dimension four.

We remark that for points that are not regular, one could extend the spin space to a four-dimensional vector space (see [17, Section 3.3] for a similar construction). However, the construction of the spin connection in Section 3.3 only works for regular points. With this in mind, it seems preferable to always restrict attention to regular points.

For a regular point x, the operator $(-x)$ on $\mathcal{H}$ has two positive and two negative eigenvalues. We denote its positive and negative spectral subspaces by S_x^+ and S_x^-, respectively. In view of (2.4), these subspaces are also orthogonal with respect to the spin scalar product,

$$S_x = S_x^+ \oplus S_x^- .$$

We introduce the *Euclidean operator* $\mathfrak{E}_x$ by

$$\mathfrak{E}_x = -x^{-1} : S_x \to S_x .$$

It is obviously invariant on the subspaces $S_x^\pm$. It is useful because it allows us to recover the scalar product of $\mathcal{H}$ from the spin scalar product,

$$\langle u, v \rangle_{\mathcal{H}}|_{S_x \times S_x} = \prec u | \mathfrak{E}_x v \succ_x . \tag{2.6}$$

Often, the precise eigenvalues of x and $\mathfrak{E}_x$ will not be relevant; we only need to be concerned about their signs. To this end, we introduce the *Euclidean sign operator* s_x as a symmetric operator on S_x whose eigenspaces corresponding to the eigenvalues ± 1 are the spaces S_x^+ and S_x^-, respectively.

In order to relate two space-time points $x, y \in M$, we define the *kernel of the fermionic operator* $P(x, y)$ by

$$P(x, y) = \pi_x \, y \;:\; S_y \to S_x , \tag{2.7}$$

where π_x is the orthogonal projection onto the subspace $S_x \subset \mathcal{H}$. The calculation

$$\begin{aligned} \prec P(x,y)\, \psi(y) \,|\, \psi(x) \succ_x &= -\langle (\pi_x \, y \, \psi(y)) \,|\, x \, \phi(x) \rangle_{\mathcal{H}} \\ &= -\langle \psi(y) \,|\, yx \, \phi(x) \rangle_{\mathcal{H}} = \prec \psi(y) \,|\, P(y,x) \, \psi(x) \succ_y \end{aligned}$$

shows that this kernel is *symmetric* in the sense that

$$P(x, y)^* = P(y, x),$$

where the star denotes the adjoint with respect to the spin scalar product. The *closed chain* is defined as the product

$$A_{xy} = P(x,y)\,P(y,x) \;:\; S_x \to S_x. \tag{2.8}$$

It is obviously symmetric with respect to the spin scalar product,

$$A_{xy}^* = A_{xy}. \tag{2.9}$$

Moreover, as it is an endomorphism of S_x, we can compute its eigenvalues. The calculation $A_{xy} = (\pi_x y)(\pi_y x) = \pi_x\, yx$ shows that these eigenvalues coincide precisely with the non-trivial eigenvalues $\lambda_1^{xy}, \ldots, \lambda_4^{xy}$ of the operator xy as considered in Definition 2.2. In this way, the kernel of the fermionic operator encodes the causal structure of M. Considering the closed chain has the advantage that instead of working in the high- or even infinite-dimensional Hilbert space $\mathcal{H}$, it suffices to consider a symmetric operator on the four-dimensional vector space S_x. Then the appearance of complex eigenvalues in Definition 2.2 can be understood from the fact that the spectrum of symmetric operators in indefinite inner product spaces need not be real, as complex conjugate pairs may appear (for details see [25]).

2.3 The connection to Dirac spinors, preparatory considerations

From the physical point of view, the appearance of indefinite inner products shows that we are dealing with a *relativistic* system. In general terms, this can be understood from the fact that the isometry group of an indefinite inner product space is non-compact, allowing for the possibility that it may contain the Lorentz group.

More specifically, we have the context of Dirac spinors on a Lorentzian manifold (M, g) in mind. In this case, the spinor bundle SM is a vector bundle, whose fibre $(S_xM, \prec.|.\succ)$ is a four-dimensional complex vector space endowed with an inner product of signature $(2,2)$. The connection to our Dirac systems is obtained by identifying this vector space with $(S_x, \prec.|.\succ_x)$ as defined by (2.3) and (2.4). But clearly, in the context of Lorentzian spin geometry one has many more structures. In particular, the Clifford multiplication associates to every tangent vector $u \in T_xM$ a symmetric linear operator on S_xM. Choosing a local frame and trivialization of the bundle, the Clifford multiplication can also be expressed in terms of Dirac matrices $\gamma^j(x)$, which satisfy the anti-communication relations

$$\{\gamma^i, \gamma^j\} = 2\, g^{ij}\, \mathbb{1}. \tag{2.10}$$

Furthermore, on the spinor bundle one can introduce the *spinorial Levi–Civita connection* ∇^{LC}, which induces on the tangent bundle an associated *metric connection*.

The goal of the present paper is to construct objects for general Dirac systems, which correspond to the tangent space, the spin connection and the metric connection in Lorentzian spin geometry and generalize these notions to the setting of a "Lorentzian quantum geometry." The key for constructing the tangent space is to observe that T_xM can be identified with the subspace of the symmetric operators on S_xM spanned by the Dirac matrices. The problem is that the anti-commutation relations (2.10) are not sufficient to distinguish this subspace, as there are many different representations of these anti-commutation relations. We refer to such a representation as a *Clifford subspace*. Thus in order to get a connection to the setting of spin geometry, we need to distinguish a specific Clifford subspace. The simplest idea for constructing the spin connection would be to use a polar decomposition of $P(x,y)$. Thus decomposing $P(x,y)$ as

$$P(x,y) = U(x)\,\rho(x,y)\,U(y)^{-1}$$

with a positive operator $\rho(x,y)$ and unitary operators $U(x)$ and $U(y)$, we would like to introduce the spin connection as the unitary mapping

$$D_{x,y} = U(x)\,U(y)^{-1} \;:\; S_y \to S_x. \tag{2.11}$$

The problem with this idea is that it is not clear how this spin connection should give rise to a corresponding metric connection. Moreover, one already sees in the simple example of a regularized Dirac sea vacuum (see Section 4) that in Minkowski space this spin connection does not reduce to the trivial connection. Thus, the main difficulty is to modify (2.11) such as to obtain a spin connection which induces a metric connection and becomes trivial in Minkowski space. This difficulty is indeed closely related to the problem of distinguishing a specific Clifford subspace.

The key for resolving these problems will be to use the Euclidean operator $\mathfrak{E}_x$ in a specific way. In order to explain the physical significance of this operator, we point out that, apart from the Lorentzian point of view discussed above, we can also go over to the *Euclidean framework* by considering instead of the spin scalar product the scalar product on $\mathcal{H}$. In view of the identity (2.6), the transition to the Euclidean framework can be described by the Euclidean operator, which motivates its name. The physical picture is that the Dirac systems of Definition 2.1 involve a regularization, which breaks the Lorentz symmetry. This fact becomes apparent in the Euclidean operator, which allows us to introduce a scalar product on spinors (2.6) which violates Lorentz invariance. The subtle point in the constructions in this paper is to use the Euclidean sign operator to distinguish certain Clifford subspaces, but in such a way that the Lorentz invariance of the resulting objects is preserved. The connection between the Euclidean operator and the regularization will become clearer in the examples in Sections 4 and 5.

We finally give a construction which will not be needed later on, but which is nevertheless useful to get a closer connection to Dirac spinors in relativistic quantum mechanics. To this end, we consider wave functions ψ, ϕ of the form (2.5) which are square integrable. Setting

$$\langle \psi | \phi \rangle = \int_M \prec \psi(x) | \phi(x) \succ_x d\mu(x), \tag{2.12}$$

the vector space of wave functions becomes an indefinite inner product space. Interpreting $P(x,y)$ as an integral kernel, we can introduce the *fermionic operator* by

$$(P\psi)(x,y) = \int_M P(x,y)\, \psi(y)\, d\mu(y).$$

Additionally imposing the idempotence condition $P^2 = P$, we obtain the *fermionic projector* as considered in [12, 14]. In this context, the inner product (2.12) reduces to the integral over Minkowski space $\int_M \overline{\psi(x)} \phi(x)\, d^4x$, where $\overline{\psi}\phi$ is the Lorentz invariant inner product on Dirac spinors.

3 Construction of a Lorentzian quantum geometry

3.1 Clifford extensions and the tangent space

We proceed with constructions in the spin space $(S_x, \prec.|.\succ)$ at a fixed space-time point $x \in M$. We denote the set of symmetric linear endomorphisms of S_x by $\text{Symm}(S_x)$; it is a 16-dimensional real vector space.

We want to introduce the Dirac matrices, but without specifying a particular representation. Since we do not want to prescribe the dimension of the resulting space-time, it is preferable to work with the maximal number of five generators (for the minimal dimensions of Clifford representations see for example [3]).

Definition 3.1. A five-dimensional subspace $K \subset \text{Symm}(S_x)$ is called a **Clifford subspace** if the following conditions hold:

(i) For any $u, v \in K$, the anti-commutator $\{u, v\} \equiv uv + vu$ is a multiple of the identity on S_x.
(ii) The bilinear form $\langle ., . \rangle$ on K defined by

$$\frac{1}{2} \{u, v\} = \langle u, v \rangle \, \mathbb{1} \quad \text{for all } u, v \in K \tag{3.1}$$

is non-degenerate.

The set of all Clifford subspaces $(K, \langle ., . \rangle)$ is denoted by $\mathcal{T}$.

Our next lemma characterizes the possible signatures of Clifford subspaces.

Lemma 3.2. *The inner product $\langle ., . \rangle$ on a Clifford subspace has either the signature $(1, 4)$ or the signature $(3, 2)$. In the first (second) case, the inner product*

$$\prec . | u \, . \succ_x \; : \; S_x \times S_x \to \mathbb{C} \tag{3.2}$$

is definite (respectively indefinite) for every vector $u \in K$ with $\langle u, u \rangle > 0$.

Proof. Taking the trace of (3.1), one sees that the inner product on K can be extended to all of $\text{Symm}(S_x)$ by

$$\langle ., . \rangle \; : \; \text{Symm}(S_x) \times \text{Symm}(S_x) \to \mathbb{C}, \;\; (A, B) \mapsto \frac{1}{4} \, \text{Tr}(AB).$$

A direct calculation shows that this inner product has signature $(8, 8)$ (it is convenient to work in the basis of $\text{Symm}(S_x)$ given by the matrices $(\mathbb{1}, \gamma^i, i\gamma^5, \gamma^5\gamma^i, \sigma^{jk})$ in the usual Dirac representation; see [6, Section 2.4]).

Since $\langle ., . \rangle$ is assumed to be non-degenerate, it has signature $(p, 5-p)$ with a parameter $p \in \{0, \ldots, 5\}$. We choose a basis $e_0, \ldots, e_4$ of K where the bilinear form is diagonal,

$$\{e_j, e_k\} = 2 s_j \, \delta_{jk} \, \mathbb{1} \quad \text{with} \quad s_0, \ldots, s_{p-1} = 1 \text{ and } s_p, \ldots, s_4 = -1. \tag{3.3}$$

These basis vectors generate a Clifford algebra. Using the uniqueness results on Clifford representations [30, Theorem 5.7], we find that in a suitable basis of S_x, the operators e_j have the basis representations

$$e_0 = c_0 \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad e_\alpha = c_\alpha \begin{pmatrix} 0 & \mathrm{i}\sigma_\alpha \\ -\mathrm{i}\sigma_\alpha & 0 \end{pmatrix}, \quad e_4 = c_4 \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{3.4}$$

with coefficients

$$c_0, \ldots, c_{p-1} \in \{1, -1\}, \quad c_p, \ldots, c_4 \in \{i, -i\}.$$

Here $\alpha \in \{1, 2, 3\}$, and σ^α are the three Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In particular, one sees that the e_j are all trace-free. We next introduce the ten bilinear operators

$$\sigma_{jk} := ie_je_k \quad \text{with} \quad 1 \leq j < k \leq 5.$$

Taking the trace and using that e_j and e_k anti-commute, one sees that the bilinear operators are also trace-free. Furthermore, using the anti-commutation relations (3.3), one finds that

$$\langle \sigma_{jk}, \sigma_{lm} \rangle = s_j s_k \, \delta_{jl} \delta_{km}.$$

Thus the operators $\{\mathbb{1}, e_j, \sigma_{jk}\}$ form a pseudo-orthonormal basis of $\mathrm{Symm}(S_x)$.

In the cases $p = 0$ and $p = 5$, the operators σ_{jk} would span a ten-dimensional definite subspace of $\mathrm{Symm}(S_x)$, in contradiction to the above observation that $\mathrm{Symm}(S_x)$ has signature $(8, 8)$. Similarly, in the cases $p = 2$ and $p = 4$, the signature of $\mathrm{Symm}(S_x)$ would be equal to $(7, 9)$ and $(11, 5)$, again giving a contradiction. We conclude that the possible signatures of K are $(1, 4)$ and $(3, 2)$.

We represent the spin scalar product in the spinor basis of (3.4) with a signature matrix S,

$$\prec .|. \succ_x = \langle .|S\, . \rangle_{\mathbb{C}^4}.$$

Let us compute S. In the case of signature $(1, 4)$, the fact that the operators e_j are symmetric gives rise to the conditions

$$[S, e_0] = 0 \quad \text{and} \quad \{S, e_j\} = 0 \quad \text{for } j = 1, \ldots, 4\,. \tag{3.5}$$

A short calculation yields $S = \lambda e_0$ for $\lambda \in \mathbb{R} \setminus \{0\}$. This implies that the bilinear form $\prec .|e_0\, . \succ_x$ is definite. Moreover, a direct calculation shows that (3.2) is definite for any vector $u \in K$ with $\langle u, u \rangle > 0$.

In the case of signature $(3, 2)$, we obtain similar to (3.5) the conditions

$$[S, e_j] = 0 \quad \text{for } j = 0, 1, 2 \quad \text{and} \quad \{S, e_j\} = 0 \quad \text{for } j = 3, 4\,.$$

It follows that $S = \mathrm{i}\lambda e_3 e_4$. Another direct calculation yields that the bilinear form (3.2) is indefinite for any $u \in K$ with $\langle u, u \rangle > 0$. $\square$

We shall always restrict attention to Clifford subspaces of signature $(1, 4)$. This is motivated physically because the Clifford subspaces of signature $(3, 2)$ only have two spatial dimensions, so that by dimensional reduction we cannot get to Lorentzian signature $(1, 3)$. Alternatively, this can be understood from the analogy to Dirac spinors, where the inner product $\overline{\psi} u^j \gamma_j \phi$ is definite for any timelike vector u. Finally, for the Clifford subspaces of signature $(3, 2)$ the constructions following Definition 3.6 would not work.

From now on, we implicitly assume that all Clifford subspaces have signature $(1, 4)$. We next show that such a Clifford subspace is uniquely determined by a two-dimensional subspace of signature $(1, 1)$.

Lemma 3.3. *Assume that $L \subset K$ is a two-dimensional subspace of a Clifford subspace K, such that the inner product $\langle ., . \rangle|_{L \times L}$ has signature $(1, 1)$. Then for every Clifford subspace $\tilde{K}$ the following implication holds:*

$$L \subset \tilde{K} \quad \Longrightarrow \quad \tilde{K} = K.$$

Proof. We choose a pseudo-orthonormal basis of L, which we denote by (e_0, e_4). Since $e_0^2 = \mathbb{1}$, the spectrum of e_0 is contained in the set $\{\pm 1\}$. The calculation $e_0(e_0 \pm \mathbb{1}) = \mathbb{1} \pm e_0 = \pm(e_0 \pm \mathbb{1})$ shows that the corresponding invariant subspaces are indeed eigenspaces. Moreover, as the bilinear form $\prec .|e_0 . \succ_x$ is definite, the eigenspaces are also definite. Thus we may choose a pseudo-orthonormal eigenvector basis $(\mathfrak{f}_1, \ldots, \mathfrak{f}_4)$ in which

$$e_0 = \pm \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

We next consider the operator e_4. Using that it anti-commutes with e_0, is symmetric and that $(e_4)^2 = -\mathbb{1}$, one easily sees that it has the matrix representation

$$e_4 = \begin{pmatrix} 0 & -V \\ V^{-1} & 0 \end{pmatrix} \quad \text{with} \quad V \in \mathrm{U}(2).$$

Thus after transforming the basis vectors $\mathfrak{f}_3$ and $\mathfrak{f}_4$ by

$$\begin{pmatrix} \mathfrak{f}_3 \\ \mathfrak{f}_4 \end{pmatrix} \to -\mathrm{i}V \begin{pmatrix} \mathfrak{f}_3 \\ \mathfrak{f}_4 \end{pmatrix}, \tag{3.6}$$

we can arrange that

$$e_4 = i \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

Now suppose that $\tilde{K}$ extends L to a Clifford subspace. We extend (e_0, e_4) to a pseudo-orthonormal basis $(e_0, e_1, \ldots, e_4)$ of $\tilde{K}$. Using that the operators e_1, e_2 and e_3 anti-commute with e_0 and e_4 and are symmetric, we see that each of these operators must be of the form

$$e_\alpha = \begin{pmatrix} 0 & A^\alpha \\ -A^\alpha & 0 \end{pmatrix} \tag{3.7}$$

with Hermitian 2×2-matrices A_α. The anti-commutation relations (3.1) imply that the A_α satisfy the anti-commutation relations of the Pauli matrices

$$\left\{ A^\alpha, A^\beta \right\} = 2\delta^{\alpha\beta}.$$

The general representation of these relations is obtained from the Pauli matrices by an SU(2)-transformation and possible sign flips,

$$A^\alpha = \pm U \sigma^\alpha U^{-1} \quad \text{with} \quad U \in \mathrm{SU}(2).$$

Since $U\sigma^\alpha U^{-1} = O^\alpha_\beta \sigma^\beta$ with $O \in \mathrm{SO}(3)$, we see that the A^α are linear combinations of the Pauli matrices. Hence the subspace spanned by the matrices e_1, e_2 and e_3 is uniquely determined by L. It follows that $\tilde{K} = K$. □

In the following corollary we choose a convenient matrix representation for a Clifford subspace.

Corollary 3.4. *For every pseudo-orthonormal basis* $(e_0, \ldots, e_4)$ *of a Clifford subspace* K*, we can choose a pseudo-orthonormal basis* $(\mathfrak{f}_1, \ldots, \mathfrak{f}_4)$ *of* S_x*,*

$$\prec \mathfrak{f}_\alpha | \mathfrak{f}_\beta \succ = s_\alpha\, \delta_{\alpha\beta} \quad \text{with} \quad s_1 = s_2 = 1 \text{ and } s_3 = s_4 = -1, \tag{3.8}$$

such that the operators e_i *have the following matrix representations,*

$$e_0 = \pm \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad e_\alpha = \pm \begin{pmatrix} 0 & \sigma_\alpha \\ -\sigma_\alpha & 0 \end{pmatrix}, \quad e_4 = i \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{3.9}$$

Proof. As in the proof of Lemma 3.3, we can choose a pseudo-orthonormal basis $(\mathfrak{f}_1, \ldots, \mathfrak{f}_4)$ of S_x satisfying (3.8) such that e_0 and e_4 have the desired representation. Moreover, in this basis the operators e_1, e_2 and e_3 are of the form (3.7). Hence by the transformation of the spin basis

$$\begin{pmatrix} \mathfrak{f}_1 \\ \mathfrak{f}_2 \end{pmatrix} \to U^{-1} \begin{pmatrix} \mathfrak{f}_1 \\ \mathfrak{f}_2 \end{pmatrix}, \quad \begin{pmatrix} \mathfrak{f}_3 \\ \mathfrak{f}_4 \end{pmatrix} \to U^{-1} \begin{pmatrix} \mathfrak{f}_3 \\ \mathfrak{f}_4 \end{pmatrix},$$

we obtain the desired representation (3.9). ☐

Our next step is to use the Euclidean sign operator to distinguish a specific subset of Clifford subspaces. For later use, it is preferable to work instead of the Euclidean sign operator with a more general class of operators defined as follows.

Definition 3.5. An operator $v \in \text{Symm}(S_x)$ is called a **sign operator** if $v^2 = \mathbb{1}$ and if the inner product $\prec . | v \,. \succ_x \,:\, S_x \times S_x \to \mathbb{C}$ is positive definite.

Clearly, the Euclidean sign operator s_x is an example of a sign operator.

Since a sign operator v is symmetric with respect to the positive definite inner product $\prec . | v \,. \succ$, it can be diagonalized. Again using that the inner product $\prec . | v \,. \succ$ is positive, one finds that the eigenvectors corresponding to the eigenvalues $+1$ and -1 are positive and negative definite, respectively. Thus we may choose a pseudo-orthonormal basis (3.8) in which v has the matrix representation $v = \text{diag}(1, 1, -1, -1)$. In this spin basis, v is represented by the matrix γ^0 (in the usual Dirac representation). Thus by adding the spatial Dirac matrices, we can extend v to a Clifford subspace. We now form the set of all such extensions.

Definition 3.6. For a given sign operator v, the set of **Clifford extensions** $\mathcal{T}^v$ is defined as the set of all Clifford subspaces containing v,

$$\mathcal{T}^v = \{K \text{ Clifford subspace with } v \in K\}.$$

After these preparations, we want to study how different Clifford subspaces or Clifford extensions can be related to each other by unitary transformations. We denote the group of unitary endomorphisms of S_x by $\mathrm{U}(S_x)$; it is isomorphic to the group $\mathrm{U}(2,2)$. Thus for given $K, \tilde{K} \in \mathcal{T}$ (or $\mathcal{T}^v$), we want to determine the unitary operators $U \in \mathrm{U}(S_x)$ such that

$$\tilde{K} = UKU^{-1}. \tag{3.10}$$

Clearly, the subgroup $\exp(i\mathbb{R}\mathbb{1}) \simeq \mathrm{U}(1)$ is irrelevant for this problem, because in (3.10) phase transformations drop out. For this reason, it is useful to divide out this group by setting

$$\mathcal{G}(S_x) = \mathrm{U}(S_x)/\exp(i\mathbb{R}\mathbb{1}). \tag{3.11}$$

We refer to $\mathcal{G}$ as the *gauge group* (this name is motivated by the formulation of spinors in curved space-time as a gauge theory; see [10]). It is a 15-dimensional non-compact Lie group whose corresponding Lie algebra is formed of all trace-free elements of $\mathrm{Symm}(S_x)$. It is locally isomorphic to the group $\mathrm{SU}(2,2)$ of $\mathrm{U}(2,2)$-matrices with determinant one. However, we point out that $\mathcal{G}$ is *not* isomorphic to $\mathrm{SU}(2,2)$, because the four-element subgroup $\mathbb{Z}_4 := \exp(\mathrm{i}\pi\mathbb{Z}\mathbb{1}/2) \subset \mathrm{SU}(2,2)$ is to be identified with the neutral element in $\mathcal{G}$. In other words, the groups are isomorphic only after dividing out this discrete subgroup, $\mathcal{G} \simeq \mathrm{SU}(2,2)/\mathbb{Z}_4$.

Corollary 3.7. *For any two Clifford subspaces $K, \tilde{K} \in \mathcal{T}$, there is a gauge transformation $U \in \mathcal{G}$ such that* (3.10) *holds.*

Proof. We choose spin bases $(\mathfrak{f}_\alpha)$ and similarly $(\tilde{\mathfrak{f}}_\alpha)$ as in Corollary 3.4 and let U be the unitary transformation describing the basis transformation. □

Next, we consider the subgroups of $\mathcal{G}$ which leave the sign operator v and possibly a Clifford subspace $K \in \mathcal{T}^v$ invariant:

$$\begin{aligned} \mathcal{G}_v &= \big\{ U \in \mathcal{G} \text{ with } UvU^{-1} = v \big\} \\ \mathcal{G}_{v,K} &= \big\{ U \in \mathcal{G} \text{ with } UvU^{-1} = v \text{ and } UKU^{-1} = K \big\}\,. \end{aligned} \tag{3.12}$$

We refer to these groups as the **stabilizer subgroups** of v and (v, K), respectively.

Lemma 3.8. *For any Clifford extension $K \in \mathcal{T}^v$, the stabilizer subgroups are related by*

$$\mathcal{G}_v = \exp(i\mathbb{R}v) \times \mathcal{G}_{v,K}.$$

Furthermore,

$$\mathcal{G}_{v,K} \simeq (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}_2 \simeq \mathrm{SO}(4),$$

where the group SO(4) *acts on any pseudo-orthonormal basis* $(v, e_1, \ldots, e_4)$ *of* K *by*

$$e_i \mapsto \sum_{j=1}^{4} O_i^j \, e_j, \quad O \in \mathrm{SO}(4). \tag{3.13}$$

Proof. The elements of $\mathcal{G}_v$ are represented by unitary operators which commute with v. Thus choosing a spin frame where

$$v = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix}, \tag{3.14}$$

every $U \in \mathcal{G}_v$ can be represented as

$$U = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \quad \text{with} \quad V_{1,2} \in \mathrm{U}(2).$$

Collecting phase factors, we can write

$$U = \mathrm{e}^{\mathrm{i}\alpha} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\beta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\beta} \end{pmatrix} \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \quad \text{with } \alpha, \beta \in \mathbb{R} \text{ and } U_{1,2} \in \mathrm{SU}(2).$$

As the two matrices in this expression obviously commute, we obtain, after dividing out a global phase,

$$\mathcal{G}_v \simeq \exp(i\mathbb{R}v) \times (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}_2, \tag{3.15}$$

where $\mathbb{Z}_2$ is the subgroup $\{\pm 1\!\!1\}$ of $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

Let us consider the group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ acting on the vectors of K by conjugation. Obviously, $UvU^{-1} = v$. In order to compute Ue_jU^{-1}, we first apply the identity

$$\mathrm{e}^{\mathrm{i}\vec{u}_1\vec{\sigma}} \, (i\rho \, 1\!\!1 + \vec{w}\vec{\sigma}) \, \mathrm{e}^{-\mathrm{i}\vec{u}_2\vec{\sigma}} = i\rho' \, 1\!\!1 + \vec{w}'\vec{\sigma}.$$

Taking the determinant of both sides, one sees that the vectors $(\rho, \vec{w})$, $(\rho', \vec{w}') \in \mathbb{R}^4$ have the same Euclidean norm. Thus the group $\mathrm{SU}(2) \times \mathrm{SU}(2)$

describes SO(4)-transformations (3.13). Counting dimensions, it follows that $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is a covering of SO(4). Next it is easy to verify that the only elements of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ which leave all γ^i, $i = 1, \ldots, 4$, invariant are multiples of the identity matrix. We conclude that $(\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}_2 \simeq \mathrm{SO}(4)$ (this can be understood more abstractly from the fact that $\mathrm{SU}(2) \times \mathrm{SU}(2) = \mathrm{Spin}(4)$; see for example [23, Chapter 1]).

To summarize, the factor $\mathrm{SU}(2) \times \mathrm{SU}(2)$ in (3.15) leaves K invariant and describes the transformations (3.13). However, the only elements of the group $\exp(i\mathbb{R}v)$, which leave K invariant are multiples of the identity. This completes the proof. □

Our method for introducing the tangent space is to form equivalence classes of Clifford extensions. To this end, we introduce on $\mathcal{T}^v$ the equivalence relation

$$K \sim \tilde{K} \quad \Longleftrightarrow \quad \text{there is } U \in \exp(i\mathbb{R}v) \text{ with } \tilde{K} = UKU^{-1}. \tag{3.16}$$

According to Corollary 3.7 and Lemma 3.8, there is only one equivalence class. In other words, for any $K, \tilde{K} \in \mathcal{T}^v$ there is an operator $U \in \exp(i\mathbb{R}v)$ such that (3.10) holds. However, we point out that the operator U is not unique. Indeed, for two choices U, U', the operator $U^{-1}U'$ is an element of $\exp(i\mathbb{R}v) \cap \mathcal{G}_{v,K}$, meaning that U is unique only up to the transformations

$$U \to \pm U \quad \text{and} \quad U \to \pm iv\, U. \tag{3.17}$$

The operator U gives rise to the so-called *identification map*

$$\phi^v_{\tilde{K},K} \,:\, K \to \tilde{K}\,,\; w \mapsto UwU^{-1}. \tag{3.18}$$

The freedom (3.17) implies that the mapping $\phi^v_{\tilde{K},K}$ is defined only up to a *parity transformation* P^v which flips the sign of the orthogonal complement of v,

$$\phi^v_{\tilde{K},K} \to P^v\, \phi^v_{\tilde{K},K} \quad \text{with} \quad P^v w = -w + 2\langle w, v\rangle\, v. \tag{3.19}$$

As the identification map preserves the inner product $\langle .,. \rangle$, the quotient space $\mathcal{T}^v/\sim$ is endowed with a Lorentzian metric. We now take v as the Euclidean sign operator, which seems the most natural choice.

Definition 3.9. The **tangent space** T_x is defined by

$$T_x = \mathcal{T}^{s_x} / \exp(i\mathbb{R}s_x).$$

It is endowed with an inner product $\langle .,. \rangle$ of signature $(1,4)$.

We point out that, due to the freedom to perform the parity transformations (3.19), the tangent space has no spatial orientation. In situations when a spatial orientation is needed, one can fix the parity by distinguishing a class of representatives.

Definition 3.10. A set of representatives $\mathfrak{U} \subset \mathfrak{T}^{s_x}$ of the tangent space is called **parity preserving** if for any two $K, \tilde{K} \in \mathfrak{U}$, the corresponding identification map $\phi^{s_x}_{\tilde{K},K}$ is of the form (3.18) with $U = \mathrm{e}^{\mathrm{i}\beta s_x}$ and $\beta \not\in \frac{\pi}{2} + \pi\mathbb{Z}$. Then the **parity preserving identification map** is defined by (3.18) with

$$U = U^{s_x}_{\tilde{K},K} := \mathrm{e}^{\mathrm{i}\beta s_x} \quad \text{and} \quad \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{3.20}$$

By identifying the elements of $\mathfrak{U}$ via the parity preserving identification maps, one can give the tangent space a spatial orientation. In Section 3.4, we will come back to this construction for a specific choice of $\mathfrak{U}$ induced by the spin connection.

3.2 Synchronizing generically separated sign operators

In this section, we will show that for two given sign operators v and $\tilde{v}$ (again at a fixed space-time point $x \in M$), under generic assumptions one can distinguish unique Clifford extensions $K \in \mathfrak{T}^v$ and $\tilde{K} \in \mathfrak{T}^{\tilde{v}}$. Moreover, we will construct the so-called synchronization map $U^{\tilde{v},v}$, which transforms these two Clifford extensions into each other.

Definition 3.11. Two sign operators $v, \tilde{v}$ are said to be **generically separated** if their commutator $[v, \tilde{v}]$ has rank four.

Lemma 3.12. *Assume that v and $\tilde{v}$ are two generically separated sign operators. Then there are unique Clifford extensions $K \in \mathfrak{T}^v$ and $\tilde{K} \in \mathfrak{T}^{\tilde{v}}$ and a unique vector $\rho \in K \cap \tilde{K}$ with the following properties:*

$$\text{(i)} \quad \{v, \rho\} = 0 = \{\tilde{v}, \rho\}, \tag{3.21}$$

$$\text{(ii)} \quad \tilde{K} = \mathrm{e}^{\mathrm{i}\rho}\, K\, \mathrm{e}^{-\mathrm{i}\rho}, \tag{3.22}$$

$$\text{(iii)} \quad \textit{If } \{v, \tilde{v}\} \textit{ is a multiple of the identity, then } \rho = 0. \tag{3.23}$$

The operator ρ depends continuously on v and $\tilde{v}$.

Proof. Our first step is to choose a spin frame where v and $\tilde{v}$ have a simple form. Denoting the spectral projector of v corresponding to the eigenvalue one by $E_+ = (\mathbb{1} + v)/2$, we choose an orthonormal eigenvector basis $(\mathfrak{f}_1, \mathfrak{f}_2)$ of the operator $E_+ \tilde{v} E_+$, i.e.,

$$E_+ \tilde{v} E_+|_{E_+(S_x)} = \mathrm{diag}(\nu_1, \nu_2) \quad \text{with } \nu_1, \nu_2 \in \mathbb{R}.$$

Setting $\mathfrak{f}_3 = (\tilde{v} - \nu_1)\mathfrak{f}_1$ and $\mathfrak{f}_4 = (\tilde{v} - \nu_2)\mathfrak{f}_2$, these vectors are clearly orthogonal to $\mathfrak{f}_1$ and $\mathfrak{f}_2$. They are both non-zero because otherwise the commutator $[\nu, \tilde{\nu}]$ would be singular. Next, being orthogonal to the eigenspace of v corresponding to the eigenvalue one, they lie in the eigenspace of v corresponding to the eigenvalue -1, and are thus both negative definite. Moreover, the following calculation shows that they are orthogonal,

$$\begin{aligned}
\prec \mathfrak{f}_3 | \mathfrak{f}_4 \succ &= \prec (\tilde{v} - \nu_1)\mathfrak{f}_1 | (\tilde{v} - \nu_2)\mathfrak{f}_2 \succ = \prec \mathfrak{f}_1 | (\tilde{v} - \nu_1)(\tilde{v} - \nu_2)\mathfrak{f}_2 \succ \\
&= \prec \mathfrak{f}_1 | \left(1 + \nu_1\nu_2 - (\nu_1 + \nu_2)\tilde{v}\right) \mathfrak{f}_2 \succ = 0,
\end{aligned}$$

where in the last step we used that $\mathfrak{f}_2$ and $\tilde{v}\mathfrak{f}_2$ are orthogonal to $\mathfrak{f}_1$. The image of $\mathfrak{f}_3$ (and similarly $\mathfrak{f}_4$) is computed by

$$\tilde{v} f_3 = \tilde{v}(\tilde{v} - \nu_1)\mathfrak{f}_1 = (1 - \nu_1 \tilde{v})\mathfrak{f}_1 = -\nu_1 \mathfrak{f}_3 + (1 - \nu_1^2)\, \mathfrak{f}_1 .$$

We conclude that, after normalizing $\mathfrak{f}_3$ and $\mathfrak{f}_4$ by the replacement $\mathfrak{f}_i \to \mathfrak{f}_i / \sqrt{-\prec f_i | f_i \succ}$, the matrix v is diagonal (3.14), whereas $\tilde{v}$ is of the form

$$\tilde{v} = \begin{pmatrix} \cosh\alpha & 0 & \sinh\alpha & 0 \\ 0 & \cosh\beta & 0 & -\sinh\beta \\ -\sinh\alpha & 0 & -\cosh\alpha & 0 \\ 0 & \sinh\beta & 0 & -\cosh\beta \end{pmatrix} \quad \text{with } \alpha, \beta > 0. \tag{3.24}$$

In the case $\alpha = \beta$, the anti-commutator $\{v, \tilde{v}\}$ is a multiple of the identity. Thus by assumption (iii) we need to choose $\rho = 0$. Then $K = \tilde{K}$ must be the Clifford subspace spanned by the matrices $e_0, \ldots, e_4$ in (3.9).

In the remaining case $\alpha \neq \beta$, a short calculation shows that any operator ρ which anti-commutes with both v and $\tilde{v}$ is a linear combination of the matrix e_4 and the matrix ie_0e_3. Since ρ should be an element of K, its square must be a multiple of the identity. This leaves us with the two cases

$$\rho = \frac{\tau}{2}\, e_4 \quad \text{or} \quad \rho = \frac{\tau}{2}\, ie_0e_3 \tag{3.25}$$

for a suitable real parameter τ. In the first case, we obtain

$$\mathrm{e}^{\mathrm{i}\rho} v\, \mathrm{e}^{-\mathrm{i}\rho} = \mathrm{e}^{2\mathrm{i}\rho} v = \begin{pmatrix} \mathbb{1}\cosh\tau & \mathbb{1}\sinh\tau \\ -\mathbb{1}\sinh\tau & -\mathbb{1}\cosh\tau \end{pmatrix}.$$

A straightforward calculation yields that the anti-commutator of this matrix with $\tilde{v}$ is a multiple of the identity if and only if

$$\cosh(\alpha - \tau) = \cosh(\beta + \tau),$$

determining τ uniquely to $\tau = (\alpha - \beta)/2$. In the second case in (3.25), a similar calculation yields the condition $\cosh(\alpha - \tau) = \cosh(\beta - \tau)$, which has no solution. We conclude that we must choose ρ as

$$\rho = \frac{\alpha - \beta}{4}\, e_4. \tag{3.26}$$

In order to construct the corresponding Clifford subspaces K and $\tilde{K}$, we first replace $\tilde{v}$ by the transformed operator $\mathrm{e}^{-\mathrm{i}\rho} \tilde{v}\, \mathrm{e}^{\mathrm{i}\rho}$. Then we are again in case $\alpha = \beta > 0$, where the unique Clifford subspace K is given by the span of the matrices $e_0, \ldots, e_4$ in (3.9). Now we can use the formula in (ii) to define $\tilde{K}$; it follows by construction that $\tilde{v} \in \tilde{K}$.

In order to prove continuity, we first note that the constructions in the two cases $\alpha = \beta$ and $\alpha \neq \beta$ obviously depend continuously on v and $\tilde{v}$. Moreover, it is clear from (3.26) that ρ is continuous in the limit $\alpha - \beta \to 0$. This concludes the proof. ☐

Definition 3.13. For generically separated signature operators $v, \tilde{v}$, we denote the unique clifford extension K in Lemma 3.12 by $K^{v,(\tilde{v})} \in \mathcal{T}^v$ and refer to it as the **Clifford extension of** v **synchronized with** $\tilde{v}$. Similarly, $K^{\tilde{v},(v)} \in \mathcal{T}^{\tilde{v}}$ is the Clifford extension of $\tilde{v}$ synchronized with v. Moreover, we introduce the **synchronization map** $U^{\tilde{v},v} := \mathrm{e}^{\mathrm{i}\rho} \in \mathrm{U}(S_x)$.

According to Lemma 3.12, the synchronization map satisfies the relations

$$U^{\tilde{v},v} = (U^{v,\tilde{v}})^{-1} \quad \text{and} \quad K^{\tilde{v},(v)} = U^{\tilde{v},v} K^{v,(\tilde{v})} U^{v,\tilde{v}}.$$

3.3 The spin connection

For the constructions in this section, we need a stronger version of Definition 2.2.

Definition 3.14. The space-time points $x, y \in M$ are said to be **properly timelike** separated if the closed chain A_{xy} has a strictly positive spectrum and if the corresponding eigenspaces are definite subspaces of S_x.

The condition that the eigenspaces should be definite ensures that A_{xy} is diagonalizable (as one sees immediately by restricting A_{xy} to the orthogonal complement of all eigenvectors). Let us verify that our definition is symmetric in x and y: Suppose that $A_{xy}u = \lambda u$ with $u \in S_x$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Then the vector $w := P(y,x)u \in S_y$ is an eigenvector of A_{yx} again to the eigenvalue λ,

$$A_{yx}\, w = P(y,x)P(x,y)\, P(y,x)\, u = P(y,x)\, A_{xy}\, u = \lambda\, P(y,x)\, u = \lambda w. \tag{3.27}$$

Moreover, the calculation

$$\begin{aligned} \lambda \prec\! u|u \!\succ &= \prec\! u|A_{xy}u\!\succ = \prec\! u\,|\,P(x,y)\,P(y,x)\,u\!\succ \\ &= \prec\! P(y,x)u\,|\,P(y,x)u\!\succ = \prec\! w|w\!\succ \end{aligned} \tag{3.28}$$

shows that w is a definite vector if and only if u is. We conclude that A_{yx} has the same eigenvalues as A_{xy} and again has definite eigenspaces.

According to (3.28), the condition in Definition 3.14 that the spectrum of A_{xy} should be positive means that $P(y,x)$ maps positive and negative definite eigenvectors of A_{xy} to positive and negative definite eigenvectors of A_{yx}, respectively. This property will be helpful in the subsequent constructions. However, possibly this condition could be weakened (for example, it seems likely that a spin connection could also be constructed in the case that the eigenvalues of A_{xy} are all negative). But in view of the fact that in the examples in Sections 4 and 5, the eigenvalues of A_{xy} are always positive in timelike directions, for our purposes Definition 3.14 is sufficiently general.

For given space-time points $x, y \in M$, our goal is to use the form of $P(x,y)$ and $P(y,x)$ to construct the *spin connection* $D_{x,y} \in \mathrm{U}(S_y, S_x)$ as a unitary transformation

$$D_{x,y} : S_y \to S_x \quad \text{and} \quad D_{y,x} = (D_{x,y})^{-1} = (D_{x,y})^* : S_x \to S_y, \tag{3.29}$$

which should have the additional property that it gives rise to an isometry of the corresponding tangent spaces.

We now give the general construction of the spin connection, first in specific bases and then in an invariant way. At the end of this section, we will list all the assumptions and properties of the resulting spin connection (see Theorem 3.20). The corresponding mapping of the tangent spaces will be constructed in Section 3.4.

Our first assumption is that the space-time points x and y should be properly timelike separated (see Definition 3.14). Combining the positive definite eigenvectors of A_{xy}, we obtain a two-dimensional positive definite invariant subspace I_+ of the operator A_{xy}. Similarly, there is a two-dimensional negative definite invariant subspace I_-. Since A_{xy} is symmetric, these invariant subspaces form an orthogonal decomposition, $S_x = I_+ \oplus I_-$. We introduce the operator $v_{xy} \in \mathrm{Symm}(S_x)$ as an operator with the property that I_+ and I_- are eigenspaces corresponding to the eigenvalues $+1$ and -1, respectively. Obviously, v_{xy} is a sign operator (see Definition 3.5). Alternatively, it can be characterized in a basis-independent way as follows.

Definition 3.15. The unique sign operator $v_{xy} \in \mathrm{Symm}(S_x)$ which commutes with the operator A_{xy} is referred to as the **directional sign operator** of A_{xy}.

We next assume that the Euclidean sign operator and the directional sign operator are generically separated at both x and y (see Definition 3.11). Then at the point x, there is the unique Clifford extension $K_{xy} := K_x^{v_{xy},(s_x)} \in \mathcal{T}_x^{v_{xy}}$ of the directional sign operator synchronized with the Euclidean sign operator (see Definitions 3.13 and 3.6, where for clarity we added the base point x as a subscript). Similarly, at y we consider the Clifford extension $K_{yx} := K_y^{v_{yx},(s_y)} \in \mathcal{T}_y^{v_{yx}}$. In view of the later construction of the metric connection (see Section 3.4), we need to impose that the spin connection should map these Clifford extensions into each other, i.e.,

$$K_{xy} = D_{x,y}\, K_{yx}\, D_{y,x}. \tag{3.30}$$

To clarify our notation, we point out that by the subscript $_{xy}$ we always denote an object at the point x, whereas the additional comma $_{x,y}$ denotes an operator which maps an object at y to an object at x. Moreover, it is natural to demand that

$$v_{xy} = D_{x,y}\, v_{yx}\, D_{y,x}. \tag{3.31}$$

We now explain the construction of the spin connection in suitably chosen bases of the Clifford subspaces and the spin spaces. We will then verify that this construction does not depend on the choice of the bases. At the end of this section, we will give a basis independent characterization of the spin connection. In order to choose convenient bases at the point x, we set $e_0 = v_{xy}$ and extend this vector to an pseudo-orthonormal basis $(e_0, \ldots, e_4)$ of K_{xy}. We then choose the spinor basis of Corollary 3.4. Similarly, at the point y we set $e_0 = v_{yx}$ and extend to a basis $(e_0, \ldots, e_4)$ of K_{yx}, which we again represent in the form (3.9). Since v_{xy} and v_{yx} are sign operators, the inner products $\prec . | v_{xy} \,. \succ_x$ and $\prec . | v_{yx} \,. \succ_y$ are positive definite, and thus these sign operators even have the representation (3.14). In the chosen matrix representations, the condition (3.31) means that $D_{x,y}$ is block diagonal. Moreover, in view of Lemma 3.8, the conditions (3.30) imply that $D_{x,y}$ must be of the form

$$D_{x,y} = \mathrm{e}^{\mathrm{i}\vartheta_{xy}} \begin{pmatrix} D^+_{x,y} & 0 \\ 0 & D^-_{x,y} \end{pmatrix} \quad \text{with } \vartheta_{xy} \in \mathbb{R} \text{ and } D^\pm_{x,y} \in \mathrm{SU}(2). \tag{3.32}$$

Next, as observed in (3.27) and (3.28), $P(y,x)$ maps the eigenspaces of v_{xy} to the corresponding eigenspaces of v_{yx}. Thus in our spinor bases, the kernel of the fermionic operator has the form

$$P(x,y) = \begin{pmatrix} P^+_{x,y} & 0 \\ 0 & P^-_{x,y} \end{pmatrix} \quad \text{and} \quad P(y,x) = \begin{pmatrix} P^+_{y,x} & 0 \\ 0 & P^-_{y,x} \end{pmatrix}, \tag{3.33}$$

where $P^\pm_{x,y}$ are invertible 2×2 matrices and $P^\pm_{y,x} = (P^\pm_{x,y})^*$ (and the star simply denotes complex conjugation and transposition).

At this point, a polar decomposition of $P^\pm_{x,y}$ is helpful. Recall that any invertible 2×2-matrix X can be uniquely decomposed in the form $X = RV$ with a positive matrix R and a unitary matrix $V \in \mathrm{U}(2)$ (more precisely, one sets $R = \sqrt{X^* X}$ and $V = R^{-1} X$). Since in (3.32) we are working with SU(2)-matrices, it is useful to extract from V a phase factor. Thus we write

$$P^s(x,y) = \mathrm{e}^{\mathrm{i}\vartheta^s_{xy}} R^s_{xy}\, V^s_{x,y} \tag{3.34}$$

with $\vartheta^s_{xy} \in \mathbb{R} \bmod 2\pi$, $R^s_{xy} > 0$ and $V^s_{x,y} \in \mathrm{SU}(2)$, where $s \in \{+,-\}$. Comparing (3.34) with (3.32), the natural ansatz for the spin connection is

$$D_{x,y} = \mathrm{e}^{\frac{\mathrm{i}}{2}(\vartheta^+_{xy} + \vartheta^-_{xy})} \begin{pmatrix} V^+_{x,y} & 0 \\ 0 & V^-_{x,y} \end{pmatrix}. \tag{3.35}$$

The construction so far suffers from the problem that the SU(2)-matrices $V^s_{x,y}$ in the polar decomposition (3.34) are determined only up to a sign, so that there still is the freedom to perform the transformations

$$V^s_{x,y} \to -V^s_{x,y}, \quad \vartheta^s_{xy} \to \vartheta^s_{xy} + \pi. \tag{3.36}$$

If we flip the signs of both $V^+_{x,y}$ and $V^-_{x,y}$, then the factor $\mathrm{e}^{\frac{\mathrm{i}}{2}(\vartheta^+_{x,y} + \vartheta^-_{x,y})}$ in (3.35) also flips its sign, so that $D_{x,y}$ remains unchanged. The relative sign of $V^+_{x,y}$ and $V^-_{x,y}$, however, does effect the ansatz (3.35). In order to fix the relative signs, we need the following assumption, whose significance will be clarified in Section 3.5 below.

Definition 3.16. The space-time points x and y are said to be **time-directed** if the phases $\vartheta^\pm_{xy}$ in (3.34) satisfy the condition

$$\vartheta^+_{xy} - \vartheta^-_{xy} \notin \frac{\mathbb{Z}\pi}{2} \, .$$

Then we can fix the relative signs by imposing that

$$\vartheta^+_{xy} - \vartheta^-_{xy} \in \left(-\frac{3\pi}{2}, -\pi \right) \cup \left(\pi, \frac{3\pi}{2} \right) \tag{3.37}$$

(the reason for this convention will become clear in Section 4.2).

We next consider the behavior under the transformations of bases. At the point x, the pseudo-orthonormal basis $(v_{xy} = e_0, e_1, \ldots, e_4)$ of K_{xy} is unique up to SO(4)-transformations of the basis vectors $e_1, \ldots, e_4$. According to Lemma 3.8, this gives rise to a $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$-freedom to transform the spin basis $\mathfrak{f}_1, \ldots, \mathfrak{f}_4$ (where U(1) corresponds to a phase transformation). At the point y, we can independently perform $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$-transformations of the spin basis. This gives rise to the freedom to transform the kernel of the fermionic operator by

$$P(x,y) \to U_x \, P(x,y) \, U_y^{-1} \quad \text{and} \quad P(y,x) \to U_y \, P(y,x) \, U_x^{-1}, \tag{3.38}$$

where

$$U_z = \mathrm{e}^{\mathrm{i}\beta_z} \begin{pmatrix} U_z^+ & 0 \\ 0 & U_z^- \end{pmatrix} \quad \text{with } \beta \in \mathbb{R} \text{ and } U_z^\pm \in \mathrm{SU}(2). \tag{3.39}$$

The phase factors $e^{\pm i\beta_z}$ shift the angles ϑ^+_{xy} and ϑ^-_{xy} by the same value, so that the difference of these angles entering Definition 3.16 are not affected. The SU(2)-matrices U_z and U_z^{-1}, on the other hand, modify the polar decomposition (3.34) by

$$V^s_{x,y} \to U^s_x \, V^s_{x,y} \, (U^s_y)^{-1} \quad \text{and} \quad R^s_{xy} \to U^s_x \, R^s_{xy} \, (U^s_x)^{-1}.$$

The transformation law of the matrices $V^s_{x,y}$ ensures that the ansatz (3.35) is indeed independent of the choice of bases. We thus conclude that this ansatz indeed defines a spin connection.

The result of our construction is summarized as follows.

Definition 3.17. Two space-time points $x, y \in M$ are said to be **spin-connectable** if the following conditions hold:

(a) The points x and y are properly timelike separated (see Definition 3.14).
(b) The directional sign operator v_{xy} of A_{xy} is generically separated from the Euclidean sign operator s_x (see Definitions 3.15 and 3.11). Likewise, v_{yx} is generically separated from s_y.
(c) The points x and y are time-directed (see Definition 3.16).

The **spin connection** D is the set of spin-connectable pairs (x, y) together with the corresponding maps $D_{x,y} \in \mathrm{U}(S_y, S_x)$ which are uniquely determined by (3.35) and (3.37),

$$D = \{((x,y), D_{x,y}) \text{ with } x, y \text{ spin-connectable}\}\,.$$

We conclude this section by compiling properties of the spin connection and by characterizing it in a basis independent way. To this end, we want to rewrite (3.34) in a way which does not refer to our particular bases. First, using (3.33) and (3.34), we obtain for the closed chain

$$A_{xy} = P(x,y)\, P(x,y)^* = \begin{pmatrix} (R^+_{xy})^2 & 0 \\ 0 & (R^-_{xy})^2 \end{pmatrix}. \tag{3.40}$$

Taking the inverse and multiplying by $P(x, y)$, the operators $R^\pm_{xy}$ drop out,

$$A_{xy}^{-\frac{1}{2}}\, P(x,y) = \begin{pmatrix} e^{i\vartheta^+_{xy}}\, V^+_{x,y} & 0 \\ 0 & e^{i\vartheta^-_{xy}}\, V^-_{x,y} \end{pmatrix}.$$

Except for the relative phases on the diagonal, this coincides precisely with the definition of the spin connection (3.35). Since in our chosen bases, the operator v_{xy} has the matrix representation (3.14), this relative phase can be removed by multiplying with the operator $\exp(i\varphi_{xy} v_{xy})$, where

$$\varphi_{xy} = -\frac{1}{2}\left(\vartheta^+_{xy} - \vartheta^-_{xy}\right). \tag{3.41}$$

Thus we can write the spin connection in the basis independent form

$$D_{x,y} = e^{i\varphi_{xy}\, v_{xy}}\, A_{xy}^{-\frac{1}{2}}\, P(x,y). \tag{3.42}$$

Obviously, the value of φ_{xy} in (3.41) is also determined without referring to our bases by using the condition (3.30). This makes it possible to reformulate our previous results in a manifestly invariant way.

Lemma 3.18. *There is $\varphi_{xy} \in \mathbb{R}$ such that $D_{x,y}$ defined by* (3.42) *satisfies the conditions* (3.29) *and*

$$(D_{x,y})^{-1} \, K_{xy} \, D_{x,y} = K_{yx}. \tag{3.43}$$

The phase φ_{xy} is determined up to multiples of $\frac{\pi}{2}$.

Definition 3.19. The space-time points x and y are said to be **time-directed** if the phase φ_{xy} in (3.42) satisfying (3.43) is not a multiple of $\frac{\pi}{4}$.

We then uniquely determine φ_{xy} by the condition

$$\varphi_{xy} \in \left(-\frac{3\pi}{4}, -\frac{\pi}{2} \right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4} \right). \tag{3.44}$$

Theorem 3.20 (characterization of the spin connection). *Assume that the points x, y are spin-connectable (see Definitions* 3.17 *and* 3.19*). Then the spin connection of Definition* 3.17 *is uniquely characterized by the following conditions:*

(i) *$D_{x,y}$ is of the form* (3.42) *with φ_{xy} in the range* (3.44)*.*
(ii) *The relation* (3.30) *holds,*

$$D_{y,x} \, K_{xy} \, D_{x,y} = K_{yx}.$$

The spin connection has the properties

$$D_{y,x} = (D_{x,y})^{-1} = (D_{x,y})^*, \tag{3.45}$$

$$A_{xy} = D_{x,y} \, A_{yx} \, D_{y,x}, \tag{3.46}$$

$$v_{xy} = D_{x,y} \, v_{yx} \, D_{y,x}. \tag{3.47}$$

Proof. The previous constructions show that the conditions (i) and (ii) give rise to a unique unitary mapping $D_{x,y} \in \mathrm{U}(S_y, S_x)$, which coincides with the spin connection of Definition 3.17. Since φ_{xy} is uniquely fixed, it follows that

$$\varphi_{yx} = -\varphi_{xy},$$

and thus it is obvious from (3.42) that the identity $D_{y,x} = D_{x,y}^{-1}$ holds.

The identity (3.46) follows from the calculation

$$\begin{aligned} D_{x,y}\, A_{yx} &= \Big(\mathrm{e}^{\mathrm{i}\varphi_{xy}\, v_{xy}}\, A_{xy}^{-\frac{1}{2}}\, P(x,y) \Big) A_{yx} = \mathrm{e}^{\mathrm{i}\varphi_{xy}\, v_{xy}}\, A_{xy}^{-\frac{1}{2}}\, A_{xy}\, P(x,y) \\ &= A_{xy} \Big(\mathrm{e}^{\mathrm{i}\varphi_{xy}\, v_{xy}}\, A_{xy}^{-\frac{1}{2}}\, P(x,y) \Big) = A_{xy}\, D_{x,y}, \end{aligned}$$

where we applied (3.42) and used that the operators A_{xy} and v_{xy} commute.

The relations (3.46) and (3.45) show that the operators A_{xy} and A_{yx} are mapped to each other by the unitary transformation $D_{y,x}$. As a consequence, these operators have the same spectrum, and $D_{y,x}$ also maps the corresponding eigenspaces to each other. This implies (3.47) (note that this identity already appeared in our previous construction; see (3.31)). □

3.4 The induced metric connection, parity-preserving systems

The spin connection induces a connection on the corresponding tangent spaces, as we now explain. Suppose that x and y are two spin-connectable space-time points. According to Lemma 3.12, the signature operators s_x and v_{xy} distinguish two Clifford subspaces at x. One of these Clifford subspaces was already used in the previous section; we denoted it by $K_{xy} := K_x^{v_{xy},(s_x)}$ (see also Definition 3.13). Now we will also need the other Clifford subspace, which we denote by $K_x^{(y)} := K_x^{s_x,(v_{xy})}$. It is an element of $\mathcal{T}^{s_x}$ and can therefore be regarded as a representative of the tangent space. We denote the corresponding synchronization map by $U_{xy} = U^{v_{xy},s_x}$, i.e.,

$$K_{xy} = U_{xy}\, K_x^{(y)}\, U_{xy}^{-1}.$$

Similarly, at the point y we represent the tangent space by the Clifford subspace $K_y^{(x)} := K_y^{s_y,(v_{yx})} \in \mathcal{T}^{s_y}$ and denote the synchronization map by $U_{yx} = U^{v_{yx},s_y}$.

Suppose that a tangent vector $u_y \in T_y$ is given. We can regard u_y as a vector in $K_y^{(x)}$. By applying the synchronization map, we obtain a vector in K_{yx},

$$u_{yx} := U_{yx}\, u_y\, U_{yx}^{-1} \in K_{yx}. \tag{3.48}$$

According to Theorem 3.20 (ii), we can now "parallel transport" the vector to the Clifford subspace K_{xy},

$$u_{xy} := D_{x,y}\, u_{yx}\, D_{y,x} \in K_{xy}. \tag{3.49}$$

Finally, we apply the inverse of the synchronization map to obtain the vector

$$u_x := U_{xy}^{-1}\, u_{xy}\, U_{xy} \in K_x^{(y)}. \tag{3.50}$$

As $K_x^{(y)}$ is a representative of the tangent space T_x and all transformations were unitary, we obtain an isometry from T_y to T_x.

Definition 3.21. The isometry between the tangent spaces defined by

$$\nabla_{x,y} \;:\; T_y \to T_x \,,\; u_y \mapsto u_x$$

is referred to as the **metric connection** corresponding to the spin connection D.

By construction, the metric connection satisfies the relation

$$\nabla_{y,x} = (\nabla_{x,y})^{-1}.$$

We would like to introduce the notion that the metric connection preserves the spatial orientation. This is not possible in general, because in view of (3.19) the tangent spaces themselves have no spatial orientation. However, using the notions of Definition 3.10 we can introduce a spatial orientation under additional assumptions.

Definition 3.22. A causal fermion system of spin dimension two is said to be **parity preserving** if for every point $x \in M$, the set

$$\mathfrak{U}(x) := \{K_x^{(y)} \text{with } y \text{ spin-connectable to } x\}$$

is parity preserving (see Definition 3.10).

Provided that this condition holds, the identification maps $\phi^{s_x}_{\tilde{K},K}$ with K, $\tilde{K} \in \mathfrak{U}(x)$ can be uniquely fixed by choosing them in the form (3.18) with U according to (3.20). Denoting the corresponding equivalence relation by $\overset{\oplus}{\sim}$, we introduce the **space-oriented tangent space** $T_x^\oplus$ by

$$T_x^\oplus = \mathfrak{U}(x) / \overset{\oplus}{\sim} .$$

Considering the Clifford subspaces $K_y^{(x)}$ and $K_x^{(y)}$ as representatives of $T_y^\oplus$ and $T_x^\oplus$, respectively, the above construction (3.48) to (3.50) gives rise to the **parity preserving metric connection**

$$\nabla_{x,y} \;:\; T_y^\oplus \to T_x^\oplus \,,\; u_y \mapsto u_x.$$

3.5 A distinguished direction of time

For spin-connectable points we can distinguish a direction of time.

Definition 3.23 (time orientation of space-time). Assume that the points $x, y \in M$ are spin-connectable. We say that y lies in the **future** of x if the phase φ_{xy} as defined by (3.42) and (3.44) is positive. Otherwise, y is said to lie in the **past** of x.

We denote the points in the future of x by $\mathcal{I}^\vee(x)$. Likewise, the points in the past of y are denoted by $\mathcal{I}^\wedge(x)$. We also introduce the set

$$\mathcal{I}(x) = \mathcal{I}^\vee(x) \cup \mathcal{I}^\wedge(x);$$

it consists of all points which are spin-connectable to x.

Taking the adjoint of (3.42) and using that $D_{x,y}^* = D_{y,x}$, one sees that $\varphi_{xy} = -\varphi_{yx}$. Hence, y lies in the future of x if and only if x lies in the past of y. Moreover, as all the conditions in Definition 3.17 are stable under perturbations of y and the phase φ_{xy} is continuous in y, we know that $\mathcal{I}^\vee(x)$ and $\mathcal{I}^\wedge(x)$ are open subsets of M.

On the tangent space, we can also introduce the notions of past and future, albeit in a completely different way. We first give the definition and explain afterwards how the different notions are related. Recall that, choosing a representative $K \in \mathcal{T}^{s_x}$ of the tangent space T_x, every vector $u \in T_x$ can be regarded as a vector in the Clifford subspace K. According to Lemma 3.2, the bilinear form $\prec .|u.\succ_x$ on S_x is definite if $\langle u, u\rangle > 0$. Using these facts, the following definition is independent of the choice of the representatives.

Definition 3.24 (time orientation of the tangent space).

$$\text{A vector } u \in T_x \text{ is called } \quad \begin{cases} \text{timelike} & \text{if } \langle u, u\rangle > 0, \\ \text{spacelike} & \text{if } \langle u, u\rangle < 0, \\ \text{lightlike} & \text{if } \langle u, u\rangle = 0. \end{cases}$$

We denote the timelike vectors by $I_x \subset T_x$.

$$\text{A vector } u \in I_x \text{ is called } \quad \begin{cases} \text{future-directed} & \text{if } \prec .|u\, .\succ_x > 0, \\ \text{past-directed} & \text{if } \prec .|u\, .\succ_x < 0. \end{cases}$$

We denote the future-directed and past-directed vectors by $I_x^\vee$ and $I_x^\wedge$, respectively.

In order to clarify the connection between these definitions, we now construct a mapping which to every point $y \in \mathcal{I}(x)$ associates a timelike tangent vector $y_x \in I_x$, such that the time orientation is preserved. To this end, for given $y \in \mathcal{I}(x)$ we consider the operator

$$L_{xy} = -iD_{x,y}\, P(y,x) \;:\; S_x \to S_x$$

and symmetrize it,

$$M_{xy} = \frac{1}{2} \left(L_{xy} + L_{xy}^* \right) \in \mathrm{Symm}(S_x).$$

The square of this operator need not be a multiple of the identity, and therefore it cannot be regarded as a vector of a Clifford subspace. But we can take the orthogonal projection $\mathrm{pr}_{K_{xy}}$ of M_{xy} onto the Clifford subspace $K_{xy} \subset \mathrm{Symm}(S_x)$ (with respect to the inner product $\langle .,. \rangle$), giving us a vector in K_{xy}. Just as in (3.50), we can apply the synchronization map to obtain a vector in $K_x^{(y)}$, which then represents a vector of the tangent space T_x. We denote this vector by y_x and refer to it as the **time-directed tangent vector** of y in T_x,

$$y_x = U_{xy}^{-1}\, \mathrm{pr}_{K_{xy}}(M_{xy})\, U_{xy} \in K_x^{(y)}. \tag{3.51}$$

Moreover, it is useful to introduce the **directional tangent vector** $\hat{y}_x$ of y in T_x by synchronizing the directional sign operator v_{xy},

$$\hat{y}_x := U_{xy}^{-1}\, v_{xy}\, U_{xy} \in K_x^{(y)}. \tag{3.52}$$

By definition of the sign operator, the inner product $\prec . | v_{xy} . \succ_x$ is positive definite. Since the synchronization map is unitary, it follows that the vector $\hat{y}_x$ is a future-directed unit vector in T_x.

Proposition 3.25. *For any $y \in \mathcal{I}(x)$, the time-directed tangent vector of y in T_x is timelike, $y_x \in I_x$. Moreover, the time orientation of the space-time points $x, y \in M$ (see Definition* 3.23*) agrees with the time orientation of $y_x \in T_x$ (see Definition* 3.24*),*

$$y \in \mathcal{I}^\vee(x) \Longleftrightarrow y_x \in I_x^\vee \quad \text{and} \quad y \in \mathcal{I}^\wedge(x) \Longleftrightarrow y_x \in I_x^\wedge.$$

Moreover,

$$y_x = \frac{1}{4}\, \sin(\varphi_{xy})\, \mathrm{Tr}\left(A_{xy}^{\frac{1}{2}} \right) \hat{y}_x. \tag{3.53}$$

Proof. From (3.42) one sees that

$$L_{xy} = -i e^{i\varphi_{xy} v_{xy}} A_{xy}^{\frac{1}{2}} \quad \text{and} \quad M_{xy} = \sin(\varphi_{xy})\, v_{xy}\, A_{xy}^{\frac{1}{2}}.$$

We again choose the pseudo-orthonormal basis $(e_0 = v_{xy}, e_1, \ldots, e_4)$ of K_{xy} and the spinor basis of Corollary 3.4. Then v_{xy} has the form (3.14), whereas A_{xy} is block diagonal (3.40). Since the matrices $e_1, \ldots e_4$ vanish on the block diagonal, the operators $e_j M_{xy}$ are trace-free for $j = 1, \ldots, 4$. Hence, the projection of M_{xy} is proportional to v_{xy},

$$\mathrm{pr}_{K_{xy}}(M_{xy}) = \frac{1}{4} \sin(\varphi_{xy}) \,\mathrm{Tr}\left(A_{xy}^{\frac{1}{2}}\right) v_{xy}.$$

By synchronizing we obtain (3.53).

The trace in (3.53) is positive because the operator A_{xy} has a strictly positive spectrum (see Definition 3.14). Moreover, in view of (3.44) and Definition 3.23, the factor $\sin(\varphi_{xy})$ is positive if and only if y lies in the future of x. Since $\hat{y}_x$ is future-directed, we conclude that $y_x \in I_x^\vee$ if and only if $y \in \mathcal{I}^\vee(x)$. $\square$

3.6 Reduction of the spatial dimension

We now explain how to reduce the dimension of the tangent space to four, with the desired Lorentzian signature $(1, 3)$.

Definition 3.26. A causal fermion system of spin dimension two is called **chirally symmetric** if to every $x \in M$ we can associate a spacelike vector $u(x) \in T_x$ which is orthogonal to all directional tangent vectors,

$$\langle u(x), \hat{y}_x \rangle = 0 \quad \text{for all } y \in \mathcal{I}(x),$$

and is parallel with respect to the metric connection, i.e.,

$$u(x) = \nabla_{x,y}\, u(y) \quad \text{for all } y \in \mathcal{I}(x).$$

Definition 3.27. For a chirally symmetric fermion system, we introduce the **reduced tangent space** T_x^{red} by

$$T_x^{\mathrm{red}} = \langle u_x \rangle^\perp \subset T_x.$$

Clearly, the reduced tangent space has dimension four and signature $(1, 3)$. Moreover, the operator $\nabla_{x,y}$ maps the reduced tangent spaces isometrically to each other. The local operator $e_5 := -\mathrm{i}u/\sqrt{-u^2}$ takes the role of the **pseudoscalar matrix**.

3.7 Curvature and the splice maps

We now introduce the curvature of the metric connection and the spin connection and explain their relation. Since our formalism should include discrete space-times, we cannot in general work with an infinitesimal parallel transport. Instead, we must take two space-time points and consider the spin or metric connection, which we defined in Sections 3.3 and 3.4 as mappings between the corresponding spin or tangent spaces. By composing such mappings, we can form the analog of the parallel transport along a polygonal line. Considering closed polygonal loops, we thus obtain the analog of a holonomy. Since on a manifold, the curvature at x is immediately obtained from the holonomy by considering the loops in a small neighborhood of x, this notion indeed generalizes the common notion of curvature to causal fermion systems.

We begin with the metric connection.

Definition 3.28. Suppose that three points $x, y, z \in M$ are pairwise spin-connectable. Then the **metric curvature** R is defined by

$$R(x,y,z) = \nabla_{x,y}\, \nabla_{y,z}\, \nabla_{z,x} \;:\; T_x \to T_x. \tag{3.54}$$

Let us analyze this notion, for simplicity for parity-preserving systems. According to (3.48)–(3.50), for a given tangent vector $u_y \in K^{(x)}_y$ we have

$$\nabla_{x,y} u_y = U u_y U^{-1} \in K^{(y)}_x \quad \text{with} \quad U = U_{xy}^{-1}\, D_{x,y}\, U_{yx}.$$

Composing with $\nabla_{z,x}$, we obtain

$$\nabla_{z,x} \nabla_{x,y} u_y = U u_y U^{-1} \in K^{(x)}_z$$

with

$$U = U_{zx}^{-1}\, D_{z,x} \left(U_{xz}\, U^{s_x}_{K^{(z)}_x, K^{(y)}_x}\, U_{xy}^{-1} \right) D_{x,y}\, U_{yx},$$

where $U^{s_x}_{K^{(z)}_x, K^{(y)}_x}$ is the unitary operator (3.20) of the identification map. We see that the composition of the metric connection can be written as the product of spin connections, joined by the product of unitary operators in the brackets, which synchronize and identify suitable Clifford extensions. We give this operator product a convenient name.

Definition 3.29. The unitary mapping

$$U_x^{(z|y)} = U_{xz}\, U^{s_x}_{K_{xz},K_{xy}}\, U_{xy}^{-1} \in \mathrm{U}(S_x)$$

is referred to as the **splice map**. A causal fermion system of spin dimension two is called **Clifford-parallel** if all splice maps are trivial.

Using the splice maps, the metric curvature can be written as

$$R(x,y,z) \;:\; K_x^{(z)} \to K_x^{(z)}\,,\; u_x \mapsto U u_x U^{-1},$$

where the unitary mapping U is given by

$$U = U_{xz}^{-1}\;\, U_x^{(z|y)}\, D_{x,y}\, U_y^{(x|z)}\, D_{y,z}\, U_z^{(y|x)}\, D_{z,x}\, U_{xz}. \tag{3.55}$$

Thus two factors of the spin connection are always joined by an intermediate splice map.

We now introduce the curvature of the spin connection. The most obvious way is to simply replace the metric connection in (3.54) by the spin connection. On the other hand, the formula (3.55) suggests that it might be a good idea to insert splice maps. As it is a-priori not clear which method is preferable, we define both alternatives.

Definition 3.30. Suppose that three points $x, y, z \in M$ are pairwise spin-connectable. Then the **unspliced spin curvature** $\mathfrak{R}^{\mathrm{us}}$ is defined by

$$\mathfrak{R}^{\mathrm{us}}(x,y,z) = D_{x,y}\, D_{y,z}\, D_{z,x} \;:\; S_x \to S_x. \tag{3.56}$$

The (spliced) **spin curvature** is introduced by

$$\mathfrak{R}(x,y,z) = U_x^{(z|y)}\, D_{x,y}\, U_y^{(x|z)}\, D_{y,z}\, U_z^{(y|x)}\, D_{z,x} \;:\; S_x \to S_x. \tag{3.57}$$

Clearly, for Clifford-parallel systems, the spliced and unspliced spin curvatures coincide. However, if the causal fermion system is not Clifford-parallel, the situation is more involved. The spliced spin curvature and the metric curvature are compatible in the sense that, after unitarily transforming to the Clifford subspace K_{xz}, the following identity holds,

$$U_{xz}\, R(x,y,z)\, U_{xz}^{-1} \;:\; K_{xz} \to K_{xz}\,,\; v \mapsto \mathfrak{R}(x,y,z)\, v\, \mathfrak{R}(x,y,z)^*.$$

Thus the metric curvature can be regarded as "the square of the spliced spin curvature."

We remark that the systems considered in Section 4 will all be Clifford parallel. In the examples in Section 5, however, the systems will *not* be Clifford parallel. In these examples, we shall see that it is indeed preferable to work with the spliced spin curvature (for a detailed explanation see Section 5.4).

We conclude this section with a construction which will be useful in Section 5. In the causal fermion systems considered in these sections, at every space-time point there is a distinguished representative of the tangent space, making it possible to introduce the following notion.

Definition 3.31. We denote the set of five-dimensional subspaces of $\text{Symm}(\mathcal{H})$ by $\mathfrak{S}_5(\mathcal{H})$; it carries the topology induced by the operator norm (2.1). A continuous mapping $\mathfrak{K}$ which to very space-time point associates a representative of the corresponding tangent space,

$$\mathfrak{K} \,:\, M \to \mathfrak{S}_5(\mathcal{H}) \quad \text{with} \quad \mathfrak{K}(x) \in \mathcal{T}^{s_x} \quad \text{for all } x \in M,$$

is referred to as a **representation map** of the tangent spaces. The system $(\mathcal{H}, \mathcal{F}, \rho, \mathfrak{K})$ is referred to as a causal fermion system with **distinguished representatives of the tangent spaces**.

If we have distinguished representatives of the tangent spaces, the spin connection can be combined with synchronization and identification maps such that forming compositions of this combination always gives rise to intermediate splice maps.

Definition 3.32. Suppose that our fermion system is parity preserving and has distinguished representatives of the tangent spaces. Introducing the splice maps $U^{|.)}_{.}$ and $U^{(.|}_{.}$ by

$$U^{|y)}_x = U^{s_x}_{\mathfrak{K}(x), K^{(y)}_x}\, U^{-1}_{xy} \quad \text{and} \quad U^{(y|}_x = \big(U^{|y)}_x\big)^* = U_{xy}\, U^{s_x}_{K^{(y)}_x, \mathfrak{K}(x)},$$

we define the **spliced spin connection** $D_{(.|.)}$ by

$$D_{(x,y)} = U^{|y)}_x\, D_{x,y}\, U^{(x|}_y \,:\, S_y \to S_x. \tag{3.58}$$

Our notation harmonizes with Definition 3.29 in that

$$U^{(y|}_x\, U^{|z)}_x = U^{(y|z)}_x. \tag{3.59}$$

Forming compositions and comparing with Definition 3.29, one readily finds that

$$D_{(x,y)}\, D_{(y,z)} = U^{|y)}_x\, D_{x,y}\, U^{(x|z)}_y\, D_{y,z}\, U^{(x|}_z.$$

Proceeding iteratively, one sees that the spin curvature (3.57) can be represented by

$$\mathfrak{R}(x,y,z) = V\, D_{(x,y)}\, D_{(y,z)}\, D_{(z,x)}\, V^* \quad \text{with} \quad V = U_x^{(z|}.$$

Thus up to the unitary transformation V, the spin curvature coincides with the holonomy of the spliced spin connection.

3.8 Causal sets and causal neighborhoods

The relation "lies in the future of" introduced in Definition 3.23 reminds of the partial ordering on a causal set. In order to explain the connection, we first recall the definition of a causal set (for details, see for example [7]).

Definition 3.33. A set C with a partial order relation $\prec$ is a **causal set** if the following conditions hold:

(i) Irreflexivity: For all $x \in C$, we have $x \not\prec x$.
(ii) Transitivity: For all $x, y, z \in C$, we have $x \prec y$ and $y \prec z$ implies $x \prec z$.
(iii) Local finiteness: For all $x, z \in C$, the set $\{y \in C \text{ with } x \prec y \prec z\}$ is finite.

Our relation "lies in the future of" agrees with (i) because the sign operators s_x and v_{xx} coincide, and therefore every space-time point x is not spin-connectable to itself. The condition (iii) seems an appropriate assumption for causal fermion systems in discrete space-time (in particular, it is trivial if M is a finite set). In the setting when space-time is a general measure space (M, μ), it is natural to replace (iii) by the condition that the set $\{y \in C \text{ with } x \prec y \prec z\}$ should have finite measure. The main difference between our setting and a causal set is that the relation "lies in the future of" is in general not transitive, so that (ii) is violated. However, it seems reasonable to weaken (ii) by a local condition of transitivity. We now give a possible definition.

Definition 3.34. A subset $U \subset M$ is called **future-transitive** if for all pairwise spin-connectable points $x, y, z \in U$ the following implication holds:

$$y \in \mathcal{I}^\vee(x) \text{ and } z \in \mathcal{I}^\vee(y) \Longrightarrow z \in \mathcal{I}^\vee(x).$$

A causal fermion system of spin dimension two is called **locally future-transitive** if every point $x \in M$ has a neighborhood U, which is future-transitive.

This definition ensures that M locally includes the structure of a causal set. As we shall see in the examples in Sections 4 and 5, Dirac sea configurations without regularization in Minkowski space or on globally hyperbolic Lorentzian manifolds are indeed locally future-transitive. However, it still needs to be investigated if Definition 3.34 applies to quantum space-times of physical interest.

4 Example: the regularized Dirac sea vacuum

As a first example, we now consider Dirac spinors in Minkowski space. Taking $\mathcal{H}$ as the space of all negative-energy solutions of the Dirac equation, we construct a corresponding causal fermion system. We show that the notions introduced in Section 3 give back the usual causal and geometric structures of Minkowski space.

We first recall the basics and fix our notation (for details see for example [6] or [12, Chapter 1]). Let $(M, \langle .,. \rangle)$ be Minkowski space (with the signature convention $(+ - - -)$) and $d\mu$ the standard volume measure (thus $d\mu = d^4x$ in a reference frame $x = (x^0, \ldots, x^3)$). Naturally identifying the spinor spaces at different space-time points and denoting them by $V = \mathbb{C}^4$, we write the free Dirac equation for particles of mass $m > 0$ as

$$(\mathrm{i}\partial\!\!\!/ - m)\, \psi := (\mathrm{i}\gamma^k \partial_k - m)\, \psi = 0, \tag{4.1}$$

where γ^k are the Dirac matrices in the Dirac representation, and $\psi : M \to V$ are four-component complex Dirac spinors. The Dirac spinors are endowed with an inner product of signature $(2, 2)$, which is usually written as $\overline{\psi}\phi$, where $\overline{\psi} = \psi^\dagger \gamma^0$ is the adjoint spinor. For notational consistency, we denote this inner product on V by $\prec .|. \succ$. The free Dirac equation has plane wave solutions, which we denote by $\psi_{\vec{k}a\pm}$ with $\vec{k} \in \mathbb{R}^3$ and $a \in \{1, 2\}$. They have the form

$$\psi_{\vec{k}a\pm}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\, \mathrm{e}^{\mp \mathrm{i}\omega t + \mathrm{i}\vec{k}\vec{x}}\, \chi_{\vec{k}a\pm}, \tag{4.2}$$

where $x = (t, \vec{x})$ and $\omega := \sqrt{|\vec{k}|^2 + m^2}$. Here the spinor $\chi_{\vec{k}a\pm}$ is a solution of the algebraic equation

$$(k\!\!\!/ - m)\chi_{\vec{k}a\pm} = 0, \tag{4.3}$$

where $\not k = k^j \gamma_j$ and $k = (\pm\omega, \vec{k})$. Using the normalization convention

$$\prec \chi_{\vec{k}a\pm} | \chi_{\vec{k}a'\pm} \succ = \pm \delta_{a,a'},$$

the projector onto the two-dimensional solution space of (4.3) can be written as

$$\frac{\not k + m}{2m} = \pm \sum_{a=1,2} |\chi_{\vec{k}a\pm} \succ \prec \chi_{\vec{k}a\pm}|. \tag{4.4}$$

The frequency $\pm\omega$ of the plane wave (4.2) is the energy of the solution. More generally, by a negative-energy solution ψ of the Dirac equation we mean a superposition of plane wave solutions of negative energy,

$$\psi(x) = \sum_{a=1,2} \int d^3k \, g_a(\vec{k}) \, \psi_{\vec{k}a-}(x). \tag{4.5}$$

Dirac introduced the concept that in the vacuum all negative-energy states should be occupied forming the so-called Dirac sea. Following this concept, we want to introduce the Hilbert space $(\mathcal{H}, \langle .,. \rangle_{\mathcal{H}})$ as the space of all negative-energy solutions, equipped with the usual scalar product obtained by integrating the probability density

$$\langle \psi | \phi \rangle_{\mathcal{H}} = 2\pi \int_{t=\text{const}} \prec \psi(t, \vec{x}) | \gamma^0 \phi(t, \vec{x}) \succ d\vec{x}. \tag{4.6}$$

Note that the plane-wave solutions $\psi_{\vec{k}a-}$ cannot be considered as vectors in $\mathcal{H}$, because the normalization integral (4.6) diverges. However, for the superposition (4.5), the normalization integral is certainly finite for test functions $g_a(\vec{k}) \in C_0^\infty(\mathbb{R}^3)$, making it possible to define $(\mathcal{H}, \langle .,. \rangle_{\mathcal{H}})$ as the completion of such wave functions. Then due to current conservation, the integral in (4.6) is time independent. For the plane-wave solutions, one can still make sense of the normalization integral in the distributional sense. Namely, a short computation gives

$$\langle \psi_{\vec{k}a-} | \psi_{\vec{k}'a'-} \rangle_{\mathcal{H}} = \frac{2\pi\omega}{m} \, \delta_{a,a'} \, \delta^3(\vec{k} - \vec{k}'). \tag{4.7}$$

The completeness of the plane-wave solutions can be expressed by the Plancherel formula

$$\psi(x) = \frac{m}{\pi} \sum_{a=1,2} \int \frac{d^3k}{2\omega} \, \psi_{\vec{k}a-}(x) \, \langle \psi_{\vec{k}a-} | \psi \rangle \quad \text{for all } \psi \in \mathcal{H}. \tag{4.8}$$

4.1 Construction of the causal fermion system

In order to construct a causal fermion system of spin dimension two, to every $x \in M$ we want to associate a self-adjoint operator $F(x) \in L(\mathcal{H})$, having at most two positive and at most two negative eigenvalues. By identifying x with $F(x)$, we then get into the setting of Definition 2.1. The idea is to define $F(x)$ as an operator which describes the correlations of the wave functions at the point x,

$$\langle \psi | F(x) \phi \rangle_{\mathcal{H}} = - \prec \psi(x) | \phi(x) \succ . \tag{4.9}$$

As the spin scalar product has signature $(2,2)$, this ansatz incorporates that $F(x)$ should be a self-adjoint operator with at most two positive and at most two negative eigenvalues. Using the completeness relation (4.8), $F(x)$ can be written in the explicit form

$$F(x)\phi = -\frac{m^2}{\pi^2} \sum_{a,a'=1,2} \int \frac{d^3k}{2\omega} \int \frac{d^3k'}{2\omega'} \psi_{\vec{k}a-} \prec \psi_{\vec{k}a-}(x) | \psi_{\vec{k}'a'-}(x) \succ \langle \psi_{\vec{k}'a'-} | \phi \rangle_{\mathcal{H}}. \tag{4.10}$$

Unfortunately, this simple method does not give rise to a well-defined operator $F(x)$. This is obvious in (4.9) because the wave functions $\psi, \phi \in \mathcal{H}$ are in general not continuous and could even have a singularity at x. Alternatively, in (4.10) the momentum integrals will in general diverge. This explains why we must introduce an ultraviolet regularization. We do it in the simplest possible way by inserting convergence generating factors,

$$\begin{aligned} F^\varepsilon(x)\,\phi := -\frac{m^2}{\pi^2} \sum_{a,a'=1,2} \int \frac{d^3k}{2\omega}\, \mathrm{e}^{-\frac{\varepsilon\omega}{2}} \int \frac{d^3k'}{2\omega'}\, \mathrm{e}^{-\frac{\varepsilon\omega'}{2}} \\ \times\, \psi_{\vec{k}a-} \prec \psi_{\vec{k}a-}(x) | \psi_{\vec{k}'a'-}(x) \succ \langle \psi_{\vec{k}'a'-} | \phi \rangle_{\mathcal{H}}, \end{aligned} \tag{4.11}$$

where the parameter $\varepsilon > 0$ is the length scale of the regularization. Note that this regularization is spherically symmetric, but the Lorentz invariance is broken. Moreover, the operator $F^\varepsilon(x)$ is no longer a local operator, meaning that space-time is "smeared out" on the scale ε.

In order to show that F^ε defines a causal fermion system, we need to compute the eigenvalues of $F^\varepsilon(x)$. To this end, it is helpful to write F^ε similar to a Gram matrix as

$$F^\varepsilon(x) = -\iota^\varepsilon_x\, (\iota^\varepsilon_x)^*, \tag{4.12}$$

where ι_x is the operator

$$\iota_x^\varepsilon \,:\, V \rightarrow \mathcal{H}\,,\; u \mapsto -\frac{m}{\pi} \sum_{a=1,2} \int \frac{d^3k}{2\omega}\, e^{-\frac{\varepsilon\omega}{2}}\, \psi_{\vec{k}a-} \prec \psi_{\vec{k}a-}(x)|u\succ, \tag{4.13}$$

and the star denotes the adjoint with respect to the corresponding inner products $\prec .|.\succ$ and $\langle .,.\rangle_{\mathcal{H}}$. From this decomposition, one sees right away that $F^\varepsilon(x)$ has at most two positive and at most two negative eigenvalues. Moreover, these eigenvalues coincide with those of the operator $-(\iota_x^\varepsilon)^*\iota_x^\varepsilon : V \rightarrow V$, which can be computed as follows:

$$\begin{aligned}
-\,(\iota_x^\varepsilon)^*\iota_x^\varepsilon\, u &= -\frac{m^2}{\pi^2} \sum_{a,a'=1,2} \iint \frac{d^3k\, d^3k'}{4\omega\omega'}\, e^{-\frac{\varepsilon(\omega+\omega')}{2}} \\
&\quad \times \psi_{\vec{k}a-}(x)\, \langle \psi_{\vec{k}a-} | \psi_{\vec{k}'a'-}\rangle_{\mathcal{H}} \prec \psi_{\vec{k}'a'-}(x)|u\succ \\
&\overset{(4.7)}{=} -\frac{m}{\pi} \sum_{a=1,2} \int \frac{d^3k}{2\omega}\, e^{-\varepsilon\omega}\, \psi_{\vec{k}a-}(x) \prec \psi_{\vec{k}a-}(x)|u\succ \qquad (4.14)\\
&\overset{(4.4)}{=} \frac{m}{\pi} \int \frac{d^3k}{2\omega}\, e^{-\varepsilon\omega}\, \frac{\not{k}+m}{2m}\, u = \frac{m}{\pi} \int \frac{d^3k}{2\omega}\, e^{-\varepsilon\omega}\, \frac{-\omega\gamma^0+m}{2m}\, u, \qquad (4.15)
\end{aligned}$$

where in the last step we used the spherical symmetry.

Proposition 4.1. *For any $\varepsilon > 0$, the operator $F^\varepsilon(x) : \mathcal{H} \rightarrow \mathcal{H}$ has rank four and has two positive and two negative eigenvalues. The mapping*

$$F \,:\, M \rightarrow \mathcal{F}\,,\; x \mapsto F^\varepsilon(x)$$

is injective. Identifying x with $F^\varepsilon(x)$ and introducing the measure $\rho^\varepsilon = F^\varepsilon_* \mu$ on $\mathcal{F}$ as the push-forward (2.2)*, the resulting tupel $(\mathcal{H}, \mathcal{F}, \rho^\varepsilon)$ is a causal fermion system of spin dimension two. Every space-time point is regular (see Definition* 2.3*).*

More specifically, the non-trivial eigenvalues $\nu_1, \ldots, \nu_4$ of the operator $F^\varepsilon(x)$ are

$$\begin{aligned}
\nu_1^\varepsilon = \nu_2^\varepsilon &= \int \frac{d^3k}{4\pi\omega}\, e^{-\varepsilon\omega}\, (-\omega+m) < 0 \\
\nu_3^\varepsilon = \nu_4^\varepsilon &= \int \frac{d^3k}{4\pi\omega}\, e^{-\varepsilon\omega}\, (\omega+m) > 0.
\end{aligned}$$

The corresponding eigenvectors $\mathfrak{f}_1^\varepsilon, \ldots, \mathfrak{f}_4^\varepsilon$ *are given by*

$$\mathfrak{f}_\alpha^\varepsilon(x) = \frac{1}{\nu_\alpha^\varepsilon}\, \iota_x^\varepsilon(\mathfrak{e}_\alpha), \tag{4.16}$$

where $(\mathfrak{e}_\alpha)$ *denotes the canonical basis of* $V = \mathbb{C}^4$.

Proof. It is obvious from (4.15) that $\mathfrak{e}_\alpha$ is an eigenvector basis of the operator $-(\iota_x^\varepsilon)^* \iota_x^\varepsilon$,

$$-(\iota_x^\varepsilon)^* \iota_x^\varepsilon\, \mathfrak{e}_\alpha = \nu_\alpha\, \mathfrak{e}_\alpha. \tag{4.17}$$

Next, the calculation

$$F^\varepsilon(x)\, (\iota_x^\varepsilon \mathfrak{e}_\alpha) = \iota_x^\varepsilon \left(-(\iota_x^\varepsilon)^* \iota_x^\varepsilon \right) \mathfrak{e}_\alpha = \nu_\alpha^\varepsilon\, (\iota_x^\varepsilon\, \mathfrak{e}_\alpha)$$

shows that the vectors $\mathfrak{f}_\alpha^\varepsilon$ are eigenvectors of $F^\varepsilon(x)$ corresponding to the same eigenvalues (our normalization convention will be explained in (4.18) below).

To prove the injectivity of F^ε, assume that $F^\varepsilon(x) = F^\varepsilon(y)$. We consider the expectation value $\langle \psi | (F^\varepsilon(x) - F^\varepsilon(y)) \phi \rangle_{\mathcal{H}}$. Since this expectation value vanishes for all ϕ and ψ, we conclude from (4.11) that

$$\prec\! \psi_{\vec{k}a-}(x) | \psi_{\vec{k}'a'-}(x) \!\succ \;=\; \prec\! \psi_{\vec{k}a-}(y) | \psi_{\vec{k}'a'-}(y) \!\succ$$

for all $a, a' \in \{1, 2\}$ and $\vec{k}, \vec{k}' \in \mathbb{R}^3$. Using (4.2), the left and right sides of this equation are plane waves of the form $\mathrm{e}^{\mathrm{i}(k-k')x}$ and $\mathrm{e}^{\mathrm{i}(k-k')y}$, respectively. We conclude that $x = y$. ☐

We now introduce for every $x \in M$ the spin space $(S_x^\varepsilon, \prec\!.|.\!\succ_x)$ by (2.3) and (2.4). By construction, the eigenvectors $\mathfrak{f}_\alpha^\varepsilon(x)$ in (4.16) form a basis of S_x^ε. Moreover, this basis is pseudo-orthonormal, as the following calculation shows:

$$\begin{aligned}
\prec\! \mathfrak{f}_\alpha^\varepsilon(x) | \mathfrak{f}_\beta^\varepsilon(x) \!\succ_x &= -\langle \mathfrak{f}_\alpha^\varepsilon(x) | F^\varepsilon(x)\, \mathfrak{f}_\beta^\varepsilon(x) \rangle_{\mathcal{H}} = -\nu_\beta^\varepsilon\, \langle \mathfrak{f}_\alpha^\varepsilon(x) | \mathfrak{f}_\beta^\varepsilon(x) \rangle_{\mathcal{H}} \\
&= -\frac{1}{\nu_\alpha^\varepsilon}\, \langle \iota_x^\varepsilon \mathfrak{e}_\alpha | \iota_x^\varepsilon \mathfrak{e}_\beta \rangle_{\mathcal{H}} = -\frac{1}{\nu_\alpha^\varepsilon} \prec\! \mathfrak{e}_\alpha | (\iota_x^\varepsilon)^* \iota_x^\varepsilon \mathfrak{e}_\beta \!\succ \\
&\overset{(4.17)}{=} \frac{\nu_\beta^\varepsilon}{\nu_\alpha^\varepsilon} \prec\! \mathfrak{e}_\alpha | \mathfrak{e}_\beta \!\succ \;=\; s_\alpha\, \delta_{\alpha\beta},
\end{aligned} \tag{4.18}$$

where we again used the notation of Corollary 3.4. It is useful to always identify the inner product space $(V, \prec\!.|.\!\succ)$ (and thus also the spinor space $S_x M$;

see before (4.1)) with the spin space $(S_x^\varepsilon, \prec .|.\succ_x)$ via the isometry $\mathfrak{J}_x^\varepsilon$ given by

$$\mathfrak{J}_x^\varepsilon \;:\; S_xM \simeq V \to S_x^\varepsilon\,,\; \mathfrak{e}_\alpha \mapsto \mathfrak{f}_\alpha^\varepsilon(x). \tag{4.19}$$

Then, as the $\mathfrak{f}^\varepsilon(x)$ form an eigenvector basis of $F^\varepsilon(x)$, the Euclidean operator takes the form

$$s_x = \gamma^0. \tag{4.20}$$

Moreover, we obtain a convenient matrix representation of the kernel of fermionic operator (2.7), which again under the identification of x with $F^\varepsilon(x)$ we now write as

$$P^\varepsilon(x,y) = \pi_{F^\varepsilon(x)}\, F^\varepsilon(y). \tag{4.21}$$

Lemma 4.2. *In the spinor basis* $(\mathfrak{e}_\alpha)$ *given by* (4.19)*, the kernel of the fermionic operator takes the form*

$$P^\varepsilon(x,y) = \int \frac{d^4k}{(2\pi)^4}\, \mathrm{e}^{-\varepsilon|k^0|}\, (\not{k}+m)\, \delta(k^2-m^2)\, \Theta(-k^0)\, \mathrm{e}^{-\mathrm{i}k(x-y)}. \tag{4.22}$$

Proof. Using (2.4), we find that $\prec .|\pi_x y.\succ_x = -\langle .|xy.\rangle_{\mathcal{H}}$. Thus, applying Proposition 4.1, we find

$$\begin{aligned}
&\prec \mathfrak{f}_\alpha^\varepsilon(x)|P^\varepsilon(x,y)\,\mathfrak{f}_\beta^\varepsilon(y)\succ_x \\
&\quad= -\langle \mathfrak{f}_\alpha^\varepsilon(x)|F^\varepsilon(x)\,F^\varepsilon(y)\,\mathfrak{f}_\beta^\varepsilon(y)\rangle_{\mathcal{H}} = -\langle F^\varepsilon(x)\,\mathfrak{f}_\alpha^\varepsilon(x)|F^\varepsilon(y)\,\mathfrak{f}_\beta^\varepsilon(y)\rangle_{\mathcal{H}} \\
&\quad= -\nu_\alpha^\varepsilon\nu_\beta^\varepsilon\,\langle \mathfrak{f}_\alpha^\varepsilon(x)|\mathfrak{f}_\beta^\varepsilon(y)\rangle_{\mathcal{H}} = -\langle \iota_x^\varepsilon\mathfrak{e}_\alpha|\iota_y^\varepsilon\mathfrak{e}_\beta\rangle_{\mathcal{H}} = -\prec \mathfrak{e}_\alpha|(\iota_x^\varepsilon)^*\iota_y^\varepsilon\mathfrak{e}_\beta\succ.
\end{aligned} \tag{4.23}$$

Identifying $\mathfrak{f}_\alpha^\varepsilon(x)$ and $\mathfrak{f}_\alpha^\varepsilon(y)$ with $\mathfrak{e}_\alpha$, we conclude that the kernel of the fermionic operator has the representation

$$\begin{aligned}
P^\varepsilon(x,y) &= -(\iota_x^\varepsilon)^*\iota_y^\varepsilon = -\frac{m}{\pi}\sum_{a=1,2}\int \frac{d^3k}{2\omega}\,\mathrm{e}^{-\varepsilon\omega}\,|\psi_{\vec{k}a-}(x)\succ\prec\psi_{\vec{k}a-}(y)| \\
&\overset{(4.2)}{=} -2m\sum_{a=1,2}\int \frac{d^3k}{2\omega\,(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}k(x-y)}\,\mathrm{e}^{-\varepsilon\omega}\,|\chi_{\vec{k}a-}(x)\succ\prec\chi_{\vec{k}a-}(y)| \\
&\overset{(4.4)}{=} \int \frac{d^3k}{2\omega\,(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}k(x-y)}\,\mathrm{e}^{-\varepsilon\omega}\,(\not{k}+m),
\end{aligned}$$

where again $k=(-\omega,\vec{k})$. Carrying out the k^0-integration in (4.22) gives the result. ∎

We point out that in the limit $\varepsilon \searrow 0$ when the regularization is removed, $P^\varepsilon(x,y)$ converges to the Lorentz invariant distribution

$$P(x,y) = \int \frac{d^4k}{(2\pi)^4}\, (\not{k}+m)\, \delta(k^2-m^2)\, \Theta(-k^0)\, \mathrm{e}^{-\mathrm{i}k(x-y)}. \tag{4.24}$$

This distribution is supported on the lower mass shell and thus describes the Dirac sea vacuum where all negative-energy solutions are occupied. It is the starting point of the fermionic projector approach (see [12, 18]).

With the spin space $(S_x^\varepsilon, \prec.|.\succ_x)$, the Euclidean operator (4.20) and the kernel of the fermionic operator (4.22), we have introduced all the objects needed for the constructions in Section 3. Before analyzing the resulting geometric structures in detail, we conclude this subsection by computing the Fourier integral in (4.22) and discussing the resulting formulas. Setting

$$\xi = y - x \tag{4.25}$$

and $t = \xi^0$, $r = |\vec{\xi}|$, $p = |\vec{k}|$, we obtain

$$\begin{aligned}
P^\varepsilon(x,y) &= (\mathrm{i}\not{\partial}_x + m) \int \frac{d^4k}{(2\pi)^4} \delta(k^2-m^2)\, \Theta(-k^0)\, \mathrm{e}^{\mathrm{i}k\xi}\, \mathrm{e}^{-\varepsilon|k^0|} \\
&= (\mathrm{i}\not{\partial}_x + m) \int \frac{d^3k}{(2\pi)^4}\, \frac{1}{2\sqrt{\vec{k}^2+m^2}}\, \mathrm{e}^{-\mathrm{i}\sqrt{\vec{k}^2+m^2}\,t - \mathrm{i}\vec{k}\vec{\xi}}\, \mathrm{e}^{-\varepsilon\sqrt{\vec{k}^2+m^2}} \\
&= (\mathrm{i}\not{\partial}_x + m) \int_0^\infty \frac{dp}{2(2\pi)^3} \int_{-1}^1 d\cos\theta\, \frac{p^2}{\sqrt{p^2+m^2}} \\
&\qquad \times \mathrm{e}^{-(\varepsilon+it)\sqrt{p^2+m^2}}\, \mathrm{e}^{-\mathrm{i}pr\cos\theta} \\
&= (\mathrm{i}\not{\partial}_x + m)\, \frac{1}{r} \int_0^\infty \frac{dp}{(2\pi)^3}\, \frac{p}{\sqrt{p^2+m^2}}\, \mathrm{e}^{-(\varepsilon+it)\sqrt{p^2+m^2}}\, \sin(pr) \\
&= (\mathrm{i}\not{\partial}_x + m) \frac{m^2}{(2\pi)^3} \frac{K_1\big(m\sqrt{r^2+(\varepsilon+it)^2}\,\big)}{m\sqrt{r^2+(\varepsilon+it)^2}},
\end{aligned} \tag{4.26}$$

where the last integral was calculated using [26, formula (3.961.1)]. Here the square root and the Bessel functions K_0, K_1 are defined using a branch cut along the negative real axis. Carrying out the derivatives, we obtain

$$P^\varepsilon(x,y) = \alpha_\varepsilon(\xi)(\not{\xi} - \mathrm{i}\varepsilon\gamma^0) + \beta_\varepsilon(\xi)\mathbb{1}$$

with the smooth functions

$$\alpha_\varepsilon(\xi) = -\mathrm{i}\frac{m^4}{(2\pi)^3}\left(\frac{K_0(z)}{z^2} + 2\,\frac{K_1(z)}{z^3}\right) \quad \text{and} \quad \beta_\varepsilon(\xi) = \frac{m^3}{(2\pi)^3}\frac{K_1(z)}{z}, \tag{4.27}$$

where we set

$$z = m\sqrt{r^2 + (\varepsilon + \mathrm{i}t)^2}.$$

Due to the regularization, $P^\varepsilon(x, y)$ is a smooth function. However, in the limit $\varepsilon \searrow 0$, singularities appear on the light cone $\{\xi^2 = 0\}$ (for details see [15, Section 4.4]). This can be understood from the fact that the Bessel functions $K_0(z)$ and $K_1(z)$ have poles at $z = 0$, leading to singularities on the light cone if $\varepsilon \searrow 0$. But using that the Bessel functions are smooth for $z \neq 0$, one also sees that away from the light cone, P^ε converges pointwise (even locally uniformly) to a smooth function. We conclude that

$$P^\varepsilon(x, y) \xrightarrow{\varepsilon \searrow 0} P(x, y) \quad \text{if } \xi^2 \neq 0 \tag{4.28}$$

and

$$P(x, y) = \alpha(\xi)\,\not{\xi} + \beta(\xi)\,\mathbb{1} \tag{4.29}$$

where the functions α and β can be written in terms of real-valued Bessel functions as

$$\begin{aligned}
\beta(\xi) &= \theta(\xi^2)\frac{m^3}{16\pi^2}\frac{Y_1(m\sqrt{\xi^2}) + \mathrm{i}\epsilon(\xi^0)J_1(m\sqrt{\xi^2})}{m\sqrt{\xi^2}} + \theta(-\xi^2)\frac{m^3}{8\pi^3}\frac{K_1(m\sqrt{-\xi^2})}{m\sqrt{-\xi^2}}, \\
\alpha(\xi) &= -\frac{2\mathrm{i}}{m}\frac{d}{d(\xi^2)}\beta(\xi)
\end{aligned} \tag{4.30}$$

(and ϵ denotes the step function $\epsilon(x) = 1$ if $x > 1$ and $\epsilon(x) = -1$ otherwise). These functions have the expansion

$$\alpha(\xi) = -\frac{i}{4\pi^3\,\xi^4} + \mathcal{O}\left(\frac{1}{\xi^2}\right) \quad \text{and} \quad \beta(\xi) = -\frac{m}{8\pi^3\,\xi^2} + \mathcal{O}\big(\log(\xi^2)\big). \tag{4.31}$$

4.2 The geometry without regularization

We now enter the analysis of the geometric objects introduced in Section 3 for given space-time points $x, y \in M$. We restrict attention to the case

$\xi^2 \neq 0$ when the space-time points are not lightlike separated. This has the advantage that, in view of the convergence (4.28), we can first consider the unregularized kernel $P(x,y)$ in the form (4.29). In Section 4.3 we can then use a continuity argument to extend the results to small $\varepsilon > 0$.

We first point out that, although we are working without regularization, the fact that we took the limit $\varepsilon \searrow 0$ of regularized objects is still apparent because the Euclidean sign operator (4.20) distinguishes a specific sign operator. This fact will enter the construction, but of course the resulting spin connection must and will be Lorentz invariant. Taking the adjoint of (4.29),

$$P(y,x) = P(x,y)^* = \overline{\alpha(\xi)}\, \not{\xi} + \overline{\beta(\xi)}\, \mathbb{1},$$

we obtain for the closed chain

$$A_{xy} = a(\xi)\, \not{\xi} + b(\xi)\, \mathbb{1} = A_{yx} \tag{4.32}$$

with the real-valued functions $a = 2\,\mathrm{Re}(\alpha\bar{\beta})$ and $b = |\alpha|^2 \xi^2 + |\beta|^2$. Subtracting the trace and taking the square, the eigenvalues of A_{xy} are computed by

$$\lambda_+ = b + \sqrt{a^2\xi^2} \quad \text{and} \quad \lambda_- = b - \sqrt{a^2\xi^2}. \tag{4.33}$$

It follows that the eigenvalues of A_{xy} are real if $\xi^2 > 0$, whereas they form a complex conjugate pair if $\xi^2 < 0$. This shows that the causal structure of Definition 2.2 agrees with the usual causal structure in Minkowski space. We next show that in the case of timelike separation, the space-time points are even properly timelike separated.

Lemma 4.3. *Let $x, y \in M$ with $\xi^2 \neq 0$. Then x and y are properly timelike separated (see Definition 3.14) if and only if $\xi^2 > 0$. The directional sign operator of A_{xy} is given by*

$$v_{xy} = \epsilon(\xi^0)\, \frac{\not{\xi}}{\sqrt{\xi^2}}. \tag{4.34}$$

Proof. In the case $\xi^2 < 0$, the two eigenvalues $\lambda_\pm$ in (4.33) form a complex conjugate pair. If they are distinct, the spectrum is not real. On the other hand, if they coincide, the corresponding eigenspace is not definite. Thus x and y are not properly timelike separated.

In the case $\xi^2 > 0$, we obtain a simple expression for a,

$$\begin{aligned} a = 2\,\mathrm{Re}(\alpha\bar{\beta})(m\sqrt{\xi^2}) &= \epsilon(\xi^0)\,\frac{2m^7}{(4\pi)^4}\,\frac{(Y_1 J_0 - Y_0 J_1)(m\sqrt{\xi^2})}{(m\sqrt{\xi^2})^3} \\ &= -\epsilon(\xi^0)\,\frac{m^3}{64\pi^5}\,\frac{1}{\xi^4}, \end{aligned} \tag{4.35}$$

where we used [1, formula (9.1.16)] for the Wronskian of the Bessel functions J_1 and Y_1. In particular, one sees that $a \neq 0$, so that according to (4.33), the matrix A_{xy} has two distinct eigenvalues.

Next, the calculation

$$\begin{aligned} \lambda_+\lambda_- = b^2 - a^2\xi^2 &= |\alpha|^4\,\xi^4 + |\beta|^4 + 2|\alpha|^2\,\xi^2\,|\beta|^2 - 4\xi^2\,\mathrm{Re}(\alpha\bar{\beta})^2 \\ &\overset{(*)}{\geq} |\alpha^4\,|\xi^4 + |\beta|^4 - 2|\alpha|^2\,\xi^2|\,\beta|^2 = \big(|\alpha^2|\xi^2 - |\beta|^2\big)^2 \geq 0 \end{aligned}$$

shows that the spectrum of A_{xy} is positive. In order to obtain a strict inequality, it suffices to show that $\mathrm{Im}(\alpha\bar{\beta}) \neq 0$ (because then the inequality in $(*)$ becomes strict). After the transformation

$$\begin{aligned} \mathrm{Im}(\alpha\bar{\beta}) &= -\frac{m^7}{(4\pi)^4}\left(\frac{(Y_0Y_1 + J_0J_1)(z)}{z^3} - 2\,\frac{(J_1^2 + Y_1^2)(z)}{z^4}\right) \\ &= -\frac{m^7}{(4\pi)^4}\,\frac{1}{2z}\,\frac{d}{dz}\left(\frac{Y_1(z)^2 + J_1(z)^2}{z^2}\right), \end{aligned} \tag{4.36}$$

where we set $z = m\sqrt{\xi^2} > 0$, asymptotic expansions of the Bessel functions yield that the function $\mathrm{Im}(\alpha\bar{\beta})$ is positive for z near zero and near infinity. The plot in figure 1 shows that this function is also positive in the intermediate range.

We now prove that the eigenspaces of A_{xy} are definite with respect to the inner product $\prec .|. \succ$ on V. First, from (4.32) it is obvious that the eigenvectors of A_{xy} coincide with those of the operator $\not{\xi}$. Thus, let $v \in V$ be an eigenvector of $\not{\xi}$, i.e., $\not{\xi} v = \pm\sqrt{\xi^2}\,v$. We choose a proper orthochronous Lorentz-transformation Λ which transforms ξ to the vector $\Lambda(\xi) = (t, \vec{0})$ with $t \neq 0$. In view of the Lorentz invariance of the Dirac equation, there is a unitary transformation $U \in \mathrm{U}(V)$ with $U\gamma^l U^{-1} = \Lambda^l_j \gamma^j$. Then the calculation

$$\begin{aligned} \pm\sqrt{\xi^2} \prec v \,|\, v \succ = \prec v \,|\, \not{\xi} v \succ &= \prec v \,|\, \eta_{ij}\xi^i\gamma^j\, v \succ = \eta_{kl} \prec v \,|\, (\Lambda^k_i \xi^i)(\Lambda^l_j \gamma^j)\, v \succ \\ &= \eta_{kl} \prec v \,|\, \Lambda(\xi)^k U\gamma^l U^{-1} v \succ = t \prec v \,|\, U\gamma^0 U^{-1} v \succ \\ &= t \prec U^{-1}v \,|\, \gamma^0 U^{-1} v \succ = t\,\langle U^{-1}v \,|\, U^{-1}v \rangle_{\mathbb{C}^4} \neq 0 \end{aligned} \tag{4.37}$$

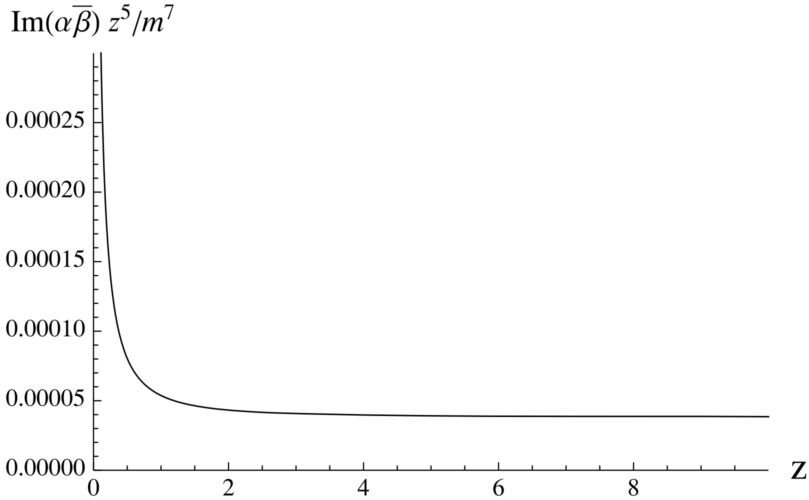

Figure 1: The Bessel functions in (4.36).

shows that $\prec v \,|\, v \succ \neq 0$, and thus v is a definite vector. We conclude that x and y are properly timelike separated.

We next show that the directional sign operator of A_{xy} is given by (4.34). The calculation (4.37) shows that the inner product $\prec . | v_{xy} . \succ$ with v_{xy} according to (4.34) is positive definite. Furthermore, the square of v_{xy} is given by

$$v_{xy}^2 = \left(\epsilon(\xi^0) \frac{\not{\xi}}{\sqrt{\xi^2}} \right)^2 = \mathbb{1},$$

showing that v_{xy} is indeed a sign operator. Since v_{xy} obviously commutes with A_{xy}, it is the directional sign operator of A_{xy}. $\square$

Let us go through the construction of the spin connection in Section 3.3. Computing the commutator of the Euclidean sign operator s_x (see (4.20)) and the directional sign operator v_{xy} (see (4.34)),

$$[v_{xy}, s_x] = \left[\epsilon(\xi^0) \frac{\not{\xi}}{\sqrt{\xi^2}}, \gamma^0 \right] = 2\epsilon(\xi^0) \frac{\vec{\xi} \cdot \vec{\gamma} \gamma^0}{\sqrt{\xi^2}},$$

one sees that these operators are generically separated (see Definition 3.11), provided that we are not in the exceptional case $\vec{\xi} \neq 0$ (for which the spin connection could be defined later by continuous continuation). Since these

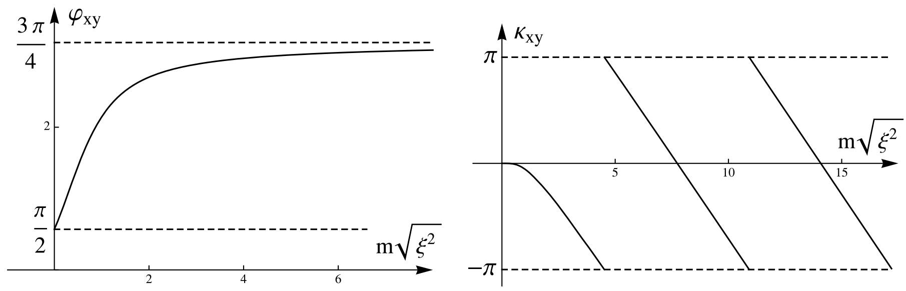

Figure 2: The phases in the spin connection in the case $\xi^0 > 0$.

two sign operators lie in the Clifford subspace K spanned by $(\gamma^0, \ldots, \gamma^3, i\gamma^5)$ (again in the usual Dirac representation), it follows that all the Clifford subspaces used in the construction of the spin connection are equal to K, i.e.,

$$K_{xy} = K_{yx} = K_x^{(y)} = K_y^{(x)} = K.$$

All synchronization and identification maps are trivial (see Definition 3.13 and (3.18)). In particular, the system is parity preserving (see Definition 3.10) and Clifford-parallel (see Definition 3.29). Choosing again the basis $(e_0 = v_{xy}, e_1, \ldots, e_4)$ of K and the spinor basis of Corollary 3.4, one sees from (4.29) and (4.34) that $P(x, y)$ is diagonal,

$$P(x,y) = \begin{pmatrix} \left(\beta + \alpha\sqrt{\xi^2}\right)\mathbb{1} & 0 \\ 0 & \left(\beta - \alpha\sqrt{\xi^2}\right)\mathbb{1} \end{pmatrix}.$$

Thus in the polar decomposition (3.34) we get

$$R_{xy}^{\pm} = \left|\beta \pm \alpha\sqrt{\xi^2}\right|, \quad \vartheta_{xy}^{\pm} = \arg\left(\beta \pm \alpha\sqrt{\xi^2}\right) \bmod \pi, \quad V_{x,y}^{\pm} \in \{\mathbb{1}, -\mathbb{1}\}.$$

Computing φ_{xy} according to (3.41) and our convention (3.44), in the case $\xi^0 > 0$, we obtain the left plot of figure 2, whereas in the case $\xi^0 < 0$ one gets the same with the opposite sign. We conclude that φ_{xy} is never a multiple of $\frac{\pi}{4}$, meaning that x and y are time-directed (see Definition 3.19). Moreover, the time direction of Definition 3.23 indeed agrees with the time orientation of Minkowski space. Having uniquely fixed φ_{xy}, the spin connection is given by (3.35) or by (3.42). A short calculation yields that D_{xy} is trivial up to a phase factor,

$$D_{x,y} = \mathrm{e}^{\mathrm{i}\kappa_{xy}}\, \mathbb{1}, \tag{4.38}$$

where the phase κ_{xy} is given by

$$\kappa_{xy} = \arg\left(\mathrm{e}^{\mathrm{i}\varphi_{xy}}\left(\beta + \alpha\sqrt{\xi^2}\right)\right) = \arg\left(\mathrm{e}^{-\mathrm{i}\varphi_{xy}}\left(\beta - \alpha\sqrt{\xi^2}\right)\right). \tag{4.39}$$

The function κ_{xy} is shown in the right plot of figure 2. One sees that the phase factor in $D_{x,y}$ oscillates on the length scale m^{-1}. We postpone the discussion of this phase to Section 4.4.

Let us consider the corresponding metric connection of Definition 3.21. We clearly identify the tangent space T_x with the vector space K. As the synchronization maps are trivial and the phases in $D_{x,y}$ drop out of (3.49), it is obvious that $\nabla_{x,y}$ reduces to the trivial connection in Minkowski space. Finally, choosing $u(x) = i\gamma^5$, the Dirac system is obviously chirally symmetric (see Definition 3.27).

Our results are summarized as follows.

Proposition 4.4. *Let $x, y \in M$ with $\xi^2 \neq 0$ and $\vec{\xi} \neq 0$. Consider the spin connection corresponding to the Euclidean signature operator* (4.20) *and the unregularized Dirac sea vacuum* (4.29)*. Then x and y are spin-connectable if and only if $\xi^2 > 0$. The spin connection $D_{x,y}$ is trivial up to a phase factor* (4.38)*. The time direction of Definition* 3.23 *agrees with the usual time orientation of Minkowksi space. The corresponding metric connection $\nabla_{x,y}$ is trivial.*

Restricting attention to pairs $(x, y) \in M \times M$ with $\xi^2 \neq 0$ and $\vec{\xi} \neq 0$, the resulting causal fermion system is parity preserving, chirally symmetric and Clifford-parallel.

4.3 The geometry with regularization

We now use a perturbation argument to extend some of the results of Proposition 4.4 to the case with regularization.

Proposition 4.5. *Consider the causal fermion systems of Proposition* 4.1*. For any $x, y \in M$ with $\xi^2 > 0$ and $\vec{\xi} \neq 0$, there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following statements hold: The points x and y are spin-connectable. The time direction of Definition* 3.23 *agrees with the usual time orientation* *of Minkowksi space. In the limit* $\varepsilon \searrow 0$*, the corresponding connections* $D^\varepsilon_{x,y}$ *and* $\nabla^\varepsilon_{x,y}$ *converge to the connections of Proposition* 4.4*,*

$$\lim_{\varepsilon \searrow 0} D^\varepsilon_{x,y} = D_{x,y}, \quad \lim_{\varepsilon \searrow 0} \nabla^\varepsilon_{x,y} = \mathbb{1}. \tag{4.40}$$

Proof. Let $x, y \in M$ with $\xi^2 > 0$ and $\vec{\xi} \neq 0$. Using the pointwise convergence (4.28), a simple continuity argument shows that for sufficiently small ε, the spectrum of A^ε_{xy} is strictly positive and the eigenspaces are definite. Thus x and y are properly timelike separated. From (3.34) and (3.41) we conclude that in a small interval $(0, \varepsilon_0)$, the phase φ^ε_{xy} depends continuously on ε and lies in the same subinterval (3.44) as the phase φ_{xy} without regularization. We conclude that for all $\varepsilon \in (0, \varepsilon_0)$, the points x and y are spin-connectable and have the same time orientation as without regularization. The continuity of the connections is obvious from (3.42). ☐

We point out that this proposition makes no statement on whether the causal fermion systems are parity preserving, chirally symmetric or Clifford-parallel. The difficulty is that these definitions are either not stable under perturbations, or else they would make it necessary to choose ε independent of x and y. To be more specific, the closed chain with regularization takes the form

$$A^\varepsilon_{xy} = a_\varepsilon \not{\xi} + b_\varepsilon \mathbb{1} + c_\varepsilon \gamma^0 - i d_\varepsilon \vec{\xi} \cdot \vec{\gamma} \gamma^0,$$

where the coefficients involve the regularized Bessel functions in (4.27),

$$\begin{aligned} a_\varepsilon &= 2\,\mathrm{Re}(\alpha_\varepsilon \overline{\beta_\varepsilon}), & b_\varepsilon &= |\alpha_\varepsilon|^2 (\xi^2 + \varepsilon^2) + |\beta_\varepsilon|^2, \\ c_\varepsilon &= 2\varepsilon\, \mathrm{Im}(\alpha_\varepsilon \overline{\beta_\varepsilon}), & d_\varepsilon &= 2\varepsilon |\alpha_\varepsilon|^2. \end{aligned}$$

A short calculation shows that for properly timelike separated points x and y, the directional sign operator is given by

$$v^\varepsilon_{xy} = \frac{a_\varepsilon \not{\xi} + c_\varepsilon \gamma^0 - \mathrm{i} d_\varepsilon \vec{\xi} \cdot \vec{\gamma} \gamma^0}{\sqrt{a_\varepsilon^2 \xi^2 + 2 a_\varepsilon c_\varepsilon \xi^0 + c_\varepsilon^2 - d_\varepsilon^2\, |\vec{\xi}|^2}},$$

and the argument of the square root is positive. A direct computation shows that the signature operators s_x and v^ε_{xy} span a Clifford subspace of signature $(1,1)$. According to Lemma 3.3, this Clifford subspace has a unique extension K, implying that $K_{xy} = K^{(y)}_x = K$. This shows that the synchronization maps are all trivial. However, as v^ε_{xy} involves a bilinear component which depends on $\vec{\xi}$, the Clifford subspaces K_{xy} and K_{xz} will in general be different, so that the identification maps (3.18) are in general non-trivial. Due to this complication, the system is no longer Clifford-parallel, and it is not obvious whether the system is parity preserving or chirally symmetric.

4.4 Parallel transport along timelike curves

The phase factor in (4.38) resembles the U(1)-phase in electrodynamics. This phase is unphysical as no electromagnetic field is present. In order to understand this seeming problem, one should note that in differential geometry, the parallel transport is always performed along a continuous curve, whereas the spin connection $D_{x,y}$ directly connects distant points. The correct interpretation is that the spin connection only gives the physically correct result if the points x and y are sufficiently close to each other. Thus in order to connect distant points x and y, one should choose intermediate points $x_1, \ldots x_N$ and compose the spin connections along neighboring points. In this way, the unphysical phase indeed disappears, as the following construction shows.

Assume that $\gamma(t)$ is a future-directed timelike curve, for simplicity parametrized by arc length, which is defined on the interval $[0,T]$ with $\gamma(0) = y$ and $\gamma(T) = x$. The Levi–Civita parallel transport of spinors along γ is trivial. In order to compare with the spin connection D^ε, we subdivide γ (for simplicity with equal spacing, although a non-uniform spacing would work just as well). Thus for any given N, we define the points $x_0, \ldots, x_N$ by

$$x_n = \gamma(t_n) \quad \text{with} \quad t_n = \frac{nT}{N}. \tag{4.41}$$

Definition 4.6. The curve γ is called **admissible** if for all sufficiently large $N \in \mathbb{N}$ there is a parameter $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $n = 1, \ldots, N$, the points x_n and x_{n-1} are spin-connectable.

If γ is admissible, we define the parallel transport $D^{N,\varepsilon}_{x,y}$ by successively composing the parallel transports between neighboring points,

$$D^{N,\varepsilon}_{x,y} := D^\varepsilon_{x_N, x_{N-1}} D^\varepsilon_{x_{N-1}, x_{N-2}} \cdots D^\varepsilon_{x_1, x_0} \;:\; V \to V.$$

Then the following theorem holds.

Theorem 4.7. *Considering the family of causal fermion systems of Proposition 4.1, the admissible curves are generic in the sense that they are dense in the C^∞-topology (meaning that for any smooth γ and every $K \in \mathbb{N}$,* *there is a series γ_ℓ of admissible curves such that $D^k\gamma_\ell \to D^k\gamma$ uniformly for all $k = 0, \ldots, K$). Choosing $N \in \mathbb{N}$ and $\varepsilon > 0$ such that the points x_n and x_{n-1} are spin-connectable for all $n = 1, \ldots, N$, every point x_n lies in the future of x_{n-1}. Moreover,*

$$\lim_{N\to\infty} \lim_{\varepsilon\searrow 0} D^{N,\varepsilon}_{x,y} = D^{\mathtt{LC}}_{x,y},$$

where we use the identification (4.19), *and $D^{\mathtt{LC}}_{x,y} : S_yM \to S_xM$ denotes the trivial parallel transport along γ.*

Proof. For any given N, we know from Proposition 4.5 that by choosing ε_0 small enough, we can arrange that for all $\epsilon \in \big(0, \epsilon_0\big)$ and all $n = 1, \ldots, N$, the points x_n and x_{n-1} are spin-connectable and x_n lies in the future of x_{n-1}, provided that the vectors $\vec{x}_n - \vec{x}_{n-1}$ do not vanish (which is obviously satisfied for a generic curve). Using (4.40) and (4.38), we obtain

$$\lim_{\varepsilon\searrow 0} D^{N,\varepsilon}_{x,y} = D_{x_N,x_{N-1}} D_{x_{N-1},x_{N-2}} \cdots D_{x_1,x_0} = \exp\left(i \sum_{n=1}^N \kappa_{x_n,x_{n-1}}\right) \mathbb{1}. \tag{4.42}$$

Combining the two equations in (4.39), one finds

$$\kappa_{xy} = \frac{1}{2}\, \arg\big(\beta^2 - \alpha^2\, \xi^2\big) \mod \pi.$$

Expanding the Bessel functions in (4.30) gives

$$\beta^2 - \alpha^2\, \xi^2 = \frac{1}{16\pi^6}\, \frac{1}{\xi^6} + \mathcal{O}\left(\frac{1}{\xi^4}\right),$$

As κ_{xy} is smooth and vanishes in the limit $y \to x$, we conclude that

$$\kappa_{xy} = \mathcal{O}(\xi^2).$$

Using this estimate in (4.42), we obtain

$$\lim_{\varepsilon\searrow 0} D^{N,\varepsilon}_{x,y} = \exp\big(i\, N\, \mathcal{O}(N^{-2})\big)\, \mathbb{1}.$$

Taking the limit $N \to \infty$ gives the result. $\square$

5 Example: the fermionic operator in a globally hyperbolic space-time

In this section we shall explore the connection between the notions of the quantum geometry introduced in Section 3 and the common objects of Lorentzian spin geometry. To this end, we consider Dirac spinors on a globally hyperbolic Lorentzian manifold (M, g) (for basic definitions see [2, 3]). For technical simplicity, we make the following assumptions:

(A) The manifold (M, g) is flat Minkowski space in the past of a Cauchy hypersurface $\mathcal{N}$.
(B) The causal fermion systems are introduced as the Cauchy development of the fermion systems in Minkowski space as considered in Section 4.1.

These causal fermion systems are constructed in Section 5.1. We proceed by analyzing the fermionic operator in the limit without regularization using its Hadamard expansion (see Sections 5.2 and 5.3). We then consider the spin connection along a timelike curve γ (see Section 5.4). We need to assume that in a neighborhood $\mathfrak{U}$ of the curve γ, the Riemann curvature tensor R is bounded pointwise in the sense that

$$\frac{\|R(x)\|}{m^2} + \frac{\|\nabla R(x)\|}{m^3} + \frac{\|\nabla^2 R(x)\|}{m^4} < c \quad \text{for all } x \in \mathfrak{U}, \tag{5.1}$$

where $\|.\|$ is a norm (for example induced by the scalar product $\prec .|\dot{\gamma}(t)\, . \succ$ on $S_{\gamma(t)}M$), and $c < 1$ is a numerical constant (which could be computed explicitly). This conditions means that curvature should be small on the Compton scale. It is a physically necessary assumption because otherwise the gravitational field would be so strong that pair creation would occur, making it impossible to speak of "classical gravity". Our main result is that the spliced spin connection goes over to the Levi–Civita connection on the spinor bundle, up to errors of the order $\|\nabla R\|/m^3$ (see Theorem 5.12). We conclude with a brief outlook (see Section 6).

5.1 The regularized fermionic operator

Let (M, g) be a globally hyperbolic Lorentzian manifold which coincides with Minkowski space in the past of a Cauchy hypersurface $\mathcal{N}$. Choosing a global time function $\mathfrak{t}$ (see [5]), M has a smooth splitting $M \cong \mathbb{R} \times \mathcal{N}$ with $\mathcal{N} = \mathfrak{t}^{-1}(\{0\})$. For consistency with Section 4, we use the conventions that the signature of g is $(+ - - -)$, and that Clifford multiplication satisfies the relation $X \cdot Y + Y \cdot X = 2g(X, Y)$. We denote the volume measure by $d\mu(x) = \sqrt{|\det g|}\, d^4x$. As a consequence of the smooth splitting, the manifold M is spin. The spinor bundle SM is endowed with a Hermitian inner product of signature $(2,2)$, which we denote by $\prec .|.\succ$. We let $\mathcal{D}$ be the Dirac operator on M, acting on sections $\psi \in \Gamma(M, SM)$ of the spinor bundle. For a given mass $m > 0$, we consider the Dirac equation on M,

$$(\mathcal{D} - m)\, \psi = 0. \tag{5.2}$$

The simplest method for constructing causal fermion systems is to replace the plane-wave solutions used in Minkowski space (see Section 4.1) by corresponding solutions of (5.2) obtained by solving a Cauchy problem. More precisely, in the past of $\mathcal{N}$ where our space-time is isometric to Minkowski space, we again introduce the plane-wave solution $\psi_{\vec{k}a-}$ (see (4.2)). Using that the Cauchy problem has a unique solution (see [2] and the integral representation (5.12) below), we can extend them to smooth solutions $\tilde{\psi}_{\vec{k}a-}$ on M by

$$(\mathcal{D} - m)\, \tilde{\psi}_{\vec{k}a-} = 0, \quad \tilde{\psi}_{\vec{k}a-}|_{\mathcal{N}} = \psi_{\vec{k}a-}. \tag{5.3}$$

In obvious generalization of (4.5) and (4.6), we can form superpositions of these solutions, on which we introduce the scalar product

$$\langle \tilde{\psi} | \tilde{\phi} \rangle_{\mathcal{H}} = 2\pi \int_{\mathfrak{t}=\text{const}} \prec \tilde{\psi}(\mathfrak{t}, x)\, |\, \nu \cdot \tilde{\phi}(\mathfrak{t}, x) \succ d\mu_{\mathcal{N}(\mathfrak{t})}(x), \tag{5.4}$$

where ν is the future-directed unit normal on $\mathcal{N}$ (note that this scalar product is independent of $\mathfrak{t}$ due to current conservation). We again denote the corresponding Hilbert space by $\mathcal{H}$.

In order to introduce a corresponding causal fermion system, we introduce the operators ι^ε_x and $F^\varepsilon(x)$ by adapting (4.13) and (4.12),

$$\begin{aligned} \iota^\varepsilon_x \,:\, S_xM \rightarrow \mathcal{H}\,,\; u \mapsto -\frac{m}{\pi} \sum_{a=1,2} \int \frac{d^3k}{2\omega}\, \mathrm{e}^{-\frac{\varepsilon\omega}{2}}\, \tilde{\psi}_{\vec{k}a-} \prec \tilde{\psi}_{\vec{k}a-}(x) | u \succ \\ F^\varepsilon(x) = -\iota^\varepsilon_x\, (\iota^\varepsilon_x)^* \,:\, \mathcal{H} \rightarrow \mathcal{H}. \end{aligned}$$

Let us verify that ι^ε_x is injective: For a given non-zero spinor $\chi \in S_xM$, we choose a wave function $\psi \in \mathcal{H}$ which is well-approximated by a WKB wave packet of large negative energy (by decreasing the energy, we can make the error of the approximations arbitrarily small). Consider the operator

$$L : \mathcal{D}(L) \subset \mathcal{H} \rightarrow \mathcal{H}, \quad (L\psi)(x) = -\frac{m}{\pi} \sum_{a=1,2} \int \frac{d^3k}{2\omega}\, \mathrm{e}^{-\frac{\varepsilon\omega}{2}}\, \tilde{\psi}_{\vec{k}a-}(x) \langle \tilde{\psi}_{\vec{k}a-} | \psi \rangle_{\mathcal{H}},$$

where $\mathcal{D}(L)$ is a suitable dense domain of definition (for example the smooth Dirac solutions with spatially compact support). As the image of L is obviously dense in $\mathcal{H}$, there is a vector $\phi \in \mathcal{D}$ such that $L\phi$ approximates ψ (again, we can make the error of this approximation arbitrarily small). Then $\langle \phi | \iota^\varepsilon_x \chi \rangle \approx \prec\!\psi(x)|\chi\!\succ_x$. By modifying the polarization and direction of the wave packet ψ, we can arrange that $\prec\!\psi(x)|\chi\!\succ_x \neq 0$.

According to (2.3), the spin space S^ε_x is defined as the image of F^ε_x. We now choose a convenient basis of S^ε_x which will at the same time give a canonical identification of S^ε_x with the differential geometric spinor space S_xM. We first choose an eigenvector basis $(\mathfrak{f}^\varepsilon_\alpha(x))_{\alpha=1,\ldots,4}$ of $S^\varepsilon_x = F^\varepsilon(x)(\mathcal{H})$ with corresponding eigenvalues

$$\nu^\varepsilon_1(x), \nu^\varepsilon_2(x) < 0 \quad \text{and} \quad \nu^\varepsilon_3(x), \nu^\varepsilon_4(x) > 0. \tag{5.5}$$

We normalize the eigenvectors according to

$$\langle \mathfrak{f}^\varepsilon_\alpha(x) \,|\, \mathfrak{f}^\varepsilon_\beta(x) \rangle_{\mathcal{H}} = \frac{1}{|\nu^\varepsilon_\alpha(x)|}\, \delta_{\alpha\beta}.$$

Then, according to (2.4), the $(\mathfrak{f}^\varepsilon_\alpha(x))$ are a pseudo-orthonormal basis of $(S^\varepsilon_x, \prec\!.|.\!\succ_x)$. Next, we introduce the vectors

$$\mathfrak{e}^\varepsilon_\alpha(x) = (\iota^\varepsilon_x)^*\, \mathfrak{f}^\varepsilon_\alpha(x) \in S_xM.$$

A short calculation shows that these vectors form a pseudo-orthonormal eigenvector basis of the operator

$$(\iota^\varepsilon_x)^* \iota^\varepsilon_x \;:\; S_xM \to S_xM,$$

corresponding to the eigenvalues $\nu^\varepsilon_\alpha(x)$. In analogy to (4.19), we always identify the spaces S_xM and S^ε_x via the mapping $\mathfrak{J}^\varepsilon_x$ defined by

$$\mathfrak{J}^\varepsilon_x \;:\; S_xM \to S^\varepsilon_x\,,\; \mathfrak{e}^\varepsilon_\alpha(x) \mapsto \mathfrak{f}^\varepsilon_\alpha(x). \tag{5.6}$$

Again identifying x with $F^\varepsilon(x)$, the kernel of the fermionic operator (2.7) takes the form (4.21). Exactly as in the proof of Lemma 4.2, we find that

$$P^\varepsilon(x,y) = -(\iota^\varepsilon_x)^* \iota^\varepsilon_y = -\frac{m}{\pi} \sum_{a=1,2} \int \frac{d^3k}{2\omega}\, \mathrm{e}^{-\varepsilon\omega}\, |\tilde{\psi}_{\vec{k}a-}(x)\!\succ\prec\!\tilde{\psi}_{\vec{k}a-}(y)|. \tag{5.7}$$

From this formula we can read off the following characterization of P^ε.

Proposition 5.1. *The kernel of the fermionic operator $P^\varepsilon(x,y)$ is the unique smooth bi-solution of* (5.2)*, i.e., in a distributional formulation*

$$P^\varepsilon\big((\mathcal{D}-m)\psi,\phi\big) = 0 = P^\varepsilon\big(\psi,(\mathcal{D}-m)\phi\big) \quad \text{for all } \psi,\phi \in \Gamma_0(M,SM),$$

with the following properties:

(i) P^ε *coincides with the regularized Dirac sea vacuum* (4.22) *if ψ and ϕ are both supported in the past of $\mathcal{N}$.*
(ii) P^ε *is symmetric in the sense that $P^\varepsilon(\psi,\phi) = P^\varepsilon(\phi,\psi)$.*

In order to keep the analysis simple, our strategy is to take the limit $\varepsilon \searrow 0$ at an early stage. In the remainder of this section, we analyze this limit for P^ε and for the Euclidean sign operator. In preparation, we recall the relation between the Dirac Green's functions and the solution of the Cauchy problem, adapting the methods in [2] to the first-order Dirac system. On a globally hyperbolic Lorentzian manifold, one can introduce the retarded Dirac Green's function, which we denote by $s^\wedge(x,y)$. It is defined as a distribution on $M \times M$, meaning that we can evaluate it with compactly supported test functions $\phi,\psi \in \Gamma_0(M,SM)$,

$$s^\wedge(\phi,\psi) = \iint_{M\times M} \prec\!\phi(x)|s^\wedge(x,y)\,\psi(y)\!\succ d\mu(x)\,d\mu(y).$$

We can also regard it as an operator on the test functions. Thus for $\psi \in \Gamma_0(M,SM)$, we set

$$s^\wedge(x,\psi) = \int_M s^\wedge(x,y)\,\psi(y)\,d\mu(y) \in \Gamma(M,SM).$$

The retarded Green's function is uniquely determined as a solution of the inhomogeneous Dirac equation

$$(\mathcal{D}_x - m)\,s^\wedge(x,\psi) = \psi(x) = s^\wedge\big(x,(\mathcal{D}-m)\psi\big) \tag{5.8}$$

subject to the support condition

$$\operatorname{supp} s^\wedge(x,.) \subset J^\wedge(x),$$

where $J^\wedge(x)$ denotes the causal past of x. The advanced Dirac Green's function $s^\vee(x,y)$ is defined similarly. It can be obtained from the retarded

Green's function by conjugation,

$$s^\wedge(x,y)^* = s^\vee(y,x), \tag{5.9}$$

where the star denotes the adjoint with respect to the Hermitian inner product on the spinor bundle.

For the construction of the Dirac Green's functions, it is useful to also consider the second-order equation

$$(\mathcal{D}^2 - m^2)\,\psi = 0. \tag{5.10}$$

Using the Lichnerowicz–Weitzenböck identity (see [3]), we can rewrite this equation as

$$\left(\Box^\nabla + \frac{\text{scal}}{4} - m^2 \right) \psi = 0,$$

where $\Box^\nabla$ denotes the Bochner Laplacian corresponding to the spinorial Levi–Civita connection. This shows that the operator in (5.10) is normally hyperbolic, ensuring the existence of the corresponding Green's function $S^\wedge$ as the unique distribution on $M \times M$, which satisfies the equation

$$(\mathcal{D}^2 - m^2)\, S^\wedge(x,\phi) = \phi(x).$$

and the support condition

$$\operatorname{supp} S^\wedge(x,.) \subset J^\wedge(x)$$

(see [2, Section 3.4]). Then the Dirac Green's function can be obtained by the identities

$$s^\wedge(\psi,\phi) = S^\wedge\big((\mathcal{D}+m)\psi,\phi\big) = S^\wedge\big(\psi,(\mathcal{D}+m)\phi\big). \tag{5.11}$$

The existence of the retarded Green's function implies that the Cauchy problem

$$(\mathcal{D} - m)\,\tilde{\psi} = 0, \quad \tilde{\psi}|_{\mathcal{N}} = \psi \in C^\infty(\mathcal{N})$$

has a unique smooth solution, as we now recall. To show uniqueness, assume that $\tilde{\psi}$ is a smooth solution of the Cauchy problem. For given x in the future of $\mathcal{N}$, we choose a test function $\eta \in C^\infty_0(M)$ which is identically equal to one in a neighborhood of the set $J^\wedge(x) \cap J^\vee(\mathcal{N})$. Moreover, for a given non-negative function $\theta \in C^\infty(\mathbb{R})$ with $\theta|_{(-\infty,0]} \equiv 0$ and $\theta|_{[1,\infty)} \equiv 1$

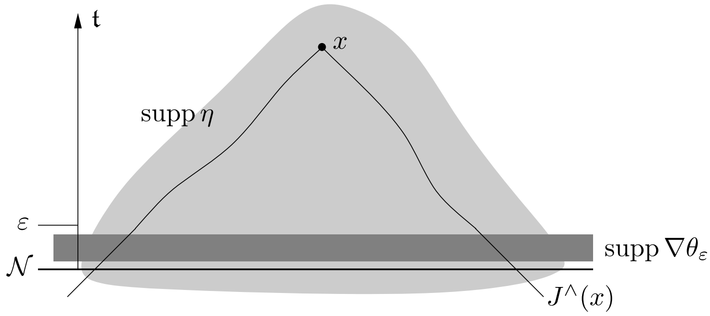

Figure 3: The cutoff functions η and θ_ε.

and sufficiently small $\varepsilon > 0$, we introduce the smooth cutoff function $\theta_\varepsilon(y) = \theta(\mathfrak{t}(y)/\varepsilon)$. Then the product $\phi := \theta_\varepsilon\, \eta\, \tilde{\psi}$ has compact support (see figure 3), and by (5.8) we obtain

$$\tilde{\psi}(x) = \phi(x) = s^\wedge\big(x, (\mathcal{D} - m)\phi\big) = s^\wedge\big(x, \mathrm{i}\eta\,(d\theta_\varepsilon)\cdot\tilde{\psi}\big).$$

Taking the limit $\varepsilon \searrow 0$, we obtain the formula

$$\tilde{\psi}(x) = \mathrm{i} \int_{\mathcal{N}} s^\wedge(x,y)\, \nu(y)\cdot\psi(y)\, d\mu_{\mathcal{N}}(y), \tag{5.12}$$

where ν is the normal of $\mathcal{N}$. This formula is an explicit integral representation of the solution in terms of the Green's function and the initial data, proving uniqueness. On the other hand, this integral representation can be used to define $\tilde{\psi}$, proving existence.

We next express $P^\varepsilon(x,y)$ in terms of Green's functions and the regularized fermionic operator of Minkowski space.

Lemma 5.2. *The regularized fermionic operator has the representation*

$$\begin{aligned} P^\varepsilon(x,y) = \iint_{\mathcal{N}\times\mathcal{N}} & s^\wedge(x,z_1)\, \nu(z_1)\cdot P^\varepsilon(z_1,z_2)\, \nu(z_2) \\ & \cdot s^\vee(z_2,y)\, d\mu_{\mathcal{N}}(z_1)\, d\mu_{\mathcal{N}}(z_2), \end{aligned} \tag{5.13}$$

with $P^\varepsilon(z_1,z_2)$ as given by (4.22).

Proof. We use (5.12) in (5.7) and apply (5.9). □

Setting ε to zero, we can use the statement of Proposition 5.1 as the definition of a distributional solution of the Dirac equation.

Definition 5.3. The distribution $P(x,y)$ is defined as the unique distributional bi-solution of (5.2),

$$P\big((\mathcal{D}-m)\psi,\phi\big) = 0 = P\big(\psi,(\mathcal{D}-m)\phi\big) \;\forall\; \psi,\phi \in \Gamma_0(M,SM), \tag{5.14}$$

with the following properties:

(i) P coincides with the regularized Dirac sea vacuum (4.22) if ψ and ϕ are both supported in the past of $\mathcal{N}$.
(ii) P is symmetric in the sense that $P(\psi,\phi) = P(\phi,\psi)$.

If the regularization is removed, P^ε goes over to P in the following sense.

Proposition 5.4.

(a) *If $\varepsilon \searrow 0$, $P^\varepsilon(x,y) \to P(x,y)$ as a distribution on $M \times M$.*
(b) *If x and y are timelike separated, $P(x,y)$ is a continuous function. In the limit $\varepsilon \searrow 0$, the function $P^\varepsilon(x,y)$ converges to $P(x,y)$ pointwise, locally uniformly in x and y.*

Proof. Part (a) is a consequence of the uniqueness of the time evolution of distributions. More specifically, suppose that ψ is a smooth solution of the Dirac equation. We choose a smooth function $\eta \in C^\infty(\mathbb{R})$ with $\eta|_{[0,\infty)} \equiv 1$ and $\eta|_{(-\infty,-1]} \equiv 0$. Then

$$(\mathcal{D}-m)\big(\eta(\mathfrak{t}(x))\,\psi(x)\big) = \big(\mathcal{D}\eta(\mathfrak{t}(x))\big)\cdot\psi(x) =: \phi(x),$$

and the function ϕ is supported in the past of $\mathcal{N}$. Using (5.8) we obtain for any x in the future of $\mathcal{N}$ that

$$\psi(x) = \eta(\mathfrak{t}(x))\,\psi(x) = s^\wedge(x,\phi) = (s^\wedge * \phi)(x). \tag{5.15}$$

Regarding the star as a convolution of distributions, this relations even holds if ψ is a distributional solution of the Dirac equation. Suppose that in the past of $\mathcal{N}$, the distribution ψ converges to zero (meaning that $\psi(\varphi) \to 0$ for every test function φ supported in the past of $\mathcal{N}$). Then, as the function ϕ is supported in the past of $\mathcal{N}$, it converges to zero as a distribution in the whole space-time. The relation (5.15) shows that ψ also converges to zero in the whole space-time. In order to prove (a), we first choose z in the past of $\mathcal{N}$ and apply the above argument to the distribution $\psi_+(x) = (P^\varepsilon - P)(x,z)$. Then according to the explicit formulas in Minkowski space (see (4.22) and (4.26)), ψ_+ converges to zero in the past of $\mathcal{N}$, and thus in the whole space-time. By symmetry, it follows that for any fixed x, the

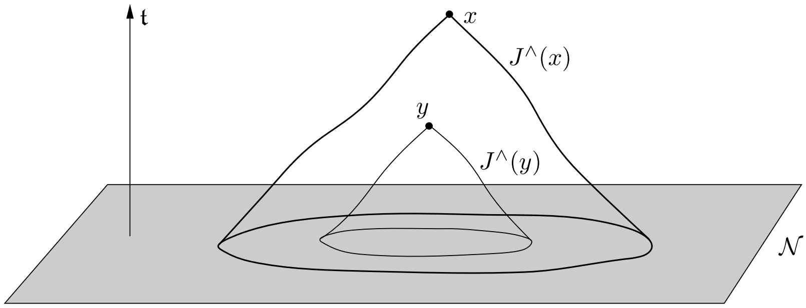

Figure 4: The singular supports of $s^\wedge(x,.)$ and $s^\vee(.,y)$.

distribution $\psi_-(z) := (P^\varepsilon - P)(z,x)$ converges to zero in the past of $\mathcal{N}$. As ψ_- is again a distributional solution of the Dirac equation, we conclude that ψ_- converges to zero in the whole space-time.

In order to prove (b), we first note that the singular support of the causal Green's functions $s^\wedge(x,.)$ and $s^\vee(.,x)$ lies on the light cone $\partial J^\wedge(x)$ centered at x (see [2, Proposition 2.4.6] and (5.11)). Thus if x and y are timelike separated, the singular supports of $s^\wedge(x,.)$ and $s^\vee(.,y)$ do not intersect (see Figure 4). Moreover, we know from (4.28) that $P^\varepsilon(z_1,z_2)$ converges as a distribution and locally uniformly away from the diagonal. Using these facts in (5.13), we conclude that $P^\varepsilon(x,y)$ converges locally uniformly to $P(x,y)$. This also implies that $P(x,y)$ is continuous. □

We remark that $P(x,y)$ is even a smooth function away from the light cone; for a proof for general bi-solutions we refer to [29, 33].

Proposition 5.5. *There is a future-directed timelike unit vector field s such that for every $x \in M$,*

$$\lim_{\varepsilon \searrow 0} s_x^\varepsilon = s(x),$$

where by $s(x)$ we mean the operator on S_xM acting by Clifford multiplication.

Proof. A short calculation gives

$$\begin{aligned} \langle \mathfrak{f}_\alpha^\varepsilon(x) \,|\, \mathfrak{f}_\beta^\varepsilon(x) \rangle_{\mathcal{H}} &= \frac{1}{\nu_\alpha^\varepsilon \nu_\beta^\varepsilon} \, \langle \iota_x^\varepsilon \mathfrak{e}_\alpha^\varepsilon | \iota_x^\varepsilon \mathfrak{e}_\beta^\varepsilon \rangle_{\mathcal{H}} = \frac{1}{\nu_\alpha^\varepsilon \nu_\beta^\varepsilon} \, \prec\!\mathfrak{e}_\alpha^\varepsilon | (\iota_x^\varepsilon)^* \iota_x^\varepsilon \mathfrak{e}_\beta^\varepsilon \!\succ \\ &= -\frac{1}{\nu_\alpha^\varepsilon \nu_\beta^\varepsilon} \prec\!\mathfrak{e}_\alpha^\varepsilon | P^\varepsilon(x,x)\, \mathfrak{e}_\beta^\varepsilon \!\succ, \end{aligned} \tag{5.16}$$

$$\begin{aligned} F^\varepsilon(x)\, \mathfrak{f}^\varepsilon_\alpha(x) &= \frac{1}{\nu^\varepsilon_\alpha} \big(-\iota^\varepsilon_x (\iota^\varepsilon_x)^*\, \iota^\varepsilon_x(\mathfrak{e}^\varepsilon_\alpha) \big) = \frac{1}{\nu^\varepsilon_\alpha}\, \iota^\varepsilon_x\, P^\varepsilon(x,x)\, \mathfrak{e}^\varepsilon_\alpha, \\ \langle \mathfrak{f}^\varepsilon_\alpha(x) \,|\, F^\varepsilon(x)\, \mathfrak{f}^\varepsilon_\beta(x) \rangle_{\mathcal{H}} &= -\frac{1}{\nu^\varepsilon_\alpha \nu^\varepsilon_\beta} \prec \mathfrak{e}^\varepsilon_\alpha | P^\varepsilon(x,x)^2\, \mathfrak{e}^\varepsilon_\beta \succ. \end{aligned} \tag{5.17}$$

Comparing (5.16) with (5.17), one sees that in our chosen basis,

$$F^\varepsilon(x) = P^\varepsilon(x,x).$$

Moreover, we know from (5.5) that $F^\varepsilon(x)$ has two positive and two negative eigenvalues. Therefore, it remains to prove that, after a suitable rescaling, $P^\varepsilon(x,x)$ converges to the operator of Clifford multiplication by a future-directed timelike unit vector $s(x) \in T_xM$, i.e.,

$$\lim_{\varepsilon \searrow 0} \varepsilon^p P^\varepsilon(x,x) = c\, s(x) \tag{5.18}$$

for suitable constants p and c.

In order to prove this claim, in the past of $\mathcal{N}$ we choose a chart where the metric is the Minkowski metric. Moreover, we choose the standard spinor frame and use the notation of Section 4. Then we can combine (5.13) with (5.11) to obtain

$$\begin{aligned} P^\varepsilon(x,x) &= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} s^\wedge(x,z_1)\, \gamma^0\, P^\varepsilon(z_1,z_2)\, \gamma^0\, s^\vee(z_2,x)\, d^3z_1\, d^3z_2 \\ &= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} S^\wedge(x,z_1)\, (-i \overleftarrow{\partial\!\!\!/}_{z_1} + m)\, \gamma^0 P^\varepsilon(z_1,z_2) \gamma^0\, (i \partial\!\!\!/_{z_2} + m) \\ &\qquad \times S^\vee(z_2,x)\, d^3z_1\, d^3z_2, \end{aligned}$$

where $z_{1\!/\!2} = (0, \vec{z}_{1\!/\!2})$, and the arrow indicates that the derivatives act to the left,

$$S^\wedge(x,z) \overleftarrow{\partial\!\!\!/}_z \equiv \frac{\partial}{\partial z^j}\, S^\wedge(x,z)\, \gamma^j.$$

We now integrate by parts the spatial derivatives of z_1 and z_2. Using the identity

$$(i \vec{\gamma} \vec{\nabla}_z + m)\, P^\varepsilon(z,y) = -i\gamma^0 \frac{\partial}{\partial z^0} P^\varepsilon(z,y) + 2m\, P^\varepsilon(z,y)$$

and its adjoint, we obtain

$$\begin{aligned} P^\varepsilon(x,x) = \iint_{\mathbb{R}^3\times\mathbb{R}^3} & S^\wedge(x,z_1) \left(-i(\overleftarrow{\partial}_{t_1} + \partial_{t_1}) + 2m\gamma^0\right) P^\varepsilon(z_1,z_2) \\ & \times \left(\mathrm{i}(\overleftarrow{\partial}_{t_2} + \partial_{t_2}) + 2m\gamma^0\right) S^\vee(z_2,x)\, d^3z_1\, d^3z_2, \end{aligned} \tag{5.19}$$

where $t_{1/2} \equiv z^0_{1/2}$ are the time components of $z_{1/2}$.

In the limit $\varepsilon \searrow 0$, the function $P^\varepsilon(z_1,z_2)$ becomes singular if $z_1 = z_2$ (see (4.26) in the case $t = r = 0$). Moreover, the singular supports of the distributions $S^\wedge(x,.)$ and $S^\vee(.,x)$ coincide (see figure 4 in the case $x = y$). As a consequence, the integral in (5.19) diverges as $\varepsilon \searrow 0$, having poles in ε. The orders of these poles can be obtained by a simple power counting. In order to analyze the structure of these poles in more detail, one performs the Hadamard expansion of the distributions $S^\wedge$ and $S^\vee$ (see [2, Section 2] or the next section of the present paper for similar calculations for the fermionic operator). Substituting the resulting formulas into (5.19), one finds that the higher orders in the Hadamard expansion give rise to lower order poles in ε. In particular, the most singular contribution to (5.19) is obtained simply by taking the first term of the Hadamard expansion of the Green's function $S^\wedge(x,z_1)$, which is a scalar multiple of the parallel transport with respect to the spinorial Levi–Civita connection along the unique null geodesic joining z_1 and x. Similarly, the Green's function $S^\vee(z_2,x)$ may be replaced by a multiple of the parallel transport along the null geodesic joining z_2 with x. Moreover, for the most singular contribution to (5.19) it suffices to consider the lowest order in m, which means that we may disregard the factors $2m\gamma^0$ in (5.19). Finally, we know that in the limit $z_1 \to z_2$, the leading contribution to $P^\varepsilon(z_1,z_2)$ is proportional to γ^0 (see (5.7) and (4.15)). Putting these facts together, the most singular contribution to $P^\varepsilon(x,x)$ is obtained simply by taking the operator γ^0 at $z_1 = z_2 = z$ and to parallel transport it along the null geodesic joining z with x. Integrating z over the set $\partial J^\wedge(x) \cap \mathcal{N}$, we obtain the desired operator of Clifford multiplication in (5.18). □

5.2 The Hadamard expansion of the fermionic operator

In this section we shall analyze the singularity structure of the distribution P introduced in Definition 5.3 by performing the so-called Hadamard expansion. In order to be able to apply the methods worked out in [2, Section 2], it is preferable to first consider the second-order equation (5.10). The following lemma relates P to a solution of (5.10).

Lemma 5.6. *Let T be the unique symmetric distributional bi-solution of the Klein–Gordon equation* (5.10)*, which coincides with the Fourier transform of the lower mass shell*

$$T(x, y) = \int \frac{d^4k}{(2\pi)^4}\, \delta(k^2 - m^2)\, \Theta(-k^0)\, \mathrm{e}^{-\mathrm{i}k(x-y)} \tag{5.20}$$

for x and y in the past of $\mathcal{N}$. Then

$$P(\psi, \phi) = T\big((\mathcal{D} + m)\psi, \phi\big). \tag{5.21}$$

Proof. We introduce the distribution P_m by

$$P_m(\psi, \phi) = \frac{1}{2m}\, T\big((\mathcal{D} + m)\psi, (\mathcal{D} + m)\phi\big).$$

Obviously, P_m is symmetric and satisfies the Dirac equation (5.14). Moreover, a short calculation using (5.20) and (4.24) shows that P_m coincides with the regularized Dirac sea vacuum (4.22) if ψ and ϕ are both supported in the past of $\mathcal{N}$. We conclude that P_m coincides with the distribution P of Definition 5.3. Obviously, P_m is also a bi-solution of the Klein–Gordon equation. Flipping the sign of m, we get another bi-solution P_{-m} of the Klein–Gordon equation. Again using (5.20) and (4.24), we find that the following combination of P_m and P_{-m} coincides with T:

$$T = \frac{1}{2m}(P_m - P_{-m}). \tag{5.22}$$

The fact that the operator P_m is a bi-solution of the Dirac equation implies that it commutes with $\mathcal{D}$,

$$P_m(\mathcal{D}\psi, \phi) = m P_m(\psi, \phi) = P_m(\psi, \mathcal{D}\phi) \tag{5.23}$$

and similarly for P_{-m}. We thus obtain

$$\begin{aligned} P(\psi, \phi) &= \frac{1}{2m} T\big((\mathcal{D} + m)\psi, (\mathcal{D} + m)\phi\big) \\ &\overset{(5.22)}{=} \frac{1}{(2m)^2}\Big(P_m\big((\mathcal{D} + m)\psi, (\mathcal{D} + m)\phi\big) \\ &\qquad - P_{-m}\big((\mathcal{D} + m)\psi, (\mathcal{D} + m)\phi\big)\Big) \end{aligned}$$

$$\stackrel{(5.23)}{=} \frac{1}{(2m)^2}\Big(P_m\big((\mathcal{D}+m)^2\psi,\phi\big) - P_{-m}\big((\mathcal{D}+m)^2\psi,\phi\big)\Big)$$
$$= \frac{1}{2m} T\big((\mathcal{D}+m)^2\psi,\phi\big) = T\big((\mathcal{D}+m)\psi,\phi\big),$$

giving the result. □

We now perform the Hadamard expansion of the distribution T using the methods of [9, 24, 33, 31, 2]. Assume that $\Omega \subset M$ is a geodesically convex subset (see [2, Definition 1.3.2]). Then for any $x, y \in \Omega$, there is a unique geodesic c in Ω joining y and x. We denote the squared length of this geodesic by

$$\Gamma(x,y) = g\big(\exp_y^{-1}(x), \exp_y^{-1}(x)\big)$$

(note that Γ is positive in timelike directions and negative in spacelike directions) and remark that the identity

$$g(\text{grad}_x \Gamma, \text{grad}_x \Gamma) = 4\Gamma \tag{5.24}$$

holds. In order to prescribe the behavior of the singularities on the light cone, we set

$$\Gamma_\varepsilon(x,y) = \Gamma + i\varepsilon\big(\mathfrak{t}(x) - \mathfrak{t}(y)\big)$$

and introduce the short notation

$$\frac{1}{\Gamma^p} = \lim_{\varepsilon \searrow 0} \frac{1}{(\Gamma_\varepsilon)^p} \quad \text{and} \quad \log\Gamma = \lim_{\varepsilon\searrow 0} \log\Gamma_\varepsilon = \log|\Gamma| - i\pi\,\epsilon\big(\mathfrak{t}(x) - \mathfrak{t}(y)\big) \tag{5.25}$$

(where ϵ is again the step function), with convergence in the distributional sense. Here the logarithm is cut along the positive real axis, with the convention

$$\lim_{\varepsilon\searrow 0} \log(1 + i\varepsilon) = -i\pi.$$

In the past of $\mathcal{N}$, this prescription gives the correct singular behavior of the distribution (4.24) on the light cone (for details see [12, equations (2.5.39)–(2.5.41)]). Using the methods of [24], it follows that this prescription holds globally. We remark that the rule (5.25) also implements the local spectral condition in [33].

In [2, Section 2], the Hadamard expansion is worked out in detail for the causal Green's functions of a normally hyperbolic operator. Adapting the methods and results in a straightforward way to the distribution T, we obtain the Hadamard expansion

$$\begin{aligned}(-8\pi^3)\, T(x,y) = \frac{\mathcal{V}}{\Gamma}\, \Pi_x^y + \frac{\log(\Gamma)}{4}\, V_x^y \\ + \Gamma \log(\Gamma)\, W_x^y + \Gamma H_x^y + \mathcal{O}(\Gamma^2 \log \Gamma)\end{aligned} \tag{5.26}$$

(the normalization constant $(-8\pi^3)$ can be read off from (4.26) and (4.31), because $T(x,y)$ coincides with β/m if x and y are in the past of $\mathcal{N}$). Here $\mathcal{V}(x,y)$ is the square root of the van Vleck–Morette determinant (see for example [31]), which in normal coordinates around y is given by

$$\mathcal{V}(x,y) = |\det(g(x))|^{-\frac{1}{4}}. \tag{5.27}$$

Moreover, $\Pi_x^y : S_yM \to S_xM$ denotes the spinorial Levi–Civita parallel transport along c. The linear mappings $V_x^y, W_x^y, H_x^y : S_yM \to S_xM$ are called Hadamard coefficients. They depend smoothly on x and y, and can be determined via the Hadamard recurrence relations [9]. The Hadamard coefficient V_x^y is given explicitly by formula (A.17) in the appendix. Writing the result of Lemma 5.6 with distributional derivatives as $P(x,y) = (\mathcal{D}_x + m)T(x,y)$, we obtain the Hadamard expansion of P by differentiation.

Corollary 5.7. *The distribution $P(x,y)$ has the Hadamard expansion*

$$\begin{aligned}(-8\pi^3)\, P(x,y) = -\frac{i\,\mathcal{V}}{\Gamma^2}\, \mathrm{grad}_x \Gamma \cdot \Pi_x^y + \frac{i}{\Gamma}\, \mathrm{grad}_x \mathcal{V} \cdot \Pi_x^y + \frac{\mathcal{V}}{\Gamma}(\mathcal{D}_x + m)\Pi_x^y \\ + \frac{i}{4\Gamma}\, \mathrm{grad}_x \Gamma \cdot V_x^y + \frac{\log(\Gamma)}{4}\, (\mathcal{D}_x + m) V_x^y \\ + i(1 + \log(\Gamma))\, \mathrm{grad}_x \Gamma \cdot W_x^y \\ + i\, \mathrm{grad}_x \Gamma \cdot H_x^y + \mathcal{O}(\Gamma \log \Gamma).\end{aligned} \tag{5.28}$$

5.3 The fermionic operator along timelike curves

Assume that $\gamma(t)$ is a future-directed, timelike curve which joins two spacetime points $p, q \in M$. For simplicity, we parametrize the curve by arc length on the interval $[0, t_{\max}]$ such that $\gamma(0) = q$ and $\gamma(t_{\max}) = p$. For any given N, we define the points $x_0, \ldots, x_N$ by

$$x_n = \gamma(t_n) \quad \text{with} \quad t_n = \frac{n}{N}\, t_{\max}. \tag{5.29}$$

Note that these points are all timelike separated, and that the geodesic distance of neighboring points is of the order $1/N$. In this section, we want to compute $P(x_{n+1}, x_n)$ in powers of $1/N$. To this end, we consider the Hadamard expansion of Corollary 5.7 and use that $0 < \Gamma(x_{n+1}, x_n) \in \mathcal{O}(1/N)$. Thus our main task is to expand the Hadamard coefficients in (5.28) in powers of $1/N$. For ease in notation, we set

$$x = x_{n+1} \quad \text{and} \quad y = x_n.$$

Possibly by increasing N, we can arrange that x and y lie in a geodesically convex subset $\Omega \subset M$. We let c be the unique geodesic in Ω joining y and x,

$$c : [0,1] \to M, \quad c(\tau) := \exp_y(\tau T) \quad \text{with } T := \exp_y^{-1}(x). \tag{5.30}$$

We also introduce the expansion parameter

$$\delta := \sqrt{\Gamma(x,y)} = \sqrt{g(T,T)} \in \mathcal{O}\left(\frac{1}{N}\right).$$

Next, we let $\{e_0 = \delta^{-1} T,\, e_1,\, e_2,\, e_3\}$ be a pseudo-orthonormal basis of $T_y M$, i.e.,

$$g(e_j, e_k) = \epsilon_j \, \delta_{jk},$$

where the signs ϵ_j are given by

$$\epsilon_j := \begin{cases} +1 & \text{if } j = 0, \\ -1 & \text{if } j = 1,2,3. \end{cases} \tag{5.31}$$

We extend this basis to a local pseudo-orthonormal frame of $T\Omega$ by

$$e_j(z) = \Lambda_z^y \, e_j, \tag{5.32}$$

where Λ_z^y denotes the Levi–Civita parallel transport in TM along the unique geodesic in Ω joining y and z. Then the following propositions hold.

Proposition 5.8. *The kernel of the fermionic operator has the expansion*

$$(-8\pi^3)\, P(x,y) = -\frac{i}{\Gamma^2}\, \mathrm{grad}_x \Gamma \cdot \Pi_x^y + \frac{m}{\Gamma}\, \Pi_x^y + \mathcal{O}(\delta^{-1}). \tag{5.33}$$

Proposition 5.9. *The closed chain has the expansion*

$$
\begin{aligned}
(-8\pi^3)^2\, A_{xy} = c(x,y)\, \mathbb{1}_{S_xM} && (5.34)\\
+ m \left(m^2 - \frac{\mathrm{scal}}{12} \right) \frac{\mathrm{Im}(\log \Gamma)}{2\Gamma^2}\, \mathrm{grad}_x \Gamma && (5.35)\\
+ i \big[\mathrm{grad}_x \Gamma, X_{xy} \big] + \big\{ \mathrm{grad}_x \Gamma, Y_{xy} \big\} && (5.36)\\
+ \mathcal{O}(\delta^{-1} \log \delta),
\end{aligned}
$$

where all operators act on S_xM. Here X_{xy} and Y_{xy} are symmetric linear operators and

$$X_{xy} = \mathcal{O}(\delta^{-3}), \quad Y_{xy} = \mathcal{O}(\delta^{-3} \log \delta).$$

The proof of Propositions 5.8 and 5.9 is given in Appendix A, where we also compute some of the Hadamard coefficients explicitly in terms of curvature expressions.

Note that the contribution (5.35) is of the order δ^{-3}, whereas (5.36) is of the order $\mathcal{O}(\delta^{-2} \log \delta)$. The term (5.35) amounts to Clifford multiplication with $\mathrm{grad}_x \Gamma$ and is thus analogous to the term $a(\xi)\, \not{\xi}$ in the closed chain (4.32) of Minkowski space. The contributions (5.36) will be discussed in detail in the next section.

5.4 The unspliced versus the spliced spin connection

In this section, we compute the unspliced and spliced spin connections and compare them. We write the results of Corollary 5.7 and Proposition 5.9 as

$$(-8\pi^3)\, P(x,y) = -\frac{\mathrm{i}}{\Gamma^2}\, \mathrm{grad}_x \Gamma \cdot \Pi^y_x + \frac{m}{\Gamma}\, \Pi^y_x + \mathcal{O}(\delta^{-1}) \tag{5.37}$$

$$
\begin{aligned}
(-8\pi^3)^2\, A_{xy} &= c_{xy} + a_{xy}\, \mathrm{grad}_x \Gamma + \mathrm{i} \big[\mathrm{grad}_x \Gamma, X_{xy} \big] \\
&\quad + \big\{ \mathrm{grad}_x \Gamma, Y_{xy} \big\} + \mathcal{O}\big(\delta^{-1} \log \delta \big), \qquad (5.38)
\end{aligned}
$$

where

$$a_{xy} \sim \delta^{-4} \quad \text{and} \quad X_{xy} \sim \delta^{-3}.$$

We want to compute the directional sign operator v_{xy} (see Definition 3.15) in an expansion in powers of δ. To this end, we first remove the commutator term in (5.38) by a unitary transformation,

$$(-8\pi^3)^2\, \mathrm{e}^{-\mathrm{i}Z_{xy}}\, A_{xy}\, \mathrm{e}^{\mathrm{i}Z_{xy}} = c_{xy} + a_{xy}\, \operatorname{grad}_x \Gamma + \big\{ \operatorname{grad}_x \Gamma, Y_{xy} \big\} + \mathcal{O}\big(\delta^{-1} \log \delta\big), \tag{5.39}$$

where we set

$$Z_{xy} = -\frac{X_{xy}}{a_{xy}} \sim \delta. \tag{5.40}$$

We let $u \in T_xM$ be a future-directed timelike unit vector pointing in the direction of $\operatorname{grad}_x \Gamma$. Then the operator u (acting by Clifford multiplication) is a sign operator (see Definition 3.5), which obviously commutes with the right-hand side of (5.39). Hence, the directional sign operator (see Definition 3.15) is obtained from u by unitarily transforming backwards,

$$v_{xy} = \mathrm{e}^{\mathrm{i}Z_{xy}}\, u\, \mathrm{e}^{-\mathrm{i}Z_{xy}} = u + i\big[Z_{xy}, u\big] + \mathcal{O}\big(\delta^2 \log^2 \delta\big). \tag{5.41}$$

In order to construct the synchronization map at x, it is convenient to work with the distinguished subspace $\mathfrak{K}(x)$ of $\operatorname{Symm}(S_xM)$ spanned by the operators of Clifford multiplication with the vectors $e_0, \ldots, e_3 \in T_xM$ and the pseudoscalar operator $e_4 = -e_0 \cdots e_3$ (thus in the usual Dirac representation, $\mathfrak{K} = \langle \gamma^0, \ldots, \gamma^3, i\gamma^5 \rangle$). The subspace $\mathfrak{K}$ is a distinguished Clifford subspace (see Definitions 3.1 and 3.31). The inner product (3.1) extends the Lorentzian metric on T_xM to $\mathfrak{K}(x)$. The space $\operatorname{Symm}(S_xM)$ is spanned by the 16 operators $\mathbb{1}$, e_j and $\sigma_{jk} = \frac{i}{2}[e_j, e_k]$ (where $j,k \in \{0, \ldots, 4\}$), giving the basis representation

$$Z_{xy} = c + \sum_{j,k=0}^{4} B^{jk} \sigma_{jk} + \sum_{j=0}^{4} w^j e_j.$$

The first summand is irrelevant as it drops out of the commutator in (5.41). The second summand gives a contribution to v_{xy}, which lies in the distinguished Clifford subspace,

$$\Delta u := i \sum_{j,k=0}^{4} \big[B^{jk} \sigma_{jk}, u\big] = 4 \sum_{j,k=0}^{4} B^{jk} u_j\, e_k \in \mathfrak{K}, \tag{5.42}$$

whereas the last summand gives a bilinear contribution

$$\mathrm{i}\big[w, u\big] \quad \text{with} \quad w := \sum_{j=0}^{4} w^j e_j.$$

We thus obtain the representation

$$v_{xy} = u + \Delta u + \mathrm{i}\big[w, u\big] + \mathcal{O}\big(\delta^2 \log^2 \delta\big). \tag{5.43}$$

We next decompose $w \in \mathfrak{K}$ as the linear combination

$$w = \alpha\, u + \beta\, s(x) + \rho \quad \text{with} \quad \rho \perp u \quad \text{and} \quad \rho \perp s(x). \tag{5.44}$$

If u and $s(x)$ are linearly dependent, we choose $\beta = 0$. Otherwise, the coefficients α and β are uniquely determined by the orthogonality conditions. Substituting this decomposition into (5.43), we obtain

$$v_{xy} = \mathrm{e}^{\mathrm{i}\rho}\, \mathrm{e}^{\mathrm{i}\beta s(x)} \,\big(u + \Delta u\big)\, \mathrm{e}^{-\mathrm{i}\beta s(x)}\, \mathrm{e}^{-\mathrm{i}\rho} + \mathcal{O}\big(\delta^2 \log^2 \delta\big). \tag{5.45}$$

Comparing with Lemma 3.12 and Definition 3.13, one finds that $\mathrm{e}^{\mathrm{i}\rho}$ is the synchronization map $U^{u,s(x)}$ at x. The mapping $\mathrm{e}^{\mathrm{i}\beta s(x)}$, on the other hand, identifies the representatives $\mathfrak{K}, K_x^{(y)} \in \mathcal{T}^{s_x}$ of the tangent space T_x (see Definition 3.9). Using the notation introduced after Definition 3.15 and at the beginning of Section 3.4, we have $U_{xy} = \mathrm{e}^{\mathrm{i}\rho}$ and

$$K_{xy} = \mathrm{e}^{\mathrm{i}\rho}\, K_x^{(y)}\, \mathrm{e}^{-\mathrm{i}\rho} \quad \text{and} \quad K_x^{(y)} = \mathrm{e}^{\mathrm{i}\beta s(x)}\, \mathfrak{K}(x)\, \mathrm{e}^{-\mathrm{i}\beta s(x)}. \tag{5.46}$$

We next compute the synchronization map at the point y. Since the matrices A_{xy} and A_{yx} have the same characteristic polynomial, we know that

$$v_{xy}\, P(x,y) = P(x,y)\, v_{yx}.$$

Multiplying by

$$(-8\pi^3)^{-1}\, P(x,y)^{-1} = \frac{\mathrm{i}}{4}\, \Gamma\, \Pi^x_y\, \mathrm{grad}_x\, \Gamma - \frac{m}{4}\, \Gamma^2\, \Pi^x_y + \mathcal{O}(\delta^5)$$

(where we used (5.37) and (5.24)), a direct calculation using (5.43) gives

$$v_{yx} = \Pi^x_y \Big(u - \Delta u - \mathrm{i}\big[w, u\big] \Big)\, \Pi^y_x + \mathcal{O}\big(\delta^2 \log^2 \delta\big).$$

Using that $s(y) = \Pi^x_y\, s(x)\, \Pi^y_x + \mathcal{O}(\delta)$, we obtain similar to (5.46)

$$\begin{aligned} K_{yx} &= \Pi^x_y\, \mathrm{e}^{-\mathrm{i}\rho} \Pi^y_x\, K_y^{(x)}\, \Pi^x_y\, \mathrm{e}^{\mathrm{i}\rho} \Pi^y_x \quad \text{and} \\ K_y^{(x)} &= \Pi^x_y\, \mathrm{e}^{-\mathrm{i}\beta s(x)} \Pi^y_x\, \mathfrak{K}(y)\, \Pi^x_y\, \mathrm{e}^{\mathrm{i}\beta s(x)} \Pi^y_x. \end{aligned} \tag{5.47}$$

We are now ready to compute the spin connections introduced in Definitions 3.17 and 3.32.

Proposition 5.10. *The unspliced and spliced spin connections are given by*

$$D_{x,y} = \Big(\mathbb{1} + (\Delta u)\cdot u + 2\mathrm{i}\,(\beta\, s(x) + \rho)\Big)\, \Pi_x^y + \mathcal{O}(\delta^2 \log^2 \delta) \tag{5.48}$$

$$D_{(x,y)} = \Big(\mathbb{1} + (\Delta u)\cdot u\Big)\, \Pi_x^y + \mathcal{O}(\delta^2 \log^2 \delta). \tag{5.49}$$

Proof. We first compute the unspliced spin connection using the characterization of Theorem 3.20. A short calculation using (5.37) and (5.43) gives

$$\begin{aligned}
(-8\pi^3)^2\, A_{xy} &= \frac{4}{\Gamma^3} + \mathcal{O}(\delta^{-4}), \\
A_{xy}^{-\frac{1}{2}}\, P(x,y) &= -\Big(\big|-8\pi^3\big|^{-1} A_{xy}^{-\frac{1}{2}}\Big)\, \Big((-8\pi^3)\, P(x,y)\Big) \\
&= \frac{i}{2}\, \Gamma^{-\frac{1}{2}}\, \mathrm{grad}_x\, \Gamma \cdot \Pi_x^y - \frac{m}{2}\, \Gamma^{\frac{1}{2}}\, \Pi_x^y + \mathcal{O}(\delta^2) \\
&= iu \cdot \Pi_x^y - \frac{m}{2}\, \Gamma^{\frac{1}{2}}\, \Pi_x^y + \mathcal{O}(\delta^2).
\end{aligned}$$

In order to evaluate the condition (ii) of Theorem 3.20, it is easiest to transform the Clifford subspaces K_{xy} and K_{yx} to the distinguished Clifford subspace $\mathfrak{K}(x)$ and $\mathfrak{K}(y)$, respectively. In view of (5.46) and (5.47), we can thus rewrite the condition (ii) of Theorem 3.20 by demanding that the unitary transformation

$$V := \mathrm{e}^{-\mathrm{i}\beta s(x)}\, \mathrm{e}^{-\mathrm{i}\rho}\, \Big(\mathrm{e}^{\mathrm{i}\varphi\, v_{xy}}\, A_{xy}^{-\frac{1}{2}}\, P(x,y)\Big)\, \Pi_y^x\, \mathrm{e}^{-\mathrm{i}\rho}\, \mathrm{e}^{-\mathrm{i}\beta s(x)}\, \Pi_x^y$$

transforms the distinguished Clifford subspaces to each other,

$$V\, \mathfrak{K}(y)\, V^{-1} = \mathfrak{K}(x). \tag{5.50}$$

The operator V is computed by

$$\begin{aligned}
V &= \mathrm{e}^{-\mathrm{i}\beta s(x)}\, \mathrm{e}^{-\mathrm{i}\rho}\, \mathrm{e}^{\mathrm{i}\varphi\, v_{xy}}\, \Big(iu - \frac{m}{2}\, \Gamma^{\frac{1}{2}}\Big)\, \mathrm{e}^{-\mathrm{i}\rho}\, \mathrm{e}^{-\mathrm{i}\beta s(x)}\, \Pi_x^y + \mathcal{O}(\delta^2) \\
&\overset{(5.45)}{=} \mathrm{e}^{\mathrm{i}\varphi\,(u+\Delta u)}\, \mathrm{e}^{-\mathrm{i}\beta s(x)}\, \mathrm{e}^{-\mathrm{i}\rho}\, \Big(iu - \frac{m}{2}\, \Gamma^{\frac{1}{2}}\Big)\, \mathrm{e}^{-\mathrm{i}\rho}\, \mathrm{e}^{-\mathrm{i}\beta s(x)}\, \Pi_x^y + \mathcal{O}(\delta^2) \\
&= \Big\{\mathrm{e}^{\mathrm{i}\varphi\,(u+\Delta u)}\, \Big(\mathrm{i}u - \frac{m}{2}\, \Gamma^{\frac{1}{2}} + 2\beta\, \langle u, s(x)\rangle\Big)\Big\}\, \Pi_x^y + \mathcal{O}(\delta^2 \log^2 \delta).
\end{aligned} \tag{5.51}$$

Now the condition (5.50) means that the curly brackets in (5.51) describe an infinitesimal Lorentz transformation on $\mathfrak{K}(x)$. Thus the brackets must only have a scalar and a bilinear contribution, but no vector contribution. This leads us to choose φ such that

$$\begin{aligned}\sin\varphi = -1 + \mathcal{O}(\delta^2 \log^2 \delta), \quad \cos\varphi = -\frac{m}{2}\,\Gamma^{\frac{1}{2}} + 2\beta\,\langle u, s(x)\rangle \\ + \mathcal{O}(\delta^2 \log^2 \delta) < 0 \end{aligned} \tag{5.52}$$

(note that this choice of φ is compatible with our convention (3.44)). It follows that

$$\begin{aligned} V &= \Big(\mathbb{1} + (\Delta u)\cdot u\Big)\Pi^y_x + \mathcal{O}(\delta^2 \log^2 \delta) \\ D_{x,y}\,\Pi^x_y &= e^{i\varphi\, v_{xy}}\, A_{xy}^{-\frac{1}{2}}\, P(x,y)\,\Pi^x_y \\ &= \mathbb{1} + (\Delta u)\cdot u + i[w,u]\,u + 2i\beta\,\langle u, s(x)\rangle\, u + \mathcal{O}(\delta^2 \log^2 \delta) \\ &\overset{(5.44)}{=} \mathbb{1} + (\Delta u)\cdot u + 2i\,(\beta\, s(x) + \rho) + \mathcal{O}(\delta^2 \log^2 \delta). \end{aligned}$$

Finally, using the notions of Definition 3.32, we obtain

$$U^{|y)}_x = e^{-i\beta s(x)}\, e^{-i\rho}, \quad U^{(x|}_y = e^{-i\rho}\, e^{-i\beta s(x)} \tag{5.53}$$

$$D_{(x,y)} = U^{|y)}_x\, D_{x,y}\, U^{(x|}_y = \mathbb{1} + (\Delta u)\cdot u + \mathcal{O}(\delta^2 \log^2 \delta), \tag{5.54}$$

completing the proof. □

The terms in the statement of the above proposition are quantified in the next lemma, which is again proven in the appendix.

Lemma 5.11. *The linear operators* Δu *and* $\beta s + \rho$ *in* (5.49) *and* (5.48) *have the expansions*

$$\Delta u = \frac{1}{6\,\delta}\left(m^2 - \frac{\mathrm{scal}}{12}\right)^{-1} \epsilon_j\,(\nabla_{e_j} R)(T, e_j)\,T + \mathcal{O}(\delta^2 \log^2 \delta) \tag{5.55}$$

$$\begin{aligned}\beta s + \rho = \left[\mathcal{O}\left(\frac{1}{m}\,\|\epsilon_j\ \mathrm{Ric}(T, e_j)\,e_j\|\right) + \mathcal{O}\left(\frac{\delta}{m^3}\,\big(\|R\|^2 + \|\nabla^2 R\|\big)\right)\right] \\ \times\left(1 + \mathcal{O}\left(\frac{\mathrm{scal}}{m^2}\right)\right) + \mathcal{O}(\delta^2 \log^2 \delta). \end{aligned} \tag{5.56}$$

Let us discuss these formulas. We first point out that all the terms in (5.55) and (5.56) are of the order $\mathcal{O}(\delta)$. Thus the corresponding correction terms in (5.48) and (5.49) are also of the order $\mathcal{O}(\delta)$. In the next section, we shall see that adding up all these correction terms along a time-like curve will give a finite deviation from the spinorial Levi-Civita parallel transport. If we assume furthermore that the Compton scale is much smaller than the length scale where curvature effects are relevant,

$$\frac{\|\nabla^2 R\|}{m^4} \ll \frac{\|\nabla R\|}{m^3} \ll \frac{\|R\|}{m^2} \ll 1, \tag{5.57}$$

then this deviation will even be small. More specifically, the term involving the Ricci tensor in (5.56) is the leading correction term. As shown in Proposition 5.10, this leading correction enters the unspliced spin connection, but drops out of the spliced spin connection. This explains why it is preferable to work with the spliced spin connection.

The above calculations also reveal another advantage of splicing: The corrections in the spliced spin connection are bilinear contributions (see (5.49) and (5.55)) and can thus be interpreted as describing an infinitesimal Lorentz transformation. However, the corrections in the unspliced spin connection (see (5.48) and (A.55)) involve vector contributions, which have the unpleasant feature that they do not leave the distinguished Clifford subspaces $\mathfrak{K}(x)$ invariant.

5.5 Parallel transport along timelike curves

We are now in the position to prove the main theorem of this section. We return to the setting at the beginning of Section 5.3 and consider a future-directed, timelike curve γ which joins two space-time points $p, q \in M$. For any given N, we again define the intermediate points $x_0, \ldots, x_N$ by (5.29). We then define the parallel transport $D^{N,\varepsilon}_{xy}$ by successively composing the spliced spin connection between neighboring points,

$$D^{N,\varepsilon}_{(p,q)} := D^{\varepsilon}_{(x_N, x_{N-1})}\, D^{\varepsilon}_{(x_{N-1}, x_{N-2})} \,\cdots\, D^{\varepsilon}_{(x_1, x_0)} \;:\; S_q \to S_p, \tag{5.58}$$

where D^ε is the spliced spin connection induced from the regularized fermionic operator P^ε. Substituting the formulas (5.49) and (5.55), one gets N correction terms $(\Delta u)\cdot u$, each of which is of the order $\delta \sim N^{-1}$. Thus in the limit $N \to \infty$, we get a finite correction, which we now compute.

Theorem 5.12. *Let (M, g) be a globally hyperbolic manifold which is isometric to Minkowski space in the past of a Cauchy-hypersurface $\mathcal{N}$. Then the admissible curves (see Definition 4.6) are dense in the C^∞-topology. Choosing $N \in \mathbb{N}$ and $\varepsilon > 0$ such that the points x_n and x_{n-1} are spin-connectable* *for all $n = 1, \ldots, N$, every point x_n lies in the future of x_{n-1}. Moreover,*

$$\lim_{N\to\infty} \lim_{\varepsilon\searrow 0} D^{N,\varepsilon}_{(p,q)} = D^{\texttt{LC}}_{p,q}\ \mathrm{Texp}\left(\frac{1}{6}\int_\gamma \left(m^2 - \frac{\mathrm{scal}}{12}\right)^{-1} \times D^{\texttt{LC}}_{q,\gamma(t)}\Big[\epsilon_j\,(\nabla_{e_j} R)\big(\dot\gamma(t), e_j\big)\,\dot\gamma(t)\Big]\cdot\dot\gamma(t)\cdot D^{\texttt{LC}}_{\gamma(t),q}\,dt\right),$$

where $\gamma(t)$ is a parametrization by arc length, and $D^{\texttt{LC}}_{p,q}$ denotes the parallel transport along γ with respect to the spinorial Levi–Civita connection, and Texp *is the time-ordered exponential (we here again identify S^ε_x and S_xM via* (5.6)*).*

Proof. Substituting the formula (5.49) into (5.58), one gets a product of N linear operators. Taking the limit $N \to \infty$ and using that differential quotients go over to differentials, one obtains a solution of the linear ordinary differential equation

$$\frac{d}{dt} D_{(\gamma(t),q)} = \left(\lim_{\delta\searrow 0} \frac{1}{\delta}\,(\Delta u)\cdot u\right)\cdot D_{(\gamma(t),q)}.$$

Here the limit $\delta \searrow 0$ can be computed explicitly using (5.55). Then the differential equation can be solved in terms of the time-ordered exponential (also called Dyson series; see [36, Sections 1.2.1 and 7.17.4]). This gives the result. ☐

This theorem shows that in the limit $\varepsilon \searrow 0$ and locally in the neighborhood of a given space-time point, the spliced spin connection reduces to the spinorial Levi–Civita connection, up to a correction term which involves line integrals of derivatives of the Riemann tensor along γ. Computing the holonomy of a closed curve, one sees that the corresponding spliced spin curvature equals the Riemann curvature, up to higher order curvature corrections.

For clarity, we point out that the above theorem does not rely on the fact that we are working with distinguished representatives of the tangent spaces. Namely, replacing (5.58) by the products of the unspliced spin connection with intermediate splice maps,

$$D^{N,\varepsilon}_{p,q} := D^\varepsilon_{x_N,x_{N-1}}\,U^{(x_N|x_{N-2})}_{x_{N-1}}\,D^\varepsilon_{x_{N-1},x_{N-2}}\,U^{(x_{N-1}|x_{N-3})}_{x_{N-2}}\,\cdots\,U^{(x_2|x_0)}_{x_{N-1}}\,D^\varepsilon_{x_1,x_0},$$

the above theorem remains true (to see this, we note that in view of (3.58) and (3.59), the parallel transports $D^{N,\varepsilon}_{p,q}$ and $D^{N,\varepsilon}_{(p,q)}$ differ only by the two factors $U^{|x_{N-1})}_{x_N}$ and $U^{(x_1|}_{x_0}$, which according to (5.53) and Lemma 5.11 converge to the identity matrix).

We now apply the above theorem to the metric connection.

Corollary 5.13. *Under the assumptions of Theorem* 5.12*, the metric connection and the Levi–Civita connection are related by*

$$\lim_{N\to\infty} \lim_{\varepsilon\searrow 0} \nabla^N_{x,y} - \nabla^{\text{LC}}_{x,y} = \mathcal{O}\left(L(\gamma)\, \frac{\|\nabla R\|}{m^2} \right) \left(1 + \mathcal{O}\left(\frac{\text{scal}}{m^2} \right) \right), \qquad (5.59)$$

where $L(\gamma)$ is the length of the curve γ, and

$$\nabla^{N,\varepsilon}_{p,q} := \nabla_{x_N,x_{N-1}}\, \nabla_{x_{N-1},x_{N-2}} \cdots \nabla_{x_1,x_0} \;:\; T_q \to T_p.$$

Proof. This follows immediately from Theorem 5.12 and the identity

$$\nabla^{N,\varepsilon}_{p,q} u_q = D^{N,\varepsilon}_{(p,q)} u_q \cdot D^{N,\varepsilon}_{(q,p)},$$

where we again identify the tangent space T_xM with the distinguished Clifford subspace $\mathfrak{K}(x)$ of $\text{Symm}(S_xM)$ (see after (5.41)). ☐

We finally discuss the notions of parity-preserving, chirally symmetric and future-transitive fermion systems (see Definitions 3.22, 3.26 and 3.34). Since our expansion in powers of δ only gives us information on $P(x,y)$ for nearby points x and y, we can only analyze local versions of these definitions. Then the expansion (5.52) shows that the fermion system without regularization is locally future-transitive and locally parity-preserving. Moreover, as the formula (5.49) only involves an even number of Clifford multiplications, the fermion system is locally chirally symmetric (with the vector field $u(x)$ in Definition 3.26 chosen as i times the pseudoscalar matrix), up to the error term specified in (5.49).

6 Outlook

We conclude by putting the previous constructions into a broader context and by mentioning possible directions of future research. We first point out that the assumptions (A) and (B) at the beginning of Section 5 should be considered only as a technical simplification. More generally, the fermionic operator can be introduced using a causality argument which gives a canonical splitting of the solution space of the Dirac equation into two subspaces. One of these subspaces extends the notion of the Dirac sea to interacting systems (see [12, Section 2.4]). Apart from the recent construction in a space-time of finite life-time [22], this method has been worked out only perturbatively in terms of the so-called causal perturbation expansion (see [19] and for linearized gravity [11, Appendix B]). This shortcoming was our motivation for the above assumptions (A) and (B), which made it possible to carry out all constructions non-perturbatively. To avoid confusion, we note that the fermionic operator constructed by solving the Cauchy problem (see Proposition 5.7) does in general *not* coincide with the physical fermionic operator obtained by the causal perturbation expansion in the same space-time (because solving the Cauchy problem for vacuum initial data is usually not compatible with the global construction in [12, equations (2.2.16) and (2.2.17)]). However, these two fermionic operators have the same singularity structure on the light cone, meaning that after removing the regularization, both fermionic operators have the same Hadamard expansion. Since in the constructions of Sections 5.2–5.5, we worked exclusively with the formulas of the Hadamard expansion, all the results in these sections immediately carry over to the physical fermionic operator.

We also point out that throughout this paper, we worked with the simplest possible regularization by a convergence generating factor $e^{-\varepsilon|k^0|}$ (see Lemma 4.2). More generally, one could consider a broader class of regularizations as introduced in [12, Section 4.1]. All our results will carry over, provided that the Euclidean operator has a suitable limit as $\varepsilon \searrow 0$ (similar to (4.20) and Lemma 5.5).

Our constructions could also be generalized to systems with several families of elementary particles (see [12, Section 2.3]). In this setting, only the largest mass will enter the conditions (5.1) and (5.57), so that it is indeed possible to describe physical systems involving fermions with an arbitrarily small or vanishing rest mass. Working with several generations also gives the freedom to perform local transformations before taking the partial trace (as is worked out in [15, Section 7.6] for axial potentials). This freedom can be used to modify the logarithmic poles of the fermionic operator on the light cone. In this context, an interesting future project is to study causal fermion systems in the presence of an electromagnetic field. We expect that the spin connection will then also include the $\mathrm{U}(1)$-gauge connection of electrodynamics.

Another direction of future research would be to study the geometry of causal fermion systems with regularization (i.e., without taking the limit

$\varepsilon \searrow 0$). It seems an interesting program to study the "quantum structure" of the resulting space-times.

From the mathematical point of view, the constructions in this paper extend the basic notions of Lorentzian spin geometry to causal fermion systems. However, most of the classical problems in geometric analysis and differential geometry have not yet been analyzed in our setting. For example, it has not yet been studied how "geodesics" are introduced in causal fermion systems, and whether such geodesics can be obtained by maximizing the "length of curves" (similar as in (5.58), such a "curve" could be a finite sequence of space-time points). Maybe the most important analytic problem is to get a connection between the geometric objects defined here and the causal action principle (see [12, 17, Section 3.5]). From the geometric point of view, our notions of connection and curvature describe the local geometry of space-time. It is a challenging open problem to explore how these local notions are related to the global geometry and topology of space-time.

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Appendix A The expansion of the Hadamard coefficients

In this section we will derive an expansion of the Hadamard coefficients in (5.28) in powers of δ. Using these expansions, we will then prove Proposition 5.8, Proposition 5.9 and Lemma 5.11. In terms of the pseudo-orthonormal frame e_j (see (5.32)), the Dirac operator on SM is given by

$$\mathcal{D}\psi = i\,\epsilon_j\, e_j \cdot \nabla_{e_j}\psi, \quad \text{with } \psi \in \Gamma(M, SM). \tag{A.1}$$

Here ∇ denotes the spinorial Levi–Civita connection, the dot denotes Clifford multiplication, and the signs ϵ_j are defined in (5.31). We denote space-time indices by Latin letters $j, k, \ldots \in \{0,1,2,3\}$, and spatial indices by Greek letters $\alpha, \beta, \ldots \in \{1,2,3\}$. Furthermore, we use Einstein's summation convention. In order to calculate the derivatives of the spinorial parallel transport Π_x^y with respect to the vectors e_j, we introduce suitable local coordinates. To this end, we consider the family of geodesics

$$c_s(t) := c(t, s_1, s_2, s_3) := \exp_y\big(tu + ts_\alpha e_\alpha\big), \tag{A.2}$$

where $u = \exp_y^{-1}(x) = \delta\, e_0$. The curve c_0 obviously coincides with the curve c defined in (5.30). The exponential map (A.2) also gives rise to local coordinates (t, s_α) around y, with corresponding local coordinate vector fields

$$T := \frac{\partial c_s}{\partial t} \quad \text{and} \quad Y_\alpha := \frac{\partial c_s}{\partial s_\alpha}. \tag{A.3}$$

The vector field T is the tangent field of the curves c_s, and in terms of e_j it is given by

$$T = \delta\, e_0 + s_\alpha\, e_\alpha. \tag{A.4}$$

Since in this appendix we always consider variations of the curve c_0, we can assume that $s_\alpha = \mathcal{O}(\delta)$, and thus

$$T = \mathcal{O}(\delta).$$

Moreover, the vector field T is timelike and the fields Y_α are spacelike. By definition of the vector fields e_j and of the spinorial parallel transport $\Pi^y_{c_s(t)}$, it also follows that

$$\nabla_T\, \Pi^y_{c_s(t)} = 0, \tag{A.5}$$

$$\nabla_T\, e_j|_{c_s(t)} = 0, \tag{A.6}$$

$$\nabla_{e_j} \Pi^y_{c_s(t)} = \mathcal{O}(\delta), \tag{A.7}$$

$$\nabla_{e_j} e_k|_{c_s(t)} = \mathcal{O}(\delta). \tag{A.8}$$

The vector fields Y_α are Jacobi fields, i.e., they are solutions of the Jacobi equation

$$\nabla_T \nabla_T Y_\alpha = R(T, Y_\alpha)\, T \tag{A.9}$$

with initial conditions

$$Y_\alpha|_{t=0} = 0 \quad \text{and} \quad \nabla_T Y_\alpha|_{t=0} = e_\alpha, \tag{A.10}$$

where R denotes the Riemann tensor on TM. This initial value problem can be solved perturbatively along each curve c_s, giving the expansion

$$Y_\alpha|_{c_s(t)} = t\, e_\alpha + \Lambda^y_{c_s(t)} \int_0^t d\tau\, \Lambda^{c_s(\tau)}_y \int_0^\tau d\sigma\, \sigma\, \Lambda^{c_s(\sigma)}_{c_s(\tau)} R(T, e_\alpha) T|_{c_s(\sigma)} + \mathcal{O}(\delta^4). \tag{A.11}$$

The spinorial curvature tensor $\mathcal{R}$ on SM is defined by the relation

$$\mathcal{R}(X,Y)\psi := \nabla_X \nabla_Y \psi - \nabla_Y \nabla_X \psi - \nabla_{[X,Y]}\psi,$$

valid for any $X, Y \in T_pM$ and $\psi \in \Gamma(M, SM)$. In the local pseudo-orthonormal frame (e_j), it takes the form

$$\mathcal{R}(X,Y)\,\psi = \frac{1}{4}\,\epsilon_j \epsilon_k\, g\big(R(X,Y)e_j, e_k\big)\; e_j \cdot e_k \cdot \psi. \tag{A.12}$$

Using (A.5) and the fact that the local coordinate vector fields T and Y_α commute, we conclude that

$$\nabla_T \nabla_{Y_\alpha} \Pi^y_{c(t)} = \mathcal{R}(T, Y_\alpha)\, \Pi^y_{c(t)}. \tag{A.13}$$

Integrating this equation gives

$$\nabla_{Y_\alpha} \Pi^y_{c(t)} = \Pi^y_{c(t)} \int_0^t \Pi_y^{c(\tau)} \mathcal{R}(T, Y_\alpha)|_{c(\tau)} \Pi^y_{c(\tau)} d\tau. \tag{A.14}$$

Using this formula, we can now derive the expansion of the Hadamard coefficient $\mathcal{D}_x \Pi^y_x$.

Lemma A.1. *The Hadamard coefficient $\mathcal{D}_x \Pi^y_x$ has the expansion*

$$\begin{aligned}
\mathcal{D}_x \Pi^y_x &= -\frac{i}{2}\,\epsilon_j \left(\int_0^1 t\,\mathrm{Ric}(T, e_j)|_{c(t)}\, dt \right)\, e_j \cdot \Pi^y_x \\
&\quad + \frac{i}{24}\,\epsilon_j \epsilon_p \epsilon_q \left(\epsilon_k\, g\big(R(e_j, T)T, e_k\big)\big|_x \int_0^1 t\, g\big(R(T, e_k)e_p, e_q\big)|_{c(t)}\, dt \right) \\
&\quad \times e_j \cdot e_p \cdot e_q \cdot \Pi^y_x + \mathcal{O}(\delta^4).
\end{aligned}$$

Proof. From (A.11) we conclude that

$$\begin{aligned}
Y_\alpha|_{c(t)} &= t e_\alpha|_{c(t)} + \frac{t^3}{6}\, R(T, e_\alpha)T|_{c(t)} + \mathcal{O}(\delta^3) \\
&= t e_\alpha|_{c(t)} + \frac{t^3}{6}\, \epsilon_k g\big(R(T, e_\alpha)T, e_k\big) e_k|_{c(t)} + \mathcal{O}(\delta^3),
\end{aligned}$$

where we performed a Taylor expansion of the integrand in (A.11) around $c(t)$. Thus

$$e_\alpha|_x = Y_\alpha - \frac{1}{6}\, g\big(R(e_\alpha, T)T, e_\beta)\big) Y_\beta + \mathcal{O}(\delta^3).$$

Next, from (A.14) we conclude that

$$\begin{aligned}
\nabla_{e_\alpha} \Pi_x^y &= \Pi_x^y \int_0^1 d\tau\, \tau \Pi_y^{c(\tau)} \mathcal{R}(T, e_\alpha)|_{c(\tau)} \Pi_{c(\tau)}^y \\
&\quad + \frac{1}{6}\, \epsilon_k\, g\big(R(e_\alpha, T)T, e_k)\big)\big|_x \Pi_x^y \int_0^1 d\tau\, \tau \Pi_y^{c(\tau)} \mathcal{R}(T, e_k)|_{c(\tau)}\, \Pi_{c(\tau)}^y \\
&\quad + \mathcal{O}(\delta^4). \qquad \text{(A.15)}
\end{aligned}$$

Representing the Dirac operator as in (A.1), we find

$$\begin{aligned}
\mathcal{D}_x \Pi_x^y &= i \Pi_x^y \int_0^1 d\tau\, \tau \Pi_y^{c(\tau)} \epsilon_j e_j \cdot \mathcal{R}(T, e_j)|_{c(\tau)} \Pi_{c(\tau)}^y \\
&\quad + \frac{i}{24}\, \epsilon_j \epsilon_p \epsilon_q \Big(\epsilon_k\, g\big(R(e_j, T)T, e_k\big)\big|_x \int_0^1 t\, g\big(R(T, e_k) e_p, e_q\big)|_{c(t)}\, dt \Big) \\
&\quad \times e_j \cdot e_p \cdot e_q \cdot \Pi_x^y + \mathcal{O}(\delta^4),
\end{aligned}$$

where we used (A.12) and the fact that the vector fields e_j are parallel along c. The result now follows from the identity

$$\epsilon_j e_j \cdot \mathcal{R}(e_j, X)\psi = \frac{1}{2} \epsilon_j \operatorname{Ric}(e_j, X) e_j \cdot \psi \quad \text{for } X \in T_pM \text{ and } \psi \in \Gamma(M, SM), \qquad \text{(A.16)}$$

which is easily verified by applying (A.12) as well as the first Bianchi identities. $\square$

We now compute the expansion of the coefficients V_x^y and $\mathcal{D}_x V_x^y$. The Hadamard recursion relations in [9, 2] yield that the first Hadamard coefficient is given by the formula

$$\begin{aligned}
V_{c(t)}^y &= -\frac{1}{t}\, \mathcal{V}(c(t), y)\, \Pi_{c(t)}^y \int_0^t d\tau\, \mathcal{V}^{-1}(c(\tau), y) \\
&\quad \times\, \Pi_y^{c(\tau)} \left(\Box_z^\nabla + \frac{\operatorname{scal}(z)}{4} - m^2 \right) \Big(\mathcal{V}(z, y)\, \Pi_z^y \Big)\Big|_{z = c(\tau)}. \qquad \text{(A.17)}
\end{aligned}$$

Note that this formula remains true if we replace the curve c by the curve c_s as defined in (A.2). For the computation of the term in (A.17) which contains the Bochner Laplacian, it is most convenient to work in local normal coordinates around y,

$$\Omega \ni p = \exp_y(x_j e_j). \tag{A.18}$$

The corresponding coordinate vector fields are given by

$$X_j := \frac{\partial}{\partial x_j}. \tag{A.19}$$

In these coordinates, the Bochner Laplacian is given by

$$\Box^\nabla \Pi^y_{c_s(t)} = -g^{jk} \nabla_{X_j} \nabla_{X_k} \Pi^y_{c_s(t)} + g^{jk} \nabla_{\nabla_{X_j} X_k} \Pi^y_{c_s(t)}, \tag{A.20}$$

where g^{jk} is the inverse matrix of $g_{jk} = g(X_j, X_k)$. Moreover, the vector fields X_j transform according to

$$X_0 = \frac{1}{\delta} T - \frac{s_\alpha}{t\delta} Y_\alpha \quad \text{and} \quad X_\alpha = \frac{1}{t} Y_\alpha, \tag{A.21}$$

where T and Y_α are the coordinate vector fields in (A.3). Also, from (A.4), (A.11) and (A.21), it follows that

$$X_j = e_j + \mathcal{O}(\delta^2). \tag{A.22}$$

More precisely, we have the following lemma for the expansion of the metric.

Lemma A.2. *In the local normal coordinates* (A.18)*, the metric g has the expansion*

$$\begin{aligned} g(X_j, X_k)|_{c_s(t)} = \epsilon_j \delta_{jk} &- \frac{t^2}{3}\, g\big(R(e_j, T)T, e_k\big)|_{c_s(t)} \\ &+ \frac{t^3}{6}\, g\big((\nabla_T R)(e_j, T)T, e_k\big)|_{c_s(t)} + \mathcal{O}(\delta^4). \end{aligned} \tag{A.23}$$

Proof. Inserting (A.11) into (A.21), we find

$$\begin{aligned} X_\alpha|_{c_s(t)} = e_\alpha|_{c_s(t)} &+ \frac{1}{t} \Lambda^y_{c_s(t)} \int_0^t d\tau\, \Lambda^{c_s(\tau)}_y \\ &\times \int_0^\tau d\sigma\, \sigma\, \Lambda^{c_s(\sigma)}_{c_s(\tau)} R(T, e_\alpha) T|_{c_s(\sigma)} + \mathcal{O}(\delta^4) \end{aligned}$$

$$
\begin{aligned}
&= e_\alpha|_{c_s(t)} + \frac{1}{t}\Lambda^y_{c_s(t)} \int_0^t d\tau\, \Lambda_y^{c_s(\tau)} \int_0^\tau d\sigma\, \sigma \Big(R(T,e_\alpha)T|_{c_s(\tau)} \\
&\quad + (\sigma-\tau)\nabla_T R(T,e_\alpha)T|_{c_s(\tau)} + \mathcal{O}(T^4)\Big) + \mathcal{O}(\delta^4) \\
&= e_\alpha|_{c_s(t)} + \frac{1}{t}\int_0^t d\tau \left(\frac{\tau^2}{2} R(T,e_\alpha)T|_{c_s(\tau)} - \frac{\tau^3}{6}\nabla_T R(T,e_\alpha)T|_{c_s(\tau)}\right) + \mathcal{O}(\delta^4) \\
&= e_\alpha|_{c_s(t)} + \frac{1}{t}\int_0^t d\tau \left(\frac{\tau^2}{2} R(T,e_\alpha)T|_{c_s(t)} \right. \\
&\quad \left. + \frac{\tau^2}{2}(\tau-t)\nabla_T R(T,e_\alpha)T|_{c_s(t)} - \frac{\tau^3}{6}\nabla_T R(T,e_\alpha)T|_{c_s(\tau)}\right) + \mathcal{O}(\delta^4) \\
&= e_\alpha|_{c_s(t)} + \frac{t^2}{6} R(T,e_\alpha)T|_{c_s(t)} - \frac{t^3}{12}(\nabla_T R)(T,e_\alpha)T|_{c_s(t)} + \mathcal{O}(\delta^4), \quad \text{(A.24)}
\end{aligned}
$$

where we expanded the integrands in a Taylor series around $c_s(t)$. Moreover, we used that $T = \mathcal{O}(\delta)$ and that T and e_j are parallel along the curve c_s. Substituting (A.4) and (A.24) into (A.21), we then find

$$
\begin{aligned}
X_0|_{c_s(t)} &= \frac{1}{\delta}T - \frac{s_\alpha}{t\delta}Y_\alpha = e_0 + \frac{s_\alpha}{\delta}e_\alpha|_{c_s(t)} - \frac{s_\alpha}{\delta}X_\alpha|_{c_s(t)} \\
&= e_0|_{c_s(t)} - \frac{t^2}{6\delta}R(T,s_\alpha e_\alpha)T|_{c_s(t)} + \frac{t^3}{12\delta}(\nabla_T R)(T,s_\alpha e_\alpha)T|_{c_s(t)} + \mathcal{O}(\delta^4) \\
&= e_0 - \frac{t^2}{6\delta}R(T,T-\delta e_0)T|_{c_s(t)} + \frac{t^3}{12\delta}(\nabla_T R)(T,T-\delta e_0)|_{c_s(t)} + \mathcal{O}(\delta^4) \\
&= e_0|_{c_s(t)} + \frac{t^2}{6}R(T,e_0)T|_{c_s(t)} - \frac{t^3}{12}(\nabla_T R)(T,e_0)T|_{c_s(t)} + \mathcal{O}(\delta^4). \quad \text{(A.25)}
\end{aligned}
$$

Thus, inserting (A.24) and (A.25) into the metric, we obtain

$$
\begin{aligned}
g(X_j,X_k) &= g(e_j,e_k) + \frac{t^2}{6}\, g\big(e_j, R(T,e_k)T\big) + \frac{1}{6}\, g\big(R(T,e_j)T, e_k\big) \\
&\quad - \frac{t^3}{12}\, g\big(e_j, (\nabla_T R)(T,e_k)T\big) - \frac{t^3}{12}\, g\big((\nabla_T R)(T,e_j)T, e_k\big) + \mathcal{O}(\delta^4) \\
&= \epsilon_j\delta_{jk} - \frac{t^2}{3}\, g\big(R(e_j,T)T,e_k\big) + \frac{t^3}{6}\, g\big((\nabla_T R)(e_j,T)T,e_k\big) + \mathcal{O}(\delta^4),
\end{aligned}
$$

where the first Bianchi identities were used in the last step. □

We now expand the function $\mathcal{V}$ and related terms in powers of δ.

Lemma A.3. *For the square root of the van Vleck–Morette determinant $\mathcal{V}$, the following expansions hold:*

$$\mathcal{V}(c_s(t), y) = 1 + \frac{t^2}{12}\,\mathrm{Ric}(T,T) - \frac{t^3}{24}(\nabla_T \,\mathrm{Ric})(T,T) + \mathcal{O}(\delta^4) \tag{A.26}$$

$$\partial_{Y_\alpha} \mathcal{V}\,|_{c_s(t)} = \frac{t^2}{6}\,\mathrm{Ric}(T, e_\alpha) + \mathcal{O}(\delta^2) \tag{A.27}$$

$$\mathrm{grad}\, \mathcal{V}\,|_{c_s(t)} = \frac{t}{6}\epsilon_j \,\mathrm{Ric}(T, e_j) X_j + \mathcal{O}(\delta^2) \tag{A.28}$$

$$\Box \mathcal{V}\,|_{c_s(t)} = -\frac{\mathrm{scal}}{6} + \mathcal{O}(\delta^2). \tag{A.29}$$

Proof. We first recall the expansion for the matrix determinant

$$\det(\mathbb{1} + A) = 1 + \mathrm{tr}(A) + \mathcal{O}(A^2).$$

From this identity and (A.23), we obtain

$$\begin{aligned} |\det(g)| &= -\det(g) \\ &= \det\Big[\mathbb{1} - \frac{t^2}{3}\,\epsilon_j\, g\big(R(e_j, T)T, e_k\big)|_{c_s(t)} \\ &\qquad + \frac{t^3}{6}\,\epsilon_j\, g\big((\nabla_T R)(e_j, T)T, e_k\big)|_{c_s(t)} + \mathcal{O}(\delta^4)\Big] \\ &= 1 + \mathrm{tr}\Big[-\frac{t^2}{3}\,\epsilon_j\, g\big(R(e_j, T)T, e_k\big) \\ &\qquad + \frac{t^3}{6}\,\epsilon_j\, g\big((\nabla_T R)(e_j, T)T, e_k\big)\Big] + \mathcal{O}(\delta^4) \\ &= 1 - \frac{t^2}{3}\,\mathrm{Ric}(T,T) + \frac{t^3}{6}\,(\nabla_T \,\mathrm{Ric})(T,T) + \mathcal{O}(\delta^4). \end{aligned}$$

Hence

$$\mathcal{V} = |\det(g)|^{-\frac{1}{4}} = 1 + \frac{t^2}{12}\,\mathrm{Ric}(T,T) - \frac{t^3}{24}\,(\nabla_T \,\mathrm{Ric})(T,T) + \mathcal{O}(\delta^4),$$

giving (A.26). Next, we calculate

$$\begin{aligned} \partial_{Y_\alpha} \mathcal{V} &= \frac{t^2}{6}\,\mathrm{Ric}(T, \nabla_{Y_\alpha} T) + \mathcal{O}(\delta^2) = \frac{t^2}{6}\,\mathrm{Ric}(T, \nabla_T Y_\alpha) + \mathcal{O}(\delta^2) \\ &= \frac{t^2}{6}\,\mathrm{Ric}(T, e_\alpha) + \mathcal{O}(\delta^2), \end{aligned}$$

proving (A.27). Using (A.23) and (A.22), the gradient of $\mathcal{V}$ is given by

$$\begin{aligned}\operatorname{grad}\mathcal{V} &= g^{jk}(\partial_{X_j}\mathcal{V})X_k = \epsilon_j(\partial_{X_j}\mathcal{V})X_j + \mathcal{O}(\delta^2)\\ &= \epsilon_j\left(\partial_{X_j}\frac{t^2}{12}\operatorname{Ric}(T,T)\right)e_j + \mathcal{O}(\delta^2).\end{aligned}$$

The derivatives with respect to X_j are computed to be

$$\partial_{X_\alpha}\frac{t^2}{12}\operatorname{Ric}(T,T) = \frac{1}{t}\partial_{Y_\alpha}\frac{t^2}{12}\operatorname{Ric}(T,T) = \frac{t}{6}\operatorname{Ric}(T,e_\alpha) + \mathcal{O}(\delta^2),$$

and

$$\begin{aligned}\partial_{X_0}\frac{t^2}{12}\operatorname{Ric}(T,T) &= \left(\frac{1}{\delta}\nabla_T - \frac{s_\alpha}{t\delta}\nabla_{Y_\alpha}\right)\frac{t^2}{12}\operatorname{Ric}(T,T)\\ &= \frac{1}{\delta}\frac{2t}{12}\operatorname{Ric}(T,T) - \frac{s_\alpha}{t\delta}\frac{t^2}{6}\operatorname{Ric}(T,\nabla_{Y_\alpha}T) + \mathcal{O}(\delta^2)\\ &= \frac{t}{6}\operatorname{Ric}\left(T, e_0 + \frac{s_\alpha}{\delta}e_\alpha\right) - \frac{t}{6}\operatorname{Ric}\left(T,\frac{s_\alpha}{\delta}e_\alpha\right) + \mathcal{O}(\delta^2)\\ &= \frac{t}{6}\operatorname{Ric}(T,e_0) + \mathcal{O}(\delta^2),\end{aligned} \tag{A.30}$$

where we used (A.4), (A.21) and (A.27). Thus

$$\operatorname{grad}\mathcal{V} = \frac{t}{6}\,\epsilon_j\,\operatorname{Ric}(T,e_j)X_j + \mathcal{O}(\delta^2),$$

which shows (A.28). Using that $g_{jk} = \epsilon_j\delta_{jk} + \mathcal{O}(\delta^2)$, we find

$$\begin{aligned}\Box\,\mathcal{V} &= -\frac{1}{\sqrt{|\det(g)|}}\,\partial_{X_j}\left(\sqrt{|\det(g)|}\,g^{jk}\partial_{X_k}\mathcal{V}\right)\\ &= -\epsilon_j\partial_{X_j}\partial_{X_j}\left(1 + \frac{t^2}{12}\operatorname{Ric}(T,T) - \frac{t^3}{24}(\nabla_T\operatorname{Ric})(T,T)\right) + \mathcal{O}(\delta^2).\end{aligned}$$

The spatial derivatives in this formula are calculated by

$$\begin{aligned}\partial_{X_\alpha}\partial_{X_\alpha}\mathcal{V} &= \frac{1}{t^2}\partial_{Y_\alpha}\partial_{Y_\alpha}\left(1 + \frac{t^2}{12}\operatorname{Ric}(T,T) - \frac{t^3}{24}(\nabla_T\operatorname{Ric})(T,T)\right) + \mathcal{O}(\delta^2)\\ &= \partial_{Y_\alpha}\left(\frac{1}{6}\operatorname{Ric}(T,e_\alpha) + \frac{t}{12}(\nabla_{e_\alpha}\operatorname{Ric})(T,T)\right.\\ &\qquad\left. - \frac{t}{24}\nabla_{Y_\alpha}\nabla_T\operatorname{Ric}(T,T)\right) + \mathcal{O}(\delta^2)\end{aligned}$$

$$
\begin{aligned}
&= \partial_{Y_\alpha} \left(\frac{1}{6}\, \mathrm{Ric}(T, e_\alpha) + \frac{t}{12}\, (\nabla_{e_\alpha} \mathrm{Ric})(T,T) \right. \\
&\quad \left. - \frac{t}{24}\, \nabla_T \nabla_{Y_\alpha} \mathrm{Ric}(T,T) \right) + \mathcal{O}(\delta^2) \\
&= \partial_{Y_\alpha} \left(\frac{1}{6}\, \mathrm{Ric}(T, e_\alpha) + \frac{t}{24}\, (\nabla_{e_\alpha} \mathrm{Ric})(T,T) \right. \\
&\quad \left. - \frac{t}{12}\, (\nabla_T \mathrm{Ric})(T, e_\alpha) \right) + \mathcal{O}(\delta^2) \\
&= \frac{1}{6}\, \mathrm{Ric}(e_\alpha, e_\alpha) + \frac{t}{4}\, (\nabla_{e_\alpha} \mathrm{Ric})(T, e_\alpha) \\
&\quad - \frac{t}{12}\, \nabla_{Y_\alpha} \nabla_T \mathrm{Ric}(T, e_\alpha) + \mathcal{O}(\delta^2) \\
&= \frac{1}{6}\, \mathrm{Ric}(e_\alpha, e_\alpha) + \frac{t}{6}\, (\nabla_{e_\alpha} \mathrm{Ric})(T, e_\alpha) \\
&\quad - \frac{t}{12}\, (\nabla_T \mathrm{Ric})(e_\alpha, e_\alpha) + \mathcal{O}(\delta^2).
\end{aligned}
$$

The derivatives with respect to X_0 are calculated similar to (A.30) and give

$$
\begin{aligned}
\partial_{X_0} \partial_{X_0} \mathcal{V} = \frac{1}{6}\, \mathrm{Ric}(e_0, e_0) + \frac{t}{6}\, (\nabla_{e_0} \mathrm{Ric})(T, e_0) \\
- \frac{t}{12}\, (\nabla_T \mathrm{Ric})(e_0, e_0) + \mathcal{O}(\delta^2).
\end{aligned}
$$

We thus obtain

$$
\begin{aligned}
\Box\, \mathcal{V} &= -\epsilon_j \left(\frac{1}{6}\, \mathrm{Ric}(e_j, e_j) + \frac{t}{6}\, (\nabla_{e_j} \mathrm{Ric})(T, e_j) - \frac{t}{12}\, (\nabla_T \mathrm{Ric})(e_j, e_j) \right) + \mathcal{O}(\delta^2) \\
&= -\frac{\mathrm{scal}}{6} - \frac{t}{6}\, \mathrm{div}(\mathrm{Ric})(T) + \frac{t}{12} \partial_T\, \mathrm{scal} + \mathcal{O}(\delta^2) = -\frac{\mathrm{scal}}{6} + \mathcal{O}(\delta^2),
\end{aligned}
$$

where in the last step we used the second Bianchi identities. □

We next derive the expansion of the Hadamard coefficient V_x^y.

Lemma A.4. *The Hadamard coefficient V_x^y has the expansion*

$$
\begin{aligned}
V_x^y &= m^2\, \Pi_x^y - \frac{\mathrm{scal}}{12}\, \Pi_x^y + \frac{\partial_T\, \mathrm{scal}}{24}\, \Pi_x^y && \text{(A.31)} \\
&\quad + \frac{\epsilon_j \epsilon_k \epsilon_l}{24}\, g\big((\nabla_{e_j} R)(T, e_j) e_k, e_l \big) e_k \cdot e_l \cdot \Pi_x^y && \text{(A.32)} \\
&\quad + \delta^2\, v^s\, \Pi_x^y + \delta^2\, v^b_{jk}\, e_j \cdot e_k \cdot \Pi_x^y + \delta^2\, v^p\, e_0 \cdot e_1 \cdot e_2 \cdot e_3 \cdot \Pi_x^y + \mathcal{O}(\delta^3), && \text{(A.33)}
\end{aligned}
$$

where the coefficients v^s, v^b_{jk} and v^p are real-valued functions.

Proof. As V_x^y is a Hadamard coefficient of the second-order equation (5.10), all contributions to V_x^y involve an even number of Clifford multiplications and only real-valued functions. As a consequence, the higher order terms can be written in the general form (A.33). In order to calculate the leading terms, we note that inserting the expansion $g^{jk} = \epsilon_j \delta_{jk} + \mathcal{O}(\delta^2)$ into the definition of the Bochner Laplacian (A.20) yields

$$\begin{aligned}
\Box^\nabla \Pi^y_{c(t)} &= -\nabla_{X_0} \nabla_{X_0} \Pi^y_x + \nabla_{X_\alpha} \nabla_{X_\alpha} \Pi^y_x + \mathcal{O}(\delta^2) = \frac{1}{t^2} \nabla_{Y_\alpha} \nabla_{Y_\alpha} \Pi^y_{c(t)} + \mathcal{O}(\delta^2) \\
&= -\frac{1}{4}\, \Pi^y_{c(t)} \int_0^t d\tau\, \frac{\tau^2}{t^2}\, \Pi_y^{c(\tau)} \epsilon_j \epsilon_k \epsilon_l\, g\big((\nabla_{e_j} R)(T, e_j)\, e_k, e_l\big) \\
&\quad \times e_k \!\cdot\! e_l \!\cdot\! \Pi^y_{c(\tau)} + \mathcal{O}(\delta^2) \\
&= -\frac{t}{12}\, \epsilon_j \epsilon_k \epsilon_l\, g\big((\nabla_{e_j} R)(T, e_j) e_k, e_l\big) e_k \!\cdot\! e_l \!\cdot\! \Pi^y_{c(t)} + \mathcal{O}(\delta^2), \qquad \text{(A.34)}
\end{aligned}$$

where we used formulas (A.7), (A.8), (A.12), (A.14) and a Taylor expansion of the integrand around $c(t)$. Inserting into the definition of V_x^y (see formula (A.17)), we obtain

$$\begin{aligned}
V_x^y &= -\,\mathcal{V}(x,y)\, \Pi^y_x \int_0^1 d\tau\ \mathcal{V}^{-1}(c(\tau), y)\, \Pi_y^{c(\tau)} \left(\Box^\nabla + \frac{\text{scal}}{4} - m^2 \right) \big(\mathcal{V}\, \Pi^y_{c(\tau)} \big) \\
&= -\Pi^y_x \int_0^1 d\tau\, \Pi_y^{c(\tau)} \left(\Box\, \mathcal{V} + \frac{\text{scal}}{4} - m^2 \right) \Pi^y_{c(\tau)} \\
&\quad + \Pi^y_x \int_0^1 d\tau\, \Pi_y^{c(\tau)} \left(2 \nabla_{\text{grad}\, \mathcal{V}} - \Box^\nabla \right) \Pi^y_{c(\tau)} + \mathcal{O}(\delta^2) \\
&= -\Pi^y_x \int_0^1 d\tau\, \Pi_y^{c(\tau)} \left(\frac{\text{scal}}{12} - m^2 \right) \Pi^y_{c(\tau)} \\
&\quad + \Pi^y_x \int_0^1 d\tau\, \Pi_y^{c(\tau)}\, \frac{\tau}{12}\, \epsilon_j \epsilon_k \epsilon_l\, g\big((\nabla_{e_j} R)(T, e_j) e_k, e_l\big) e_k \!\cdot\! e_l \!\cdot\! \Pi^y_{c(\tau)} + \mathcal{O}(\delta^2) \\
&= \left(m^2 - \frac{\text{scal}}{12} + \frac{\partial_T\, \text{scal}}{24} \right) \Pi^y_x \\
&\quad + \frac{\epsilon_j \epsilon_k \epsilon_l}{24}\, g\big((\nabla_{e_j} R)(T, e_j) e_k, e_l\big) e_k \!\cdot\! e_l \!\cdot\! \Pi^y_x + \mathcal{O}(\delta^2),
\end{aligned}$$

where we used (A.26), (A.29), (A.7), (A.28), (A.34) and again performed a Taylor expansion of the integrands around x. $\Box$

The expansion of the Hadamard coefficient $\mathcal{D}_x V_x^y$ is given in the next lemma.

Lemma A.5. *The Hadamard coefficient $\mathcal{D}_x V_x^y$ has the expansion*

$$\mathcal{D}_x V_x^y = i\,\delta\, d_j^v\, e_j \cdot \Pi_x^y + i\,\delta\, d_{jkl}^a\, e_j \cdot e_k \cdot e_l \cdot \Pi_x^y + \mathcal{O}(\delta^2),$$

where the coefficients d_j^v and d_{jkl}^a are real-valued functions.

Proof. We apply the Dirac operator to the expansion of Lemma A.4. The derivatives of the factors Π_x^y can be computed using Lemma A.1 and (A.15), giving contributions of the form

$$\mathcal{D}_x V_x^y \asymp i\,\delta\, d_j^v\, e_j \cdot \Pi_x^y + i\,\delta\, d_{jkl}^a e_j \cdot e_k \cdot e_l \cdot \Pi_x^y + \mathcal{O}(\delta^2). \tag{A.35}$$

If the derivative acts on the scalar curvature in the second summand in (A.31) or on the factor T in the last summand in (A.31), the resulting terms can be rewritten using the second Bianchi identities as

$$\mathcal{D}_x V_x^y \asymp -i\frac{\epsilon_j}{12}\, \mathrm{div}(\mathrm{Ric})(e_j) e_j \cdot \Pi_x^y. \tag{A.36}$$

On the other hand, if the derivative acts on the scalar curvature in the last summand in (A.31), we get terms of the form (A.35). Similarly, if the Riemann tensor in (A.32) is differentiated, we again get terms of the form (A.35). Moreover, differentiating the factor T in (A.32) gives the contribution

$$\mathcal{D}_x V_x^y \asymp i\frac{\epsilon_j}{12}\, \mathrm{div}(\mathrm{Ric})(e_j) e_j \cdot \Pi_x^y, \tag{A.37}$$

which cancels against the term (A.36). Finally, we need to be concerned about differentiating the error term in (A.32). Noting that all the contributions to V_x^y involve an even number of Clifford multiplications and only real-valued functions, applying the Dirac operator obviously gives terms of the form (A.35). $\square$

The last relevant Hadamard coefficients W_x^y and H_x^y can be expanded as follows.

Lemma A.6. *The Hadamard coefficients W_x^y and H_x^y have the expansion*

$$W_x^y = w^s\, \Pi_x^y + w_{jk}^b\, e_j \cdot e_k \cdot \Pi_x^y + w^p\, e_0 \cdot e_1 \cdot e_2 \cdot e_3 \cdot \Pi_x^y + \mathcal{O}(\delta), \tag{A.38}$$

$$H_x^y = h^s\, \Pi_x^y + h_{jk}^b\, e_j \cdot e_k \cdot \Pi_x^y + h^p\, e_0 \cdot e_1 \cdot e_2 \cdot e_3 \cdot \Pi_x^y + \mathcal{O}(\delta), \tag{A.39}$$

where all coefficients are real-valued functions.

Proof. As W_x^y and H_x^y are Hadamard coefficients of the second-order equation (5.10), all contributions to W_x^y and H_x^y involve an even number of Clifford multiplications and only real-valued functions. Thus, W_x^y and H_x^y can be written in the form (A.38) and (A.39), respectively. □

We now come to the proof of the propositions stated in Section 5.3.

Proof of Proposition 5.8. We rewrite the results of Lemmas A.1, A.4, A.5 and A.6 in the form

$$\begin{aligned}
\mathcal{D}_x \Pi_x^y &= \mathrm{i}\,\delta\, c_j^s\, e_j \cdot \Pi_x^y + \mathrm{i}\,\delta^3\, c_{jkl}^a\, e_j \cdot e_k \cdot e_l \cdot \Pi_x^y + \mathcal{O}(\delta^4), \\
V_x^y &= v^s\, \Pi_x^y + \delta\, \tilde{v}_{kl}\, e_k \cdot e_l \cdot \Pi_x^y + \delta^2\, v_{jk}^b\, e_j \cdot e_k \cdot \Pi_x^y \\
&\quad + \delta^2\, v^p\, e_0 \cdot e_1 \cdot e_2 \cdot e_3 \cdot \Pi_x^y + \mathcal{O}(\delta^3), \\
\mathcal{D}_x V_x^y &= \mathrm{i}\,\delta\, d_j^v\, e_j \cdot \Pi_x^y + \mathrm{i}\,\delta\, d_{jkl}^a\, e_j \cdot e_k \cdot e_l \cdot \Pi_x^y + \mathcal{O}(\delta^2), \\
W_x^y &= w^s\, \Pi_x^y + w_{jk}^b\, e_j \cdot e_k \cdot \Pi_x^y + w^p\, e_0 \cdot e_1 \cdot e_2 \cdot e_3 \cdot \Pi_x^y + \mathcal{O}(\delta), \\
H_x^y &= h^s\, \Pi_x^y + h_{jk}^b\, e_j \cdot e_k \cdot \Pi_x^y + h^p\, e_0 \cdot e_1 \cdot e_2 \cdot e_3 \cdot \Pi_x^y + \mathcal{O}(\delta),
\end{aligned} \tag{A.40}$$

where all coefficients are real-valued functions. Here each factor δ corresponds to a factor T in the resulting explicit formulas (for details see [27]). The coefficients v^s and $\tilde{v}_{jk}$ are given by

$$v^s = m^2 - \frac{\mathrm{scal}}{12} + \delta\, \tilde{v}^s \quad \text{and} \quad \tilde{v}_{kl} = \frac{1}{\delta}\, \frac{\epsilon_j \epsilon_k \epsilon_l}{24}\, g\big((\nabla_{e_j} R)(T, e_j) e_k, e_l\big), \tag{A.41}$$

where $\tilde{v}^s$ is a real-valued function. Inserting this formulas into the Hadamard expansion (5.28), we find

$$\begin{aligned}
(-8\pi^3)\, P(x,y) = &-\frac{\mathrm{i}\,\mathcal{V}}{\Gamma^2}\, \mathrm{grad}_x\, \Gamma \cdot \Pi_x^y + \frac{\mathrm{i}}{\Gamma}\, \mathrm{grad}_x\, \mathcal{V} \cdot \Pi_x^y \\
&+ \frac{\mathcal{V}}{\Gamma} \Big(m + \mathrm{i}\,\delta\, c_j^s\, e_j + \mathrm{i}\,\delta^3\, c_{jkl}^a\, e_j \cdot e_k \cdot e_l \Big) \cdot \Pi_x^y + \frac{\mathrm{i}}{4\Gamma}\, \mathrm{grad}_x\, \Gamma \\
&\cdot \Big(v^s + \delta\, \tilde{v}_{kl}\, e_k \cdot e_l + \delta^2\, v_{jk}^b\, e_j \cdot e_k + \delta^2\, v^p\, e_0 \cdot e_1 \cdot e_2 \cdot e_3 \Big) \cdot \Pi_x^y \\
&+ \frac{m}{4} \big[\log|\Gamma| - \mathrm{i}\pi\, \epsilon\big(\mathfrak{t}(x) - \mathfrak{t}(y)\big) \big] \Big(v^s + \delta\, \tilde{v}_{kl}\, e_k \cdot e_l \Big) \cdot \Pi_x^y \\
&+ \frac{1}{4} \big[\log|\Gamma| - \mathrm{i}\pi\, \epsilon\big(\mathfrak{t}(x) - \mathfrak{t}(y)\big) \big] \Big(i\,\delta\, d_j^v\, e_j + i\,\delta\, d_{jkl}^a\, e_j \cdot e_k \cdot e_l \Big) \\
&\cdot \Pi_x^y + \mathrm{i} \big[1 + \log|\Gamma| - \mathrm{i}\pi\, \epsilon\big(\mathfrak{t}(x) - \mathfrak{t}(y)\big) \big]\, \mathrm{grad}_x\, \Gamma \\
&\cdot \Big(w^s + w_{jk}^b\, e_j \cdot e_k + w^p\, e_0 \cdot e_1 \cdot e_2 \cdot e_3 \Big) \cdot \Pi_x^y
\end{aligned}$$

$$+\,\mathrm{i}\;\mathrm{grad}_x\,\Gamma\cdot\Big(h^s + h^b_{jk}\,e_j\cdot e_k + h^p\,e_0\cdot e_1\cdot e_2\cdot e_3\Big)$$
$$\cdot\Pi^y_x + \mathcal{O}(\delta^2\log\delta). \tag{A.42}$$

The Clifford relations immediately yield the identities

$$\mathrm{grad}_x\,\Gamma\cdot\big(f_{jk}\,e_j\cdot e_k\big) = \delta\,f_j\,e_j + \delta\,f_{jkl}\,e_j\cdot e_k\cdot e_l,$$
$$\mathrm{grad}_x\,\Gamma\cdot\big(f\,e_0\cdot e_1\cdot e_2\cdot e_3\big) = \delta\,f_{jkl}\,e_j\cdot e_k\cdot e_l,$$

where all coefficients are real-valued functions. Using these identities in (A.42) and combining terms which are of the same order in δ and contain the same number of Clifford multiplications, we obtain

$$(-8\pi^3)\,P(x,y)$$
$$= -\frac{\mathrm{i}}{\Gamma^2}\,\mathrm{grad}_x\,\Gamma\cdot\Pi^y_x + \frac{m}{\Gamma}\,\Pi^y_x + \frac{\mathrm{i}}{\Gamma}\,\delta\,p^{(1)}_j\,e_j\cdot\Pi^y_x + m\log|\Gamma|\,v^s\,\Pi^y_x \tag{A.43}$$
$$+\frac{\mathrm{i}}{4\Gamma}\,\delta\,\tilde{v}_{kl}\;\mathrm{grad}_x\,\Gamma\cdot e_k\cdot e_l\cdot\Pi^y_x - \frac{\mathrm{i}\pi}{4}\,\epsilon\big(\mathfrak{t}(x)-\mathfrak{t}(y)\big)m\,v^s\,\Pi^y_x \tag{A.44}$$
$$+\log|\Gamma|\,\delta\,\Big(\mathrm{i}\,p^{(2)}_j\,e_j + \frac{m}{4}\,\tilde{v}_{kl}\,e_k\cdot e_l + i\,p^{(3)}_{jkl}\,e_j\cdot e_k\cdot e_l\Big)\cdot\Pi^y_x \tag{A.45}$$
$$+\,\delta\,\pi\,\epsilon\big(\mathfrak{t}(x)-\mathfrak{t}(y)\big)\left(p^{(4)}_j\,e_j - \frac{\mathrm{i}\,m}{4}\,\tilde{v}_{kl}\,e_k\cdot e_l + p^{(5)}_{jkl}\,e_j\cdot e_k\cdot e_l\right)\cdot\Pi^y_x \tag{A.46}$$
$$+\,\delta\Big(\mathrm{i}\,p^{(6)}_j\,e_j + i\,p^{(7)}_{jkl}\,e_j\cdot e_k\cdot e_l\Big)\cdot\Pi^y_x + \mathcal{O}(\delta^2\log\delta). \tag{A.47}$$

Here all coefficients $p^{(1)}_j,\ldots,p^{(7)}_{jkl}$ are real-valued functions of the order $\mathcal{O}(\delta^0)$. Using the Clifford relations, the composition of three Clifford multiplications can be rewritten as as vector and axial components,

$$e_j\cdot e_k\cdot e_l = \big(g_{jk}e_l + g_{kl}e_j - g_{jl}e_k\big) + i\epsilon_n\,\epsilon_{jkln}\,e_5\cdot e_n \tag{A.48}$$

(where ϵ_{jkln} is the totally anti-symmetric tensor, and $e_5 = ie_0e_1e_2e_3$ denotes the pseudoscalar matrix; see [6, Appendix A]). Thus, in (A.45), (A.46) and (A.47) the resulting vector components can be combined with the corresponding vector components in these lines. The resulting axial component in (A.45) can be written as

$$(-8\pi^3)\,P(x,y) \asymp \log|\Gamma|\,\delta\,a_j\,e_5\cdot e_j\cdot\Pi^y_x \tag{A.49}$$

with real coefficients a_j. Moreover, from (5.28) and the previous calculations one sees that there is no other contribution to $P(x,y)$ of this form. As the expression $\delta\,a_j\,e_5\cdot e_j$ is linear in δ and smooth in x and y, it is odd under permutations of x and y (this can also be understood from the fact that the linear factor δ corresponds to a factor T in the resulting explicit formulas; for details see [27]). Also using the identity $(e_5 \cdot e_j)^* = e_5 \cdot e_j$, we obtain

$$(-8\pi^3)\, P(x,y) = (-8\pi^3)\, P(y,x)^* = -\log(\Gamma)\, \delta\, a_j\, e_5 \cdot e_j \cdot \Pi_x^y.$$

Comparing with (A.49), we conclude that the coefficients a_j vanish. For the same reason, the term in (A.47) containing three Clifford multiplications reduces to a vectorial contribution. Finally, the axial contribution in (A.46) resulting from the decomposition (A.48) can be written in the form

$$(-8\pi^3)\, P(x,y) \asymp \mathrm{i}\, \delta\, \epsilon\big(\mathfrak{t}(x) - \mathfrak{t}(y)\big)\, a_j\, e_5 \cdot e_j \cdot \Pi_x^y \tag{A.50}$$

with real coefficients a_j, and from (5.28) and the previous calculations one sees that there is no other contribution to $P(x,y)$ of this form. However, the term (A.50) is odd under conjugation but even when interchanging x and y. Therefore, we conclude that the coefficients a_j vanish. We thus obtain the following expansion of the kernel of the fermionic operator,

$$\begin{aligned}
(-8\pi^3)\, P(x,y) = &-\frac{\mathrm{i}}{\Gamma^2}\, \mathrm{grad}_x \Gamma \cdot \Pi_x^y + \frac{m}{\Gamma}\, \Pi_x^y + \frac{\mathrm{i}}{\Gamma}\, \delta\, p_j^{(1)}\, e_j \cdot \Pi_x^y \\
&+ \frac{m}{4} \log|\Gamma|\, v^s\, \Pi_x^y + \frac{\mathrm{i}}{4\Gamma}\, \delta\, \tilde{v}_{kl}\, \mathrm{grad}_x \Gamma \cdot e_k \cdot e_l \cdot \Pi_x^y \\
&- \frac{\mathrm{i}\pi}{4}\, \epsilon\big(\mathfrak{t}(x) - \mathfrak{t}(y)\big) m\, v^s\, \Pi_x^y \\
&+ \log|\Gamma|\, \delta \left(\mathrm{i}\, \tilde{p}_j^{(2)}\, e_j + \frac{m}{4}\, \tilde{v}_{kl}\, e_k \cdot e_l \right) \cdot \Pi_x^y \\
&+ \delta\, \pi\, \epsilon\big(\mathfrak{t}(x) - \mathfrak{t}(y)\big) \left(\tilde{p}_j^{(4)}\, e_j - \frac{i\, m}{4}\, \tilde{v}_{kl}\, e_k \cdot e_l \right) \cdot \Pi_x^y \\
&+ \mathrm{i}\, \delta\, \tilde{p}_j^{(6)}\, e_j \cdot \Pi_x^y + \mathcal{O}(\delta^2 \log \delta),
\end{aligned} \tag{A.51}$$

where $\tilde{p}_j^{(2)}$, $\tilde{p}_j^{(4)}$ and $\tilde{p}_j^{(6)}$ are real-valued functions. The first two terms in this expansion show that Proposition 5.8 holds. The other terms will be needed to calculate the expansion of the closed chain. □

Proof of Proposition 5.9. Using the expansion (A.51), we compute

$$
\begin{aligned}
(-8\pi^3)^2 A_{yx} &= (-8\pi^3)^2 P(x,y)^* \, P(x,y) \\
&= c(x,y)\, \mathbb{1}_{S_yM} + \frac{\pi\, m}{2\Gamma^2}\, v^s\, \epsilon\big(\mathfrak{t}(x)-\mathfrak{t}(y)\big)\, \Pi^y_x \,\mathrm{grad}_x \Gamma \cdot \Pi^y_x \\
&\quad + \frac{\mathrm{i}m}{4\Gamma^2}\, (\delta + \delta \log|\Gamma|)\, \Pi^y_x \left\{ \mathrm{grad}_x \Gamma, \tilde{v}_{kl}\, e_k \cdot e_l \right\} \cdot \Pi^y_x \\
&\quad + \frac{\mathrm{i}\pi}{\Gamma^2}\, \delta\, \epsilon\big(\mathfrak{t}(x)-\mathfrak{t}(y)\big)\, \Pi^y_x \left[\mathrm{grad}_x \Gamma, \tilde{p}^{(4)}_j\, e_j \right] \cdot \Pi^y_x \\
&\quad + \frac{m\pi}{4\Gamma^2}\, \delta\, \epsilon\big(\mathfrak{t}(x)-\mathfrak{t}(y)\big)\, \Pi^y_x \left[\mathrm{grad}_x \Gamma, \tilde{v}_{kl}\, e_k \cdot e_l \right] \cdot \Pi^y_x \\
&\quad + \mathcal{O}(\delta^{-1} \log \delta) \\
&= c(x,y)\, \mathbb{1}_{S_yM} \\
&\quad - m \left(m^2 - \frac{\mathrm{scal}}{12} + \delta\, \tilde{v}^s \right) \frac{\pi\, \epsilon\big(\mathfrak{t}(x)-\mathfrak{t}(y)\big)}{2\Gamma^2}\, \mathrm{grad}_y \Gamma \cdot \mathbb{1}_{S_yM} \\
&\quad - \frac{m}{4\Gamma^2}\, (\delta + \delta \log|\Gamma|) \left\{ \mathrm{grad}_x \Gamma, i\tilde{v}_{kl}\, e_k \cdot e_l \right\} \cdot \mathbb{1}_{S_yM} \\
&\quad - \frac{\mathrm{i}\pi}{\Gamma^2}\, \delta\, \epsilon\big(\mathfrak{t}(x)-\mathfrak{t}(y)\big) \left[\mathrm{grad}_x \Gamma, \tilde{p}^{(4)}_j\, e_j \right] \cdot \mathbb{1}_{S_yM} \\
&\quad + \mathrm{i}\frac{m\pi}{4\Gamma^2}\, \delta\, \epsilon\big(\mathfrak{t}(x)-\mathfrak{t}(y)\big) \left[\mathrm{grad}_x \Gamma, i\tilde{v}_{kl}\, e_k \cdot e_l \right] \cdot \mathbb{1}_{S_yM} \\
&\quad + \mathcal{O}(\delta^{-1} \log \delta),
\end{aligned}
$$

where we used that $\tilde{v}_{kl} = -\tilde{v}_{lk}$ according to (A.41). Moreover, we used the formula for v^s in (A.41) as well as the identities

$$
\Pi^x_y\, e_j \cdot \Pi^y_x = e_j \cdot \mathbb{1}_{S_yM} \quad \text{and} \quad \Pi^x_y \,\mathrm{grad}_x \Gamma \cdot \Pi^y_x = -\,\mathrm{grad}_y \Gamma \cdot \mathbb{1}_{S_yM}.
$$

Now the operators X_{yx} and Y_{yx} defined by

$$
X_{yx} := \frac{\pi}{\Gamma^2}\, \delta\, \epsilon\big(\mathfrak{t}(x)-\mathfrak{t}(y)\big) \left(\frac{\mathrm{i}\, m}{4}\, \tilde{v}_{kl}\, e_k \cdot e_l - \tilde{p}^{(4)}_j\, e_j \right), \tag{A.52}
$$

$$
Y_{yx} := -\frac{m}{4\Gamma^2} \Big((\delta + \delta \log|\Gamma|)\, \mathrm{i}\tilde{v}_{kl}\, e_k \cdot e_l - \delta\, \pi\, \tilde{v}^s\, \epsilon\big(\mathfrak{t}(x)-\mathfrak{t}(y)\big) \Big) \tag{A.53}
$$

are obviously symmetric, linear operators on S_yM. Moreover, X_{yx} is of the order $\mathcal{O}(\delta^{-3} \log \delta)$, whereas Y_{yx} is of the order $\mathcal{O}(\delta^{-3})$. Interchanging x and y completes the proof. $\square$

We finally prove the expansions (5.55) and (5.56) in Lemma 5.11.

Proof of Lemma 5.11. From (5.35), we conclude that the coefficient a_{xy} in (5.38) is given by

$$a_{xy} = m \left(m^2 - \frac{\text{scal}}{12} \right) \frac{\text{Im}(\log \Gamma)}{2\Gamma^2}.$$

Thus we obtain from (A.52) and (A.41) that the operator Z_{xy} in (5.40) is given by

$$Z_{xy} = \frac{1}{2} \left(m^2 - \frac{\text{scal}}{12} \right)^{-1} \left[\frac{\epsilon_j \epsilon_k \epsilon_l}{12} \, g\big((\nabla_{e_j} R)(T, e_j) e_k, e_l \big) \, \frac{i}{2} [e_k, e_l] - \frac{4\delta}{m} \tilde{p}_j^{(4)} \, e_j \right]. \tag{A.54}$$

Moreover, since x lies in the future of y and $\text{grad}_x \Gamma$ is normalized according to (5.24), the future-directed timelike unit vector u introduced after (5.40) is given by

$$u = \frac{1}{2\delta} \, \text{grad}_x \Gamma = \frac{1}{\delta} \, T.$$

Therefore, the vector Δu introduced in (5.42) is given by

$$\Delta u = \frac{1}{6\,\delta} \left(m^2 - \frac{\text{scal}}{12} \right)^{-1} \epsilon_j (\nabla_{e_j} R)(T, e_j) T,$$

proving (5.55).

The operator w in (5.43) is given by the vectorial part of (A.54), i.e.,

$$w = -\frac{2}{m} \left(m^2 - \frac{\text{scal}}{12} \right)^{-1} \delta \, \tilde{p}_j^{(4)} \, e_j. \tag{A.55}$$

A short review of the proofs of Propositions 5.8 and 5.9 yields that the functions $\tilde{p}_j^{(4)}$ are combinations of the real-valued functions appearing in the expansion of the Hadamard coefficients $\mathcal{D}_x V_x^y$, W_x^y and H_x^y in (A.40). These functions are calculated explicitly in [27]. They are of the order

$$\mathcal{O}\left(\frac{m^2}{\delta} \, \|\epsilon_j \, \text{Ric}(T, e_j) \, e_j\| \right) + \mathcal{O}\big(\|R\|^2 + \|\nabla^2 R\| \big).$$

Inserting into formula (A.55), we conclude that w is of the order

$$\left[\mathcal{O}\left(\frac{1}{m} \, \|\epsilon_j \, \text{Ric}(T, e_j) \, e_j\| \right) + \mathcal{O}\left(\frac{\delta}{m^3} \, \big(\|R\|^2 + \|\nabla^2 R\| \big) \right) \right] \left(1 + \mathcal{O}\left(\frac{\text{scal}}{m^2} \right) \right).$$

Now (5.56) follows immediately from the representation (5.44). $\square$

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