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Quasiconformal realizations of $E_{6(6)}$, $E_{7(7)}$, $E_{8(8)}$ and SO$(n+3, m+3)$, $N \geqslant 4$ supergravity and spherical vectors

Murat Günaydin[1] **and Oleksandr Pavlyk**[2]

[1]Institute for Gravitation and the Cosmos, Physics Department, Penn State University, University Park, PA 16802, USA
murat@phys.psu.edu
[2]Wolfram Research Inc., 100 Trade Center Dr., Champaign, IL 61820, USA
pavlyk@wolfram.com

Abstract

After reviewing the underlying algebraic structures we give a unified realization of split exceptional groups $F_{4(4)}$, $E_{6(6)}$, $E_{7(7)}$, $E_{8(8)}$ and of SO$(n+3, m+3)$ as quasiconformal groups that is covariant with respect to their (Lorentz) subgroups SL$(3, \mathbb{R})$, SL$(3, \mathbb{R}) \times$ SL$(3, \mathbb{R})$, SL$(6, \mathbb{R})$, $E_{6(6)}$ and SO$(n, m) \times$ SO$(1, 1)$, respectively. We determine the spherical vectors of quasiconformal realizations of all these groups twisted by a unitary character ν. We also give their quadratic Casimir operators and determine their values in terms of ν and the dimension n_V of the underlying Jordan algebras. For $\nu = -(n_V + 2) + i\rho$ the quasiconformal action induces unitary representations on the space of square integrable

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functions in $(2n_V + 3)$ variables, that belong to the principle series. For special discrete values of ν the quasiconformal action leads to unitary representations belonging to the discrete series and their continuations. The manifolds that correspond to "quasiconformal compactifications" of the respective $(2n_V + 3)$ dimensional spaces are also given. We discuss the relevance of our results to $N = 8$ supergravity and to $N = 4$ Maxwell–Einstein supergravity theories and , in particular, to the proposal that three and four dimensional U-duality groups act as spectrum generating quasiconformal and conformal groups of the corresponding four and five dimensional supergravity theories, respectively.

Contents

1 Introduction

Earliest studies of unitary representations of four-dimensional U-duality groups of extended supergravity theories were given in [1–3]. These representations were constructed using oscillators that transform in the same representations of the U-duality groups as the vector field strengths plus their magnetic duals. These works were motivated by the idea that in a quantum theory global symmetries must be realized unitarily on the spectrum and the composite scenarios that attempted to connect maximal $N = 8$ supergravity with observation [4–6].[1] In a composite scenario proposed in [4] it was conjectured that SU(8) local symmetry of $N = 8$ supergravity becomes dynamical and acts as a family unifying grand unified theory (GUT) which contains SU(5) GUT as well as a family group SU(3). A similar scenario leads to E_6 GUT with a family group $U(1)$ in the exceptional supergravity theory [7] whose U-duality group is $E_{7(-25)}$ in $d = 4$. However with the discovery of possible counter terms at higher loops it was argued that divergences would eventually spoil the finiteness properties of $N = 8$ supergravity. After the work of Green and Schwarz on anomaly cancellation in superstring theory [8] attempts at composite scenarios in supergravity were abandoned. However, recent discovery of unexpected cancellations of divergences in supergravity theories [9–17] has brought back the question of finiteness of $N = 8$ supergravity as well as of exceptional supergravity.

Over the last decade or so, there has been a great deal more work done on unitary representations of U-duality groups of extended supergravity theories. The renewed interest in unitary realizations of U-duality groups was due partly to the proposals that certain extensions of U-duality groups may act as spectrum generating symmetry groups of supergravity theories. Based on geometric considerations involving orbits of extremal black hole

[1] For further references on the subject, see [6].

solutions in $N=8$ supergravity and $N=2$ Maxwell–Einstein–supergravity theories (MESGTs) with symmetric scalar manifolds, it was suggested that four dimensional U-duality groups act as spectrum generating conformal symmetry groups of the corresponding five dimensional supergravity theories [18–22]. This proposal was then extended to the proposal that three-dimensional U-duality groups act as spectrum generating quasiconformal groups of the corresponding four-dimensional supergravity theories [19–22]. Quasiconformal realization of U-duality group $E_{8(8)}$ of three-dimensional maximal supergravity given in [19] is the first known geometric realization of $E_{8(8)}$ and its quantization leads to the minimal unitary representation of $E_{8(8)}$ [23]. Remarkably, quasiconformal realizations exist for different real forms of all noncompact groups and their quantizations yield directly the minimal unitary representations of the respective groups [23–26]. Furthermore, the quasiconformal method gives a unified approach to the minimal unitary representations of all noncompact groups and extends also to supergroups [26]. For symplectic groups $\mathrm{Sp}(2m,\mathbb{R})$ these minimal unitary representations are simply the singleton representations.

Many results have been obtained over recent years that support the proposals that four- and three-dimensional U-duality groups act as spectrum generating conformal and quasiconformal groups of five- and four-dimensional supergravity theories with symmetric scalar manifolds, respectively. The work relating black hole solutions in four and five dimensions (4d/5d lift) [27–30] is consistent with the proposal that four-dimensional U-duality groups act as spectrum generating conformal symmetry groups of five-dimensional supergravity theories from which they descend. Furthermore, the work of [31, 32] on using solution generating techniques to relate the known black hole solutions of five-dimensional ungauged supergravity theories to each other and generate new solutions using symmetry groups of the corresponding three-dimensional supergravity theories and related work on gauged supergravity theories [33] are in accord with these proposals.

A concrete framework for implementing the proposal that three-dimensional U-duality groups act as spectrum generating quasiconformal groups was formulated in [34–36] for spherically symmetric stationary Bogomolny-Prasad-Sommerfield (BPS) black holes. This framework is based on the fact that the attractor equations [37, 38] of spherically symmetric stationary black holes of four-dimensional supergravity theories are equivalent to the equations describing the geodesic motion of a fiducial particle on the moduli space $\mathcal{M}_3^*$ of three-dimensional supergravity theories obtained by reduction on a time-like circle.[2] A related analysis on non-BPS extremal black holes in theories with symmetric target manifolds was carried out

[2]This was first observed in [39] and used in [40, 41] to construct static and rotating black holes in heterotic string theory.

in [42, 43]. For $N = 2$ MESGTs defined by Euclidean Jordan algebras of degree three the manifolds $\mathcal{M}_3^*$ are para-quaternionic symmetric spaces

$$\mathcal{M}_3^* = \frac{\mathrm{QConf}(J)}{\mathrm{Conf}(J) \times \mathrm{SU}(1,1)},$$

where QConf$J$ and Conf$J$ are the quasiconformal and conformal groups of the Jordan algebra J, respectively. When one quantizes the fiducial particle's motion one is led to quantum mechanical wave functions that provide the basis of a unitary representation of QConf(J). BPS black holes correspond to a special class of geodesics and the twistor space $\mathcal{Z}_3$ of $\mathcal{M}_3^*$ can be identified with the BPS phase space. Then the spherically symmetric stationary BPS black hole solutions of $N = 2$ MESGT's are described by holomorphic curves in $\mathcal{Z}_3$ [34–36, 44]. One finds that the action of three-dimensional U-duality group QConf(J) on the natural complex coordinates of the twistor space is precisely of the quasiconformal form [36]. Therefore the unitary representations of QConf(J) relevant for BPS black holes of $N = 2$ MESGTs are those induced by holomorphic quasiconformal actions of QConf(J) on the corresponding twistor spaces $\mathcal{Z}_3$, which belong in general to quaternionic discrete series representations of QConf(J) [36].

Another result in support of the proposal that three-dimensional U-duality groups act as spectrum generating groups of the corresponding four-dimensi onal theories comes from the connection established in [45] between the harmonic superspace (HSS) formulation of $N = 2$, $d = 4$ supersymmetric quaternionic Kähler sigma models that couple to $N = 2$ supergravity and the minimal unitary representations of their isometry groups. One finds that for $N = 2$ sigma models with quaternionic symmetric target spaces of the form[3]

$$\frac{\mathrm{QConf}(J)}{\widetilde{\mathrm{Conf}}(J) \times \mathrm{SU}(2)}$$

there exists a one-to-one mapping between the quartic Killing potentials that generate the isometry group QConf(J) under Poisson brackets in the HSS formulation, and the generators of the minimal unitary representation of QConf(J) obtained by quantization of its quasiconformal realization. Therefore the "fundamental spectrum" of the quantum theory must fit into the minimal unitary representation of QConf(J) and the full spectrum is obtained by tensoring of the minimal unitary representation.

In [36] unitary representations of two quaternionic groups of rank two, namely SU$(2, 1)$ and $G_{2(2)}$, induced by their geometric quasiconformal actions were studied in great detail. They are the isometry groups of four and five dimensional simple $N = 2$ supergravity theories dimensionally reduced

[3] $\widetilde{\mathrm{Conf}}(J)$ is the compact form of Conf(J).

on tori to three dimensions, respectively. Unitary representations induced by the geometric quasiconformal action include the quaternionic discrete series representations that were studied in mathematics literature using other methods [46]. In the study of unitary representations of $\mathrm{SU}(2,1)$ and $G_{2(2)}$, in particular of quaternionic discrete series, studied in [36] spherical vectors of maximal compact subgroups under their quasiconformal actions play an essential role. In a recent paper [47] we gave a unified quasiconformal realization of three-dimensional U-duality groups $\mathrm{QConf}(J)$ of all $N=2$ MESGTs with symmetric scalar manifolds defined by Euclidean Jordan algebras of degree three in a basis covariant with respect to their five-dimensional U-duality groups. These three-dimensional U-duality groups are $F_{4(4)}$, $E_{6(2)}$, $E_{7(-5)}$, $E_{8(-24)}$ and $\mathrm{SO}(n_V+2,4)$ and their five dimensional U-duality groups are $\mathrm{SL}(3,\mathbb{R})$, $\mathrm{SL}(3,\mathbb{C})$, $\mathrm{SU}^*(6)$, $E_{6(-26)}$ and $\mathrm{SO}(1,1)\times \mathrm{SO}(n_V-1,1)$, respectively.[4] We gave their quadratic Casimir operators and determined their values and most importantly presented the spherical vectors of all these quasiconformal groups in a unified manner, which are essential for the construction of the quaternionic discrete series representations.

In this paper we extend the results of [47] to the split exceptional groups $E_{6(6)}$, $E_{7(7)}$, $E_{8(8)}$ and $\mathrm{SO}(n+3,m+3)$, which are the quasiconformal groups of split non-Euclidean Jordan algebras of degree three.[5] More specifically, in Section 2 we review the necessary background regarding Euclidean and non-Euclidean Jordan algebras of degree three and their rotation (automorphism) and Lorentz (reduced structure) groups. The U-duality symmetries of maximal supergravity in five, four and three dimensions are simply the Lorentz, conformal and quasiconformal groups of the split exceptional Jordan algebra $J_3^{\mathbb{O}_S}$. The corresponding symmetry groups of $N=4$ (16 supercharges) MESGTs are determined by the non-simple Jordan algebras $\mathbb{R}\oplus\Gamma_{(5,n)}$. In Section 3 we review the conformal symmetry groups of the relevant Jordan algebras. In Section 4 we present the unified quasiconformal realizations of $E_{6(6)}$, $E_{7(7)}$, $E_{8(8)}$ and $\mathrm{SO}(n+3,m+3)$ twisted by a unitary character ν and their commutation relations as well as their quadratic Casimir operators. We determine the values of the Casimir operators as a function of ν and the dimension n_V of the underlying Jordan algebra J. From this we determine the values of ν for which the quasiconformal action induces unitary representations on the space of square integrable functions in $(2n_V+3)$ variables, that belong to the principle series. In Section 5

[4]Of course, the rank two quaternionic quasiconformal groups can be obtained as a trivial limit of the general unified formulation.

[5]Actually, maximally split orthogonal groups correspond to the case $n=m$. For $n=1$ one gets the quasiconformal group of a Euclidean Jordan algebra.

we present the spherical vectors of these quasiconformal groups in a unified manner and discuss each group separately. We also present the compact spaces corresponding to the "quasiconformal compactification" of the $(2n_V + 3)$-dimensional spaces on which the quasiconformal groups $\mathrm{QConf}(J)$ act for all Jordan algebras of degree three, Euclidean as well as split. In Section 6 we point out the connection between real and split exceptional Jordan algebras of degree three and the quaternionic Jordan algebras of degree four and discuss the similarities and differences between the exceptional $N = 2$ supergravity and maximal $N = 8$ supergravity as they relate to quaternionic Jordan algebras of rank four.

2 Euclidean (compact) and non-Euclidean (noncompact) Jordan algebras of degree three

Referring to the monograph [48] for details and references on the subject we shall give a brief review of Jordan algebras in this section, focussing mainly on Jordan algebras of degree three.

A Jordan algebra over a field $\mathbb{F}$, which we take to generally to be the real numbers $\mathbb{R}$, is an algebra, J , with a symmetric product $\circ$

$$X \circ Y = Y \circ X \in J, \quad \forall X, Y \in J, \tag{2.1}$$

such that the Jordan identity holds:

$$X \circ (Y \circ X^2) = (X \circ Y) \circ X^2, \tag{2.2}$$

where $X^2 \equiv (X \circ X)$. Hence a Jordan algebra is commutative and in general not associative algebra. They were introduced by Pascual Jordan in his attempt to generalize the formalism of quantum mechanics and finite-dimensional simple Jordan algebras were classified by him, von Neumann and Wigner [49].

A Jordan algebra J is said to be Euclidean if for any two elements X and Y of J the condition

$$X \circ X + Y \circ Y = 0$$

implies that both X and Y must vanish. Since the automorphism groups of Euclidean Jordan algebras are compact they are also referred to as compact. Otherwise the Jordan algebra is referred to as noncompact or non-Euclidean. One can in general define a norm form, $\mathbf{N} : J \to \mathbb{R}$ over J that satisfies the composition property [50]

$$\mathbf{N}(\{X,Y,X\}) = \mathbf{N}^2(X)\mathbf{N}(Y), \tag{2.3}$$

where $\{X,Y,Z\}$ is the Jordan triple product defined as

$$\{X,Y,Z\} = X\circ(Y\circ Z) + Z\circ(Y\circ X) - (X\circ Z)\circ Y. \tag{2.4}$$

The degree, p, of the norm form as well as of J is defined by the homogeneity condition $\mathbf{N}(\lambda X) = \lambda^p \mathbf{N}(X)$, where $\lambda \in \mathbb{R}$.

2.1 Euclidean Jordan algebras of degree three and $5D$, $N=2$ MESGTs

As was shown in [51], there exists a one-to-one correspondence between Euclidean Jordan algebras of degree three and the $5D$, $N=2$ MESGTs whose scalar manifolds are symmetric spaces such that G is a symmetry of their Lagrangian. In these theories the symmetric C-tensor that describes the $F\wedge F\wedge A$ type coupling[6]

$$C_{IJK}\epsilon_{\mu\nu\lambda\rho\sigma}F^{I\mu\nu}F^{J\lambda\rho}A^{J\sigma}$$

of all the vector fields including the graviphoton is identified with the symmetric tensor that defines the cubic norm of the corresponding Euclidean Jordan algebra J of degree three. Their scalar manifolds are of the form

$$\mathcal{M}_5(J) = \frac{\mathrm{Str}_0(J)}{\mathrm{Aut}(J)}, \tag{2.5}$$

where $\mathrm{Str}_0(J)$ is the Lorentz (reduced structure) group of J and $\mathrm{Aut}(J)$ is its rotation (automorphism) group. For Euclidean Jordan algebras the rotation (automorphism) groups are compact.

There exists an infinite family of nonsimple Jordan algebras of degree three which are the direct sum of a one-dimensional Jordan algebra $\mathbb{R}$ and a Jordan algebra $\Gamma_{(1,n-1)}$ associated with a quadratic form of Lorentzian signature:

$$J = \mathbb{R}\oplus\Gamma_{(1,n-1)} \tag{2.6}$$

which is referred to as the generic Jordan family. A simple realization of

[6]We should note that a given $N=2$ MESGT in five dimensions is uniquely determined by the C-tensor.

$\Gamma_{(1,n-1)}$ is provided by $2^{[n/2]} \times 2^{[n/2]}$ Dirac gamma matrices γ^i $(i, j, \ldots = 1, \ldots, (n-1))$ of an $(n-1)$-dimensional Euclidean space together with the identity matrix $\gamma^0 = \mathbf{1}$ and the Jordan product $\circ$ being one half the anti-commutator:

$$\begin{aligned} \gamma^i \circ \gamma^j &= \frac{1}{2}\{\gamma^i, \gamma^j\} = \delta^{ij}\gamma^0, \\ \gamma^0 \circ \gamma^0 &= \frac{1}{2}\{\gamma^0, \gamma^0\} = \gamma^0, \\ \gamma^i \circ \gamma^0 &= \frac{1}{2}\{\gamma^i, \gamma^0\} = \gamma^i. \end{aligned} \tag{2.7}$$

The quadratic norm of a general element $\mathbb{X} = X_0\gamma^0 + X_i\gamma^i$ of $\Gamma_{(1,n-1)}$ is defined as

$$\mathbf{Q}(\mathbb{X}) = \frac{1}{2^{[n/2]}} \operatorname{Tr} \mathbb{X}\bar{\mathbb{X}} = X_0 X_0 - X_i X_i,$$

where

$$\bar{\mathbb{X}} \equiv X_0\gamma^0 - X_i\gamma^i.$$

The norm of a general element $\xi \oplus \mathbb{X}$ of the nonsimple Jordan algebra $J = \mathbb{R} \oplus \Gamma_{(1,n-1)}$ is simply given by

$$\mathbf{N}(\xi \oplus \mathbb{X}) = \xi \mathbf{Q}(\mathbb{X}), \tag{2.8}$$

where $\xi \in \mathbb{R}$.

The scalar manifolds of corresponding 5D, $N = 2$ MESGTs are

$$\mathcal{M} = \frac{\mathrm{SO}(n-1,1)}{\mathrm{SO}(n-1)} \times \mathrm{SO}(1,1)$$

In addition to the generic infinite family there exist four simple Euclidean Jordan algebras of degree three. They are generated by Hermitian (3×3)-matrices over the four division algebras $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

$$J = \begin{pmatrix} \alpha & Z & \bar{Y} \\ \bar{Z} & \beta & X \\ Y & \bar{X} & \gamma \end{pmatrix},$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and $X, Y, Z \in \mathbb{A}$ with the product being one half the anti-commutator. They are denoted as $J_3^{\mathbb{R}}$, $J_3^{\mathbb{C}}$, $J_3^{\mathbb{H}}$, $J_3^{\mathbb{O}}$, respectively, and the corresponding $N = 2$ MESGts are called “magical supergravity theories.”

They have the 5D scalar manifolds:

$$\begin{aligned} J_3^{\mathbb{R}}: &\quad \mathcal{M} = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3), \\ J_3^{\mathbb{C}}: &\quad \mathcal{M} = \mathrm{SL}(3,\mathbb{C})/\mathrm{SU}(3), \\ J_3^{\mathbb{H}}: &\quad \mathcal{M} = \mathrm{SU}^*(6)/U\mathrm{Sp}(6), \\ J_3^{\mathbb{O}}: &\quad \mathcal{M} = E_{6(-26)}/F_4. \end{aligned} \tag{2.9}$$

The cubic norm form, $\mathbf{N}$, of the simple Jordan algebras of degree three is given by the "determinant" of the corresponding Hermitian (3×3)-matrices (modulo an overall scaling factor):

$$\mathbf{N}(J) = \alpha\beta\gamma - \alpha X\bar{X} - \beta Y\bar{Y} - \gamma Z\bar{Z} + 2\mathrm{Re}(XYZ), \tag{2.10}$$

where $\mathrm{Re}(XYZ)$ denotes the real part of XYZ and bar denotes conjugation in the underlying division algebra.

For a real quaternion $X \in \mathbb{H}$ we have

$$\begin{aligned} X &= X_0 + X_1 j_1 + X_2 j_2 + X_3 j_3, \\ \bar{X} &= X_0 - X_1 j_1 - X_2 j_2 - X_3 j_3, \\ X\bar{X} &= X_0^2 + X_1^2 + X_2^2 + X_3^2, \end{aligned} \tag{2.11}$$

where the imaginary units j_i satisfy

$$j_i j_j = -\delta_{ij} + \epsilon_{ijk} j_k. \tag{2.12}$$

For a real octonion $X \in \mathbb{O}$ we have

$$\begin{aligned} X &= X_0 + X_1 j_1 + X_2 j_2 + X_3 j_3 + X_4 j_4 + X_5 j_5 + X_6 j_6 + X_7 j_7, \\ \bar{X} &= X_0 - X_1 j_1 - X_2 j_2 - X_3 j_3 - X_4 j_4 - X_5 j_5 - X_6 j_6 - X_7 j_7, \\ X\bar{X} &= X_0^2 + \sum_{A=1}^{7} (X_A)^2. \end{aligned} \tag{2.13}$$

Seven imaginary units of real octonions satisfy

$$j_A j_B = -\delta_{AB} + \eta_{ABC} j_C, \tag{2.14}$$

where η_{ABC} is completely antisymmetric and in the conventions of [52] take on the values

$$\eta_{ABC} = 1 \Leftrightarrow (ABC) = (123), (471), (572), (673), (624), (435), (516). \tag{2.15}$$

as indicated in Figure 1. The automorphism group of the division algebra of octonions is the compact group G_2.

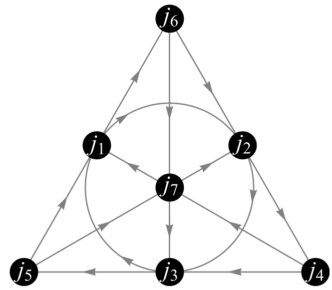

Figure 1: Multiplication table of imaginary units of real octonions $\mathbb{O}$. Three imaginary units on each side, height and circle correspond to the imaginary units of a quaternion subalgebra. The arrows represent the positive directions for multiplication, e.g., $j_1 j_2 = -j_2 j_1 = j_3$ and $j_6 j_2 = -j_2 j_6 = j_4$, etc.

We should note that the simple Jordan algebras $J_3^{\mathbb{A}}$ of degree 3 have nonsimple subalgebras generated by elements of the form

$$J = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & X \\ 0 & \bar{X} & \gamma \end{pmatrix}$$

that are isomorphic to generic Jordan algebras $(\mathbb{R} \oplus \Gamma_{(1,2)})$, $(\mathbb{R} \oplus \Gamma_{(1,3)})$, $(\mathbb{R} \oplus \Gamma_{(1,5)})$ and $(\mathbb{R} \oplus \Gamma_{(1,9)})$ for $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$, respectively.

2.2 Non-Euclidean Jordan algebras of degree three and $5D$, $N \geqslant 4$ supergravity theories

In the generic infinite family of nonsimple Jordan algebras of degree three, $\mathbb{R} \oplus \Gamma$, one can take the quadratic form defining the Jordan algebra Γ of degree two to be of arbitrary signature different from Minkowskian, which result in non compact or non-Euclidean Jordan algebras. If the quadratic norm form has signature (n, m) we shall denote the Jordan algebra as $\Gamma_{(n,m)}$. $\Gamma_{(n,m)}$ is realized by $2^{[(n+m)/2]} \times 2^{[(n+m)/2]}$ Dirac gamma matrices γ^i $(i, j, \ldots = 1, \ldots, (n + m - 1))$ together with the identity matrix $\gamma^0 = \mathbf{1}$ and the Jordan product $\circ$ being one-half the anticommutator:

$$\begin{aligned} \gamma^i \circ \gamma^j &= \frac{1}{2}\{\gamma^i, \gamma^j\} = \eta^{ij}\gamma^0, \\ \gamma^0 \circ \gamma^0 &= \frac{1}{2}\{\gamma^0, \gamma^0\} = \gamma^0, \\ \gamma^i \circ \gamma^0 &= \frac{1}{2}\{\gamma^i, \gamma^0\} = \gamma^i, \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} \eta^{ij} &= \delta^{ij}, \quad \text{for } i,j,.. = 1,2,..,(n-1), \\ \eta^{ij} &= -\delta^{ij}, \quad \text{for } i,j,.. = n, n+1,..,(n+m-1). \end{aligned} \tag{2.17}$$

The quadratic norm of a general element $\mathbb{X} = X_0\gamma^0 + X_i\gamma^i$ of $\Gamma_{(n,m)}$ is given by

$$\mathbf{Q}(\mathbb{X}) = \frac{1}{2^{[n/2]}}\mathrm{Tr}\mathbb{X}\bar{\mathbb{X}} = X_0X_0 + \eta^{ij}X_iX_j,$$

where

$$\bar{\mathbb{X}} \equiv X_0\gamma^0 - X_i\gamma^i.$$

The norm of a general element $\xi \oplus \mathbb{X}$ of non-simple Jordan algebra $J = \mathbb{R} \oplus \Gamma_{(n,m)}$ is simply given by

$$\mathbf{N}(\xi \oplus \mathbb{X}) = \xi\mathbf{Q}(\mathbb{X}), \tag{2.18}$$

where $\xi \in \mathbb{R}$. The invariance group of the cubic norm is

$$\mathrm{Str}_0(\mathbb{R} \oplus \Gamma_{(n,m)}) = \mathrm{SO}(1,1) \times \mathrm{SO}(n,m). \tag{2.19}$$

If one replaces the underlying division algebra $\mathbb{A}$ of three of the four simple Euclidean Jordan algebras $J_3^{\mathbb{A}}$ by the corresponding split composition algebras $\mathbb{A}_S$ one obtains non-Euclidean simple Jordan algebras $J_3^{\mathbb{A}_S}$ for $\mathbb{A}_S = \mathbb{C}_S, \mathbb{H}_S$ and $\mathbb{O}_S$.

For split octonions $\mathbb{O}_S$, four of the seven "imaginary units" square to $+1$, while the other three square to -1. If we denote the split imaginary units as j^{s}_μ $(\mu = 4,5,6,7)$ and the imaginary units of the real quaternion subalgebra as j_i, $(i = 1,2,3)$ we have:

$$\begin{aligned} j^{\mathrm{s}}_\mu j^{\mathrm{s}}_\nu &= \delta_{\mu\nu} - \eta_{\mu\nu i}j_i, \\ j_i j_j &= -\delta_{ij} + \epsilon_{ijk}j_k, \\ j_i j^{\mathrm{s}}_\mu &= \eta_{i\mu\nu}j^{\mathrm{s}}_\nu, \end{aligned} \tag{2.20}$$

where η_{ABC} $(A,B,C = 1,2,\dots,7)$ are the structure constant of the real octonion algebra $\mathbb{O}$ defined above. For a split octonion

$$O_{\mathrm{s}} = o_0 + o_1j_1 + o_2j_2 + o_3j_3 + o_4j^{\mathrm{s}}_4 + o_5j^{\mathrm{s}}_5 + o_6j^{\mathrm{s}}_6 + o_7j^{\mathrm{s}}_7$$

the norm is

$$O_{\mathrm{s}}\bar{O}_{\mathrm{s}} = o_0^2 + o_1^2 + o_2^2 + o_3^2 - o_4^2 - o_5^2 - o_6^2 - o_7^2,$$

where $\bar{O}_{\rm s} = o_0 - o_1 j_1 - o_2 j_2 - o_3 j_3 - o_4 j_4^{\rm s} - o_5 j_5^{\rm s} - o_6 j_6^{\rm s} - o_7 j_7^{\rm s}$. The norm has the invariance group SO(4, 4). The automorphism group of split octonions is the exceptional group $G_{2(2)}$ with the maximal compact subgroup SU(2) × SU(2).

The automorphism group of the split exceptional Jordan algebra defined by 3 × 3 split octonionic Hermitian matrices of the form

$$J^{\rm s} = \begin{pmatrix} \alpha & Z^{\rm s} & \bar{Y}^{\rm s} \\ \bar{Z}^{\rm s} & \beta & X^{\rm s} \\ Y^{\rm s} & \bar{X}^{\rm s} & \gamma \end{pmatrix} \tag{2.21}$$

is the noncompact group $F_{4(4)}$ with the maximal compact subgroup $U\mathrm{Sp}(6)$ $\times\mathrm{SU}(2)$ and its reduced structure group is $E_{6(6)}$ with the maximal compact subgroup $U\mathrm{Sp}(8)$.

The split exceptional Jordan algebra has a subalgebra generated by elements of the form

$$J^{\rm s} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & X^{\rm s} \\ 0 & \bar{X}^{\rm s} & \gamma \end{pmatrix} \tag{2.22}$$

which is isomorphic to $(\mathbb{R} + \Gamma_{(5,5)})$, whose reduced structure and automorphism groups are SO(5, 5) × SO(1, 1) and SO(4, 5), respectively.

The split quaternion algebra $\mathbb{H}^{\rm s}$ has two "imaginary units" $j_m^{\rm s}$ $(m = 2, 3)$ that square to +1:

$$\begin{aligned} j_m^{\rm s} j_n^{\rm s} &= \delta_{mn} - \epsilon_{mnk} j_k, \\ (j_1)^2 &= -1, \\ j_1 j_m^{\rm s} &= \epsilon_{1mn} j_n^{\rm s} \end{aligned} \tag{2.23}$$

For a split quaternion

$$Q_{\rm s} = q_0 + q_1 j_1 + q_2 j_2^{\rm s} + q_3 j_3^{\rm s},$$

the norm is

$$Q_{\rm s} \bar{Q}_{\rm s} = q_0^2 + q_1^2 - q_2^2 - q_3^2,$$

where $\bar{Q}_{\rm s} = q_0 - q_1 j_1 - q_2 j_2^{\rm s} - q_3 j_3^{\rm s}$ and it is invariant under SO(2, 2). The automorphism group of the split Jordan algebra $J_3^{\mathbb{H}_S}$ is $\mathrm{Sp}(6, \mathbb{R})$ with the maximal compact subgroup SU(3) × $U(1)$. Its reduced structure group is $\mathrm{SL}(6, \mathbb{R})$.

The split complex numbers have an "imaginary unit" that squares to +1 and the norm has SO(1,1) invariance. The automorphism group of split complex Jordan algebra $J_3^{\mathbb{C}_s}$ is $\mathrm{SL}(3,\mathbb{R})$ and its reduced structure group is

$$\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$$

The invariant tensor C_{IJK} defining the cubic norm of the Jordan algebra $(\mathbb{R} \oplus \Gamma_{(5,n)})$ can be identified with the C-tensor in the $F \wedge F \wedge A$ coupling

$$C_{IJK} F^I \wedge F^J \wedge A^K = C_{IJK} \epsilon_{\mu\nu\lambda\sigma\rho} F^{I\mu\nu} F^{J\lambda\sigma} A^{K\rho} \tag{2.24}$$

of $(n+5)$ vector fields of $N=4$ MESGTs that describe the coupling of n vector multiplets to $N=4$ supergravity in five dimensions. Scalar manifolds of these theories are symmetric spaces

$$\mathcal{M}_5 = \frac{\mathrm{SO}(5,n) \times \mathrm{SO}(1,1)}{\mathrm{SO}(5) \times \mathrm{SO}(n)}, \tag{2.25}$$

where the isometry group $\mathrm{SO}(5,n) \times \mathrm{SO}(1,1)$ is simply the reduced structure group of the Jordan algebra $(\mathbb{R} \oplus \Gamma_{(5,n)})$.

The bosonic field content of $N=6$ simple supergravity is the same as that of $N=2$ MESGT defined by the Euclidean Jordan algebra $J_3^{\mathbb{H}}$ [7]. Hence the scalar manifold of $N=6$ supergravity is

$$\mathcal{M}_5(N=6) = \frac{\mathrm{Str}_0(J_3^{\mathbb{H}})}{\mathrm{Aut}(J_3^{\mathbb{H}})} = \frac{\mathrm{SU}^*(6)}{U\mathrm{Sp}(6)}. \tag{2.26}$$

Therefore its invariant C-tensor is simply the one given by the cubic norm of $J_3^{\mathbb{H}}$, which is a Euclidean Jordan algebra.

As for $N=8$ supergravity in five dimensions its C-tensor is simply the one given by the cubic norm of the split exceptional Jordan algebra $J_3^{\mathbb{O}_S}$ defined over split octonions $\mathbb{O}_S$. $E_{6(6)}$ is the invariance group of the C-tensor as well as of the full maximal supergravity in five dimensions whose scalar manifold is

$$\mathcal{M}_5(N=8) = \frac{E_{6(6)}}{U\mathrm{Sp}(8)}. \tag{2.27}$$

The five-dimensional $N=8$ supergravity can be truncated to $N=4$ MESGT describing the coupling of five vector multiplets to $N=4$ supergravity with the scalar manifold

$$\frac{\mathrm{SO}(1,1) \times \mathrm{SO}(5,5)}{\mathrm{SO}(5) \times \mathrm{SO}(5)}. \tag{2.28}$$

The C-tensor of this truncated theory coincides with the symmetric tensor that defines the cubic norm of the Jordan subalgebra $(\mathbb{R} \oplus \Gamma_{(5,5)})$ of $J_3^{\mathbb{O}_S}$ given above.

3 Conformal groups of Jordan algebras and U-duality groups in $d = 4$

The proposal [53] to define generalized spacetimes coordinatized by elements of a Jordan algebra J identifies the automorphism $\mathrm{Aut}(J)$ and reduced structure groups $\mathrm{Str}_0(J)$ of the Jordan algebra J with the rotation and Lorentz groups of the corresponding spacetime, respectively. The Lorentz group and dilatations generate the structure group $\mathrm{Str}(J)$ of J, which extends to the generalized conformal group $\mathrm{Conf}(J)$ of the Jordan algebra J. $\mathrm{Conf}(J)$ can be identified with the invariance group of the light cone defined by the norm form of the Jordan algebra J [19, 53–55]. For Euclidean Jordan algebras of dimensions n defined by a quadratic norm form with a Lorentzian signature the rotation, Lorentz and conformal groups are simply $\mathrm{SO}(n-1)$, $\mathrm{SO}(n-1,1)$ and $\mathrm{SO}(n,2)$, respectively.

Lie algebra $\mathfrak{conf}(J)$ of the generalized conformal group $\mathrm{Conf}(J)$ has a natural three-grading with respect to the dilatation generator $\mathcal{R}$. Choosing a basis e_I for the Jordan algebra and labelling the translations and special conformal generators as T_I and K^I, respectively, we have

$$\mathfrak{conf}(J) = T_I \oplus R_I^J \oplus K^I, \tag{3.1}$$

where $I, J, .. = 1, 2, \ldots, \dim(J) = n_V$. Traceless components L_I^J of R_I^J are the Lorentz group generators and the trace part is proportional to the dilatation generator $\mathcal{R}$:

$$\begin{aligned} \mathcal{R} &= \frac{1}{n_V} R_K^K, \\ R_I^J &= L_I^J + \delta_I^J \mathcal{R}. \end{aligned} \tag{3.2}$$

In the chosen basis e_I for the Jordan algebra an element $\mathbf{x} \in J$ can be written as $\mathbf{x} = e_I q^I = \tilde{e}^I q_I$. Then the action of the generators of $\mathfrak{conf}(J)$ on J can be written as differential operators acting on the "coordinates" q^I.[7]

[7]Note that there are, in general, two inequivalent actions of the reduced structure group: one on Jordan algebra and another one on its conjugate (or dual). The tilde refers to the conjugate basis such that $q_I p^I$ is invariant under the action of reduced structure group $\mathrm{Str}_0(J)$. For details on this issue see [55].

These generators can be twisted by a unitary character λ and take the simple form

$$T_I = \frac{\partial}{\partial q^I},$$

$$R^I_J = -\Lambda^{IK}_{JL} q^L \frac{\partial}{\partial q^K} - \lambda \delta^I_J, \tag{3.3}$$

$$K^I = \frac{1}{2} \Lambda^{IK}_{JL} q^J q^L \frac{\partial}{\partial q^K} + \lambda q^I. \tag{3.4}$$

They satisfy the commutation relations

$$[T_I, K^J] = -R^J_I, \tag{3.5}$$

$$[R^J_I, T_K] = \Lambda^{JL}_{IK} T_L, \tag{3.6}$$

$$[R^J_I, K^K] = -\Lambda^{JL}_{IK} K^L, \tag{3.7}$$

where Λ^{IJ}_{KL} are the structure constants of the Jordan triple product

$$\{e_I, \tilde{e}^K, e_J\} = \Lambda^{KL}_{IJ} e_L, \tag{3.8}$$

$$\{\tilde{e}^K, e_I, \tilde{e}^L\} = \Lambda^{KL}_{IJ} \tilde{e}^J. \tag{3.9}$$

The generators of rotation (automorphism) group Aut(J) are

$$A_{IJ} = R^J_I - R^I_J. \tag{3.10}$$

For Jordan algebras of degree three the structure constants can be written as

$$\Lambda^{IJ}_{KL} := \delta^I_K \delta^J_L + \delta^I_L \delta^J_K - \frac{4}{3} C^{IJM} C_{KLM}, \tag{3.11}$$

where C_{IJK} is the symmetric tensor that defines the cubic norm of J and satisfies the "adjoint identity" [51]:

$$C^{IJK} C_{J(MN} C_{PQ)K} = \delta^I_{(M} C_{NPQ)}. \tag{3.12}$$

Since the C-tensor is an invariant of the Lorentz (reduced structure) group we have

$$C_{IJK} = C^{IJK}. \tag{3.13}$$

The conformal groups of non-Euclidean Jordan algebras $J_3^{\mathbb{C}_s}$, $J_3^{\mathbb{H}_s}$, $J_3^{\mathbb{O}_s}$ and $(\mathbb{R} \oplus \Gamma_{(m,n)})$ are listed in table 1. The conformal group of $J_3^{\mathbb{O}_s}$ is $E_{7(7)}$ which is the U-duality group of maximal supergravity in $d = 4$. Similarly, the conformal groups $\mathrm{SO}(6, n+1) \times \mathrm{SU}(1,1)$ of the Jordan algebras $(\mathbb{R} \oplus \Gamma_{(5,n)})$ are the U-duality groups of $N = 4$ MESGTs in $d = 4$.

Table 1: Below we give the automorphism ($\mathrm{Aut}(J)$), reduced structure $\mathrm{Str}_0(J)$, conformal ($\mathrm{Conf}(J)$) and quasiconformal groups ($\mathrm{QConf}(J)$) associated with non-Euclidean Jordan algebras of degree three.

J	$\mathrm{Aut}(J)$	$\mathrm{Str}_0(J)$	$\mathrm{Conf}(J)$	$\mathrm{QConf}(J)$
$J_3^{\mathbb{C}_s}$	$\mathrm{SL}(3,\mathbb{R})$	$\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$	$\mathrm{SL}(6,\mathbb{R})$	$E_{6(6)}$
$J_3^{\mathbb{H}_s}$	$\mathrm{Sp}(6,\mathbb{R})$	$\mathrm{SL}(6,\mathbb{R})$	$\mathrm{SO}(6,6)$	$E_{7(7)}$
$J_3^{\mathbb{O}_s}$	$F_{4(4)}$	$E_{6(6)}$	$E_{7(7)}$	$E_{8(8)}$
$\mathbb{R} \oplus \Gamma_{(n,m)}$	$\mathrm{SO}(n-1,m)$	$\mathrm{SO}(1,1) \times \mathrm{SO}(n,m)$	$\mathrm{SO(n+1,m+1)} \times \mathrm{SU}(1,1)$	$\mathrm{SO}(n+3,3+m)$

4 Quasiconformal groups associated with non-Euclidean Jordan algebras of degree three

Quasiconformal realizations of Lie groups was first formulated over Freudenthal triple systems associated with Lie groups extended by an extra singlet coordinate [19]. Given a simple Lie algebra $\mathfrak{g}$ one can associate a Freudenthal triple system $\mathcal{F}$ with it via the Freudenthal–Kantor construction [56] using the five-grading of $\mathfrak{g}$

$$\mathfrak{g} = g^{-2} \oplus g^{-1} \oplus g^{0} \oplus g^{+1} \oplus g^{+2} \tag{4.1}$$

such that grade ± 2 subspaces are one dimensional. The generators of grade $+1$ and by conjugation also of -1 subspaces are labelled by the elements of the underlying Freudenthal triple systems (FTS):

$$\mathfrak{g} = \tilde{K} \oplus \tilde{U}_A \oplus S_{AB} \oplus U_B \oplus K, \tag{4.2}$$

where $A, B \in \mathcal{F}$. Here we shall focus on quasiconformal realizations of groups associated with FTSs defined by non-Euclidean Jordan algebras J of degree three in a basis covariant with respect to their Lorentz (reduced structure) groups. The elements of a FTS $\mathcal{F}(J)$ defined over J can be represented as formal 2×2 "matrices"

$$X = \begin{pmatrix} \alpha & \mathbf{x} \\ \mathbf{y} & \beta \end{pmatrix}, \tag{4.3}$$

where $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in J$. We shall write X simply as $X \equiv (\alpha, \beta, \mathbf{x}, \mathbf{y})$ for convenience. Every FTS admits a skew-symmetric bilinear form. Given two elements $X = (\alpha, \beta, \mathbf{x}, \mathbf{y})$ and $Y = (\gamma, \delta, \mathbf{w}, \mathbf{z})$ of $\mathcal{F}(J)$ their skew symmetric bilinear form is

$$\langle X, Y \rangle \equiv \alpha\delta - \beta\gamma + (\mathbf{x}, \mathbf{z}) - (\mathbf{y}, \mathbf{w}), \tag{4.4}$$

where $(\mathbf{x}, \mathbf{z})$ is the symmetric bilinear form over J given by the trace Tr:

$$(\mathbf{x}, \mathbf{z}) \equiv \mathrm{Tr}(\mathbf{x} \circ \mathbf{z}). \tag{4.5}$$

In the normalization and conventions of [19] the quartic norm $\mathcal{Q}_4(X)$ of $X \in \mathcal{F}(J)$ is given by

$$\mathcal{Q}_4(X) \equiv \frac{1}{48} \langle (X, X, X), X \rangle, \tag{4.6}$$

where (X, Y, Z) denotes the Freudenthal triple product. The automorphism group of the FTS $\mathcal{F}(J)$ is isomorphic to the conformal group of the Jordan algebra J:

$$\mathrm{Aut}(\mathcal{F}(J)) \cong \mathrm{Conf}(J) \tag{4.7}$$

We note that under the action of $\mathrm{Aut}(\mathcal{F}(J))$ the elements $X = (\alpha, \beta, \mathbf{x}, \mathbf{y}) \in \mathcal{F}(J)$ transform linearly, which is not to be confused with the nonlinear action of the conformal group $\mathrm{Conf}(J)$ on J. Under the Lorentz subgroup $\mathrm{Str}_0(J)$ of $\mathrm{Aut}(\mathcal{F}(J))$ the Jordan components $\mathbf{x}$ and $\mathbf{y}$ of X transform in conjugate (dual) representations.

For $5d$ supergravity theories whose C-tensors are given by the norm forms of Jordan algebras J of degree three one-to-one correspondence between the vector fields and the elements of J gets extended to a one-to-one correspondence between the vector field strengths and their magnetic duals and elements of the Freudenthal triple system $\mathcal{F}(J)$:

$$(A^I_\mu \leftrightarrow J) \Longrightarrow \left(\begin{pmatrix} F^0 & F^I \\ \tilde{F}^I & \tilde{F}^0 \end{pmatrix} \longrightarrow \mathcal{F}(J) \right),$$

where F^0 denotes the field strength of the vector field that comes from the 5d graviton. Field strengths F^I and their magnetic duals $\tilde{F}^I$ transform in conjugate representations under the Lorentz group $\mathrm{Str}_0(J)$. Since the automorphism group of a Freudenthal triple system $\mathcal{F}(J)$ defined over a Jordan algebra J of degree three is isomorphic to the four-dimensional U-duality group $\mathrm{Aut}(\mathcal{F}(J)) = \mathrm{Conf}(J)$ of corresponding supergravity theories the original formulation of [19] is covariant with respect to $\mathrm{Conf}(J)$.

Consider now the vector space $\mathcal{T}$ of a FTS extended by an extra coordinate. We shall denote vectors in this space as $\mathcal{X} = (X, x) \in \mathcal{T}$ where X

belongs to the FTS and x is the extra coordinate. The action of Lie algebra of the quasiconformal group, associated with a FTS $\mathcal{F}$, on the vector space $\mathcal{T}$ is given by [19, 25]:

$$\begin{aligned} K(X) = 0, \quad U_A(X) = A, \quad S_{AB}(X) = (A, B, X), \\ K(x) = 2, \quad U_A(x) = \langle A, X\rangle, \quad S_{AB}(x) = 2\langle A, B\rangle x, \\ \tilde{U}_A(X) = \frac{1}{2}(X, A, X) - Ax, \\ \tilde{U}_A(x) = -\frac{1}{6}\langle (X, X, X), A\rangle + \langle X, A\rangle x, \\ \tilde{K}(X) = -\frac{1}{6}(X, X, X) + Xx, \\ \tilde{K}(x) = \frac{1}{6}\langle (X, X, X), X\rangle + 2x^2, \end{aligned} \tag{4.8}$$

where $A, B \in \mathcal{F}$.

The quartic norm over the space $\mathcal{T}$ is defined as

$$\mathcal{N}_4(\mathcal{X}) := \mathcal{Q}_4(X) - x^2 \tag{4.9}$$

where $\mathcal{Q}_4(X)$ is the quartic invariant of $X \in \mathcal{F}$. Quartic "symplectic distance" $d(\mathcal{X}, \mathcal{Y})$ between any two points $\mathcal{X} = (X, x)$ and $\mathcal{Y} = (Y, y)$ in $\mathcal{T}$ is defined as the quartic norm of "symplectic difference"

$$\delta(\mathcal{X}, \mathcal{Y}) := (X - Y, x - y + \langle X, Y\rangle) \tag{4.10}$$

of two vectors in $\mathcal{T}$

$$d(\mathcal{X}, \mathcal{Y}) := \mathcal{N}_4(\delta(\mathcal{X}, \mathcal{Y}) = \mathcal{Q}_4(X - Y) - (x - y + \langle X, Y\rangle)^2. \tag{4.11}$$

The quasiconformal group action defined above leaves invariant light-like separations [19]

$$d(\mathcal{X}, \mathcal{Y}) = 0. \tag{4.12}$$

In other words, the quasiconformal group is the invariance group of the light-cone with respect to the quartic distance function (4.11). We shall refer to the submanifold with base point $\mathcal{X}$ in the space $\mathcal{T}$ defined by the condition (4.12) as the "quartic light-cone".[8] Quasiconformal realization of a simple Lie algebra $\mathfrak{g}$ over a FTS $\mathcal{F}$ extended by an extra singlet coordinate

[8] By an abuse of terminology we shall sometimes refer to the distance function (4.11) also as the quartic light cone.

carries over to the complexification of $\mathfrak{g}$. Therefore by taking different real sections one can obtain quasiconformal realizations of different real forms of the corresponding group G.

The quartic light-cone (4.11) is manifestly invariant under the Heisenberg symmetry group corresponding to "symplectic translations" generated by U_A and K in (4.8). "Symplectic special conformal generators" $\tilde{U}_A$ and $\tilde{K}$ also form an Heisenberg subalgebra and their action on the quartic light-cone $d(\mathcal{X},\mathcal{Y})$ results in overall multiplicative factors [19, 36]

$$d(\mathcal{X},\mathcal{Y}) \Longrightarrow f(\mathcal{X},\mathcal{Y}) d(\mathcal{X},\mathcal{Y}) \tag{4.13}$$

which proves that light-like separations are left invariant under the full quasiconformal group action.

For supergravity theories whose five- and four-dimensional U-duality symmetry groups are the Lorentz and Conformal groups of a Jordan algebra J of degree three the U-duality groups of corresponding 3d supergravity theories are isomorphic to the quasiconformal groups $\mathrm{QConf}(J)$ of the Jordan algebras J.

4.1 Quasiconformal Lie algebras of non-Euclidean Jordan algebras of degree three twisted by a unitary character

We shall denote the basis vectors of $\mathcal{F}(J)$ as follows:

$$\begin{pmatrix} \alpha & \mathbf{x} \\ \mathbf{y} & \beta \end{pmatrix} = \alpha \mathbf{e}_0 + \beta \tilde{\mathbf{e}}^0 + x^I \mathbf{e}_I + y_I \tilde{\mathbf{e}}^I, \tag{4.14}$$

where $I = 1, \ldots, n_V = \dim(J)$ and $\mathbf{x}$ and $\mathbf{y}$ transform in conjugate representations of the Lorentz group $\mathrm{Str}_0(J)$.

The quasiconformal Lie algebra associated with a Jordan algebra J of degree three which we denote interchangeably as $\mathrm{QConf}(\mathcal{F}(J))$ or as $\mathrm{QConf}(J)$ can be given a 7×5 graded decomposition that is covariant with respect to the reduced structure group $\mathrm{Str}_0(J)$ as shown in table 2. With applications to supergravity theories in mind we shall label the elements $X, Y, ..$ of FTS $\mathcal{F}(J)$ in terms of coordinates $(q_0,\, q_I)$ and momenta $(p^0,\, p^I)$ as follows[9]

$$X = q_0 \tilde{\mathbf{e}}^0 + q_I \tilde{\mathbf{e}}^I + p^I \mathbf{e}_I + p^0 \mathbf{e}_0. \tag{4.15}$$

[9] $p^I q_I = (\mathbf{e}_I p^I, q_I \tilde{\mathbf{e}}^I)$ and $(p^\sharp)_I (q^\sharp)^I = ({p^\sharp}_I \tilde{\mathbf{e}}^I, {q^\sharp}^I \mathbf{e}_I)$.

Table 2: Below we give the 7×5 grading of the quasiconformal Lie algebra QConf(J) associated with the Freudenthal triple system $\mathcal{F}(J)$ defined over a Jordan algebra J of degree 3. The vertical five-grading is determined by $\mathcal{D} = -\Delta$ that commutes with the Lorentz group generators L_I^J and with $\mathcal{R}$. Horizontal seven-grading is determined by $\mathcal{R}$. The generators $\tilde{R}^I, R_I^J$ and R_I generate the automorphism group Aut$(\mathcal{F}(J))$ under which the generators (U_0, U_I, V^I, V^0) as well as $(\tilde{U}_0, \tilde{U}_I, \tilde{V}^I, \tilde{V}^0)$ transform linearly in a symplectic representation.

$$
\begin{array}{ccccccccc}
 & & & & K & & & & \\
U_0 & & U_I & & | & & V^I & & V^0 \\
--- & \tilde{R}^I & --- & & (\mathcal{D} \oplus L_I^J \oplus \mathcal{R}) & & --- & R_J & --- \\
\tilde{U}_0 & & \tilde{U}_I & & | & & \tilde{V}^I & & \tilde{V}^0 \\
 & & & & \tilde{K} & & & &
\end{array}
$$

We shall normalize the basis elements and cubic norm (C-tensor) such that the quartic invariant is given by

$$
\begin{aligned}
I_4(X) &= (p^0 q_0 - p^I q_I)^2 - \frac{4}{3} C_{IJK} p^J p^K C^{ILM} q_L q_M \\
&\quad + \frac{4}{3\sqrt{3}} p^0 C^{IJK} q_I q_J q_K + \frac{4}{3\sqrt{3}} q_0 C_{IJK} p^I p^J p^K \\
&= (p^0 q_0 - p^I q_I)^2 - \frac{4}{3} (p^\sharp)_I (q^\sharp)^I \\
&\quad + \frac{4}{3\sqrt{3}} p^0 \mathcal{N}(q) + \frac{4}{3\sqrt{3}} q_0 \mathcal{N}(p),
\end{aligned} \tag{4.16}
$$

where

$$
\begin{aligned}
\mathcal{N}(q) &\equiv C^{IJK} q_I q_J q_K, \quad (q^\sharp)^I \equiv C^{IJK} q_J q_K, \\
\mathcal{N}(p) &\equiv C_{IJK} p^I p^J p^K, \quad (p^\sharp)_I \equiv C_{IJK} p^J p^K.
\end{aligned}
$$

The basis vectors $\mathbf{e}_I$ ($\tilde{\mathbf{e}}^I$) of the Jordan algebra J (and its conjugate $\tilde{J}$) are normalized such that

$$
(\tilde{e}^J, \tilde{e}^I) = \operatorname{Tr} \tilde{e}^I \circ \tilde{e}^J = \eta^{IJ}, \tag{4.17}
$$

$$
(e_I, e_J) = \operatorname{Tr} e_I \circ e_J = \eta_{IJ}, \tag{4.18}
$$

$$
(e_I, \tilde{e}^J) = \operatorname{Tr} e_I \circ \tilde{e}^J = \delta_I^J, \tag{4.19}
$$

where η_{IJ} is diagonal and equal to δ_{IJ} for Euclidean (compact) elements and equal to $-\delta_{IJ}$ for non Euclidean (noncompact) elements and will be explicitly given below. Furthermore, we will label the basis elements of the Jordan algebra J such that e_1, e_2 and e_3 are the three irreducible idempotents of J and the identity element $\mathbb{I}$ is simply

$$\mathbb{I} = e_1 + e_2 + e_3. \tag{4.20}$$

The action of the generators of quasiconformal group QConf(J) on the space $\mathcal{T} = \mathcal{F}(J) \oplus \mathbb{R}$ with coordinates q_0, q_I, p^0, p^I of $\mathcal{F}(J)$ plus an extra singlet coordinate $x \in \mathbb{R}$, twisted by a unitary character ν, is given by the following differential operators:

$$K = \partial_x, \tag{4.21}$$

$$U_0 = \partial_{p^0} + q_0 \partial_x, \tag{4.22}$$

$$U_I = -\partial_{p^I} + q_I \partial_x, \tag{4.23}$$

$$V^0 = \partial_{q_0} - p^0 \partial_x, \tag{4.24}$$

$$V^I = \partial_{q_I} + p^I \partial_x, \tag{4.25}$$

$$R_I = -\sqrt{2} C_{IJK} p^K \partial_{q_K} - \sqrt{\frac{3}{2}} (p^0 \partial_{p^I} + q_I \partial_{q_0}), \tag{4.26}$$

$$\tilde{R}^I = \sqrt{2} C^{IJK} q_J \partial_{p^K} + \sqrt{\frac{3}{2}} (q_0 \partial_{q_I} + p^I \partial_{p^0}), \tag{4.27}$$

$$\begin{aligned} R_I{}^J = {} & \frac{3}{2} \delta_I{}^J (p^0 \partial_{p^0} - q_0 \partial_{q_0}) \\ & + \frac{3}{2} \left(\delta_I{}^N \delta_K{}^J - \frac{4}{3} C_{IKL} C^{JNL} \right) (q_N \partial_{q_K} - p^K \partial_{p^N}), \end{aligned} \tag{4.28}$$

$$\mathcal{R} = \frac{1}{n_V} R_I{}^I = \frac{3}{2} (p^0 \partial_{p^0} - q_0 \partial_{q_0}) + \frac{1}{2} (p^I \partial_{p^I} - q_I \partial_{q^I}), \tag{4.29}$$

$$\Delta == -\mathcal{D} = -(p^0 \partial_{p^0} + p^I \partial_{p^I} + q_0 \partial_{q_0} + q_I \partial_{q_I} - \nu) - 2x \partial_x, \tag{4.30}$$

$$\begin{aligned} \tilde{K} = {} & x(p^0 \partial_{p^0} + p^I \partial_{p^I} + q_0 \partial_{q_0} + q_I \partial_{q_I} - \nu) + (x^2 + I_4) \partial_x \\ & + \frac{1}{2} \left(\frac{\partial I_4}{\partial p^0} \partial_{q_0} - \frac{\partial I_4}{\partial q_0} \partial_{p^0} + \frac{\partial I_4}{\partial q_I} \partial_{p^I} - \frac{\partial I_4}{\partial p^I} \partial_{q_I} \right). \end{aligned} \tag{4.31}$$

The vertical five grading is determined by the adjoint action of $\mathcal{D}$

$$\left[\mathcal{D}, \begin{pmatrix} U_0 \\ U_I \\ V^I \\ V^0 \end{pmatrix} \right] = \begin{pmatrix} U_0 \\ U_I \\ V^I \\ V^0 \end{pmatrix}. \tag{4.32}$$

The vertical grade -1 generators are obtained from the grade $+1$ generators by commutation with the grade -2 generator $\tilde{K}$

$$\tilde{U}_0 = [U_0, \tilde{K}], \tag{4.33}$$
$$\tilde{V}^0 = [V^0, \tilde{K}], \tag{4.34}$$
$$\tilde{U}_I = [U_I, \tilde{K}], \tag{4.35}$$
$$\tilde{V}^I = [V^I, \tilde{K}] \tag{4.36}$$

and satisfy

$$\left[\mathcal{D}, \begin{pmatrix} \tilde{U}_0 \\ \tilde{U}_I \\ \tilde{V}^I \\ \tilde{V}^0 \end{pmatrix}\right] = - \begin{pmatrix} \tilde{U}_0 \\ \tilde{U}_I \\ \tilde{V}^I \\ \tilde{V}^0 \end{pmatrix}. \tag{4.37}$$

The remaining nonvanishing commutation relations of Lie algebra of QConf(J) are as follows:

$$[K, \tilde{K}] = \Delta, \tag{4.38}$$
$$[\Delta, K] = -2K, \tag{4.39}$$
$$[\Delta, \tilde{K}] = 2\tilde{K}, \tag{4.40}$$
$$[U_I, V^J] = -2\delta_I^J K, \tag{4.41}$$
$$[U_0, V^0] = -2K, \tag{4.42}$$
$$[K, \tilde{U}_0] = U_0, \tag{4.43}$$
$$[K, \tilde{U}_I] = U_I, \tag{4.44}$$
$$[K, \tilde{V}^I] = V^I, \tag{4.45}$$
$$[K, \tilde{V}^0] = V^0, \tag{4.46}$$
$$[\tilde{U}_I, \tilde{V}^J] = -2\delta_I^J \tilde{K}, \tag{4.47}$$
$$[\tilde{U}_0, \tilde{V}^0] = -2\tilde{K}, \tag{4.48}$$
$$[U_0, \tilde{V}^0] = -2\mathcal{R} + \mathcal{D}, \tag{4.49}$$
$$[V^0, \tilde{U}_0] = -2\mathcal{R} - \mathcal{D}, \tag{4.50}$$
$$[R_I^J, R_K] = \frac{3}{2}\Lambda_{IK}^{JL} R_L, \tag{4.51}$$
$$[R_I^J, \tilde{R}^L] = -\frac{3}{2}\Lambda_{IL}^{JK} \tilde{R}^L, \tag{4.52}$$
$$[R_I^J, U_K] = \frac{3}{2}\Lambda_{IK}^{JL} U_L - \frac{3}{2}\delta_I^J U_K, \tag{4.53}$$

$$[R_I^J, V^K] = -\frac{3}{2}\Lambda_{IL}^{JK} V^L + \frac{3}{2}\delta_I^J V^K, \tag{4.54}$$

$$[R_I^J, \tilde{U}_K] = \frac{3}{2}\Lambda_{IK}^{JL} \tilde{U}_L - \frac{3}{2}\delta_I^J \tilde{U}_K, \tag{4.55}$$

$$[R_I^J, \tilde{V}^K] = -\frac{3}{2}\Lambda_{IL}^{JK} \tilde{V}^L + \frac{3}{2}\delta_I^J \tilde{V}^K, \tag{4.56}$$

$$[R_I, \tilde{R}^J] = -R_I^J, \tag{4.57}$$

$$[U_0, \tilde{V}^I] = 2\sqrt{\frac{2}{3}}\tilde{R}^I, \tag{4.58}$$

$$[\tilde{U}_0, V^I] = -2\sqrt{\frac{2}{3}}\tilde{R}^I, \tag{4.59}$$

$$[V^0, \tilde{U}_I] = -2\sqrt{\frac{2}{3}}R_I, \tag{4.60}$$

$$[\tilde{V}^0, U_I] = -2\sqrt{\frac{2}{3}}R_I, \tag{4.61}$$

$$[U_I, \tilde{V}^J] = \frac{4}{3}R_I^J - \delta_I^J(\Delta + 2\mathcal{R}), \tag{4.62}$$

$$[\tilde{U}_I, V^J] = -\frac{4}{3}R_I^J - \delta_I^J(\Delta - 2\mathcal{R}), \tag{4.63}$$

$$[U_I, \tilde{U}_J] = -\frac{4}{3}\sqrt{2}C_{IJK}\tilde{R}^K, \tag{4.64}$$

$$[V^I, \tilde{V}^J] = -\frac{4}{3}\sqrt{2}C^{IJK}R_K, \tag{4.65}$$

$$[V^I, R_J] = -\sqrt{\frac{3}{2}}\delta_J^I V^0, \tag{4.66}$$

$$[\tilde{V}^I, R_J] = -\sqrt{\frac{3}{2}}\delta_J^I \tilde{V}^0, \tag{4.67}$$

$$[\tilde{U}_I, \tilde{R}^J] = -\sqrt{\frac{3}{2}}\delta_I^J \tilde{U}_0, \tag{4.68}$$

$$[U_I, \tilde{R}^J] = -\sqrt{\frac{3}{2}}\delta_I^J U_0, \tag{4.69}$$

where

$$\Lambda_{KL}^{IJ} := \delta_K^I \delta_L^J + \delta_L^I \delta_K^J - \frac{4}{3}C^{IJM}C_{KLM}. \tag{4.70}$$

There is a distinguished $\mathrm{SL}(3, \mathbb{R})$ subgroup of $\mathrm{QConf}(J)$ whose centralizer is the Lorentz group $\mathrm{Str}_0(J)$ generated by L_I^J, whose generators are K, $\tilde{K}$, U_0, $\tilde{U}_0$, V^0, $\tilde{V}^0$, $\mathcal{R}$ and $\mathcal{D}$. Its maximal compact subgroup $\mathrm{SO}(3) \subset \mathrm{SL}(3, \mathbb{R})$

is generated by

$$T_1 := \frac{1}{\sqrt{2}}(U_0 - \tilde{V}^0), \tag{4.71}$$

$$T_2 := \frac{1}{\sqrt{2}}(V^0 + \tilde{U}_0), \tag{4.72}$$

$$T_3 := -(K + \tilde{K}) \tag{4.73}$$

that satisfy

$$[T_i, T_j] = \epsilon_{ijk} T_k, \tag{4.74}$$

where $i, j, k = 1, 2, 3$. The generators of the maximal compact subgroup K of $\mathrm{QConf}(J)$ are

$$\begin{aligned}
&(U_I - \eta_{IJ}\tilde{V}^J),\\
&(V^I + \eta^{IJ}\tilde{U}_J),\\
&(R_I + \eta_{IJ}\tilde{R}^J),\\
&(R_I^J - \eta_{IK}\eta^{JL}R_L^K),\\
&(\tilde{U}_0 + V^0),\\
&(-U_0 + \tilde{V}^0),\\
&(K + \tilde{K}),
\end{aligned} \tag{4.75}$$

where $I, J, \ldots = 1, 2, \ldots, n_V$ where n_V is the dimension of J.[10]

4.2 Quadratic Casimir operators of quasiconformal Lie algebras

The generators L_I^J of the reduced structure (Lorentz) group of a Jordan algebra are given by the traceless components of R_I^J:

$$L_I^J = R_I^J - \frac{1}{n_V}\delta_I^J (R_K^K) = R_I^J - \delta_I^J \mathcal{R}. \tag{4.76}$$

The quadratic Casimir operator of the quasiconformal group $\mathrm{QConf}(J)$ of a simple Jordan algebra J of degree three can then be written in a general form involving a single parameter α, valid both for Euclidean and

[10] For the supergravity theories whose C-tensors are determined by a Jordan algebra, n_V is the number of vector fields in five dimensions. Hence the notation.

non-Euclidean J:

$$\begin{aligned}\mathcal{C}_2 = \alpha L_I^J L_J^I - \frac{4}{3}(\mathcal{R}^2 + \tilde{R}^I R_I + R_I \tilde{R}^I) + (U_0 \tilde{V}^0 + U_I \tilde{V}^I + \tilde{V}^0 U_0 + \tilde{V}^I U_I) \\ - (\tilde{U}_0 V^0 + \tilde{U}_I V^I + V^0 \tilde{U}_0 + V^I \tilde{U}_I) - 2(K\tilde{K} + \tilde{K}K) + \Delta^2, \qquad (4.77)\end{aligned}$$

where α takes on the following values for different quasiconformal Lie groups QCG(J):

$$\begin{aligned}\alpha(F_{4(4)}) &= \frac{16}{45}, \\ \alpha(E_{6(6)}) &= \frac{8}{27}, \\ \alpha(E_{7(7)}) &= \frac{2}{9}, \\ \alpha(E_{8(8)}) &= \frac{4}{27}.\end{aligned} \qquad (4.78)$$

As for the quasiconformal groups $\mathrm{QConf}(\mathbb{R} \oplus \Gamma_{(n,m)}) = \mathrm{SO}(n+3, m+3)$ associated with the generic family of reducible Jordan algebras $J = \mathbb{R} \oplus \Gamma_{(n,m)}$ the quadratic Casimir can be written in the form:

$$\begin{aligned}&\mathcal{C}_2(\mathrm{SO}(n+3, m+3)) \\ &\quad = \frac{4}{9} R_I^J R_J^I - \frac{4}{3}(\tilde{R}^I R_I + R_I \tilde{R}^I) - \frac{(n+m-2)}{9}(R_2^2 + R_3^3)^2 \\ &\quad + (U_0 \tilde{V}^0 + U_I \tilde{V}^I + \tilde{V}^0 U_0 + \tilde{V}^I U_I) \\ &\quad - (\tilde{U}_0 V^0 + \tilde{U}_I V^I + V^0 \tilde{U}_0 + V^I \tilde{U}_I) - 2(K\tilde{K} + \tilde{K}K) + \Delta^2. \qquad (4.79)\end{aligned}$$

The dimension of $J = \mathbb{R} \oplus \Gamma_{(n,m)}$ is

$$n_V = n + m + 1. \qquad (4.80)$$

The quadratic Casimir operators for all the quasiconformal groups QConf(J) reduce to a c-number whose value can be expressed universally in terms of the twisting parameter and the dimension n_V of the Jordan algebra J as

$$\mathcal{C}_2(\mathrm{QCG}(J)) = \nu(\nu + 2n_V + 4). \qquad (4.81)$$

As a consequence the representations induced by the quasiconformal action of QConf(J) on the space of square integrable functions $L^2(p^I, q_I, x)$ of $(n_V + 1)$ coordinates q_I, $(n_V + 1)$ momenta and the singlet coordinate x are unitary representations belonging to the principle series under the scalar product

$$\langle f|g\rangle = \int \bar{f}(p,q,x)g(p,q,x)\,dp\,dq\,dx \tag{4.82}$$

for

$$\nu = -(n_V+2)+\mathrm{i}\rho, \tag{4.83}$$

where $\rho \in \mathbb{R}$. For special discrete values of the twisting parameter ν one obtains representations belonging to the discrete series and their continuations. Typically, these representations arise as submodules of the Verma modules obtained by the action of noncompact generators on the spherical vectors of quasiconformal group actions. For the quasiconformal groups of Euclidean Jordan algebras these special representations include the quaternionic discrete representations and their continuations [36,57], which can be realized over the space of holomorphic functions of complexified quasiconformal coordinates. These holomorphic coordinates can be identified with the natural complex coordinates of the twistor spaces associated with the quaternionic symmetric spaces

$$\frac{\mathrm{QConf}(J)}{\widetilde{\mathrm{Conf}}(J)\times \mathrm{SU}(2)}$$

when J is Euclidean. For rank two quaternionic quasiconformal groups $\mathrm{SU}(2,1)$ and $G_{2(2)}$ these representations were studied in [36].

5 Spherical vectors of quasiconformal groups associated with split non-Euclidean Jordan algebras of degree three

A Jordan algebra J of degree three admits three mutually orthogonal irreducible idempotents $\mathbb{P}_1$,$\mathbb{P}_2$,$\mathbb{P}_3$:

$$\begin{aligned} \mathbb{P}_i \circ \mathbb{P}_j &= \delta_{ij}\mathbb{P}_i, \\ Tr(\mathbb{P}_i) &= 1, \quad i,j,.. = 1,2,3. \end{aligned} \tag{5.1}$$

As stated earlier in our labelling of the basis elements of split Jordan algebras of degree three we identify e_i with $\mathbb{P}_i$ for $i=1,2,3$. By the action of the automorphism group $\mathrm{Aut}(J)$ one can "diagonalize" a general element $\mathbf{x} \in J$

$$\mathrm{Aut}(J):\ \mathbf{x} \longrightarrow \mathbf{x}_D = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3. \tag{5.2}$$

The cubic norm of $\mathbf{x}$ is therefore

$$\mathcal{N}(\mathbf{x}) = 3\sqrt{3}(\lambda_1\lambda_2\lambda_3). \tag{5.3}$$

This holds true for Euclidean as well as for split non-Euclidean Jordan algebras of degree three (assuming $n_V \geqslant 3$). The quasiconformal group of the Jordan subalgebra generated by the irreducible idempotents is $\mathrm{SO}(4,4)$ which is the U-duality group of the STU model and was studied in [47]. This Jordan subalgebra is isomorphic to the Euclidean algebra $(\mathbb{R} \oplus \Gamma_{(1,1)})$, which is a subalgebra of all the generic family of Jordan algebras $(\mathbb{R} \oplus \Gamma_{(m,n)})$ for $m, n > 1$. The simple split non-Euclidean Jordan algebras all have $J_3^{\mathbb{R}}$ as a subalgebra and hence its quasiconformal group $F_{4(4)}$ is a subgroup of the split exceptional quasiconformal groups $E_{6(6)}$, $E_{7(7)}$ and $E_{8(8)}$ of the split simple Jordan algebras $J_3^{\mathbb{C}_s}$, $J_3^{\mathbb{H}_s}$ $J_3^{\mathbb{O}_s}$. Now the quasiconformal groups of all Euclidean Jordan algebras of degree three are of the quaternionic real form, in particular the groups $\mathrm{SO}(4,4)$ and $F_{4(4)}$.

Unitary representations induced by the action of quaternionic quasiconformal groups with unitary character ν include the quaternionic discrete series representations of Gross and Wallach [46] as was shown explicitly for rank two cases in [36]. The explicit expressions for the spherical vectors of quasiconformal realizations of $\mathrm{SU}(2,1)$ and $G_{2(2)}$ were essential to establish this result [36]. The quaternionic discrete series representations and their continuations appear as submodules in the Verma modules generated by the action of noncompact generators on the spherical vectors for special discrete values of the parameter ν that is the twisting parameter in the quasiconformal group action. The spherical vectors for all quaternionic quasiconformal groups defined by Euclidean Jordan algebras of degree three were given in [47]. In this section we shall extend these results to quasiconformal groups of all split non-Euclidean Jordan algebras of degree three. The study of discrete series representations induced by the corresponding quasiconformal group actions will be subjects of separate studies.

Now the spherical vector of quasiconformal group action of $\mathrm{QConf}(J)$ twisted by a unitary character ν is a function $\Phi_\nu(p, q, x)$ of $2n_v + 3$ variables q_0,q_I,p^0,p^I and x that is annihilated by all the generators $\mathfrak{K}_M$ of the maximal compact subgroup K of $\mathrm{QConf}(J)$:

$$\mathfrak{K}_M \Phi_\nu(p, q, x) = 0. \tag{5.4}$$

In [47] we presented the spherical vectors of all quasiconformal groups associated with Euclidean Jordan algebras of degree three. The spherical vector of a general quasiconformal group $\mathrm{QConf}(J)$ associated with a split non-Euclidean Jordan algebra J can be written in the form:

$$\Phi_\nu(p,q,x) = \left[(1+x^2+I_2^S - I_4)^2 - (I_2^S)^2 + 8I_4 + \frac{1}{2}I_6^S + 8xJ_4^S + \frac{4}{81}H_4^S\right]^{\frac{\nu}{4}}, \tag{5.5}$$

where

$$I_2^S = (p^0)^2 + (q_0)^2 + p^I\eta_{IJ}p^J + q_I\eta^{IJ}q_J, \tag{5.6}$$

$$\begin{aligned} I_4 &= (p^0q_0 - p^Iq_I)^2 - \frac{4}{3}C_{IJK}p^Jp^KC^{ILM}q_Lq_M \\ &\quad + \frac{4}{3\sqrt{3}}p^0C^{IJK}q_Iq_Jq_K + \frac{4}{3\sqrt{3}}q_0C_{IJK}p^Ip^Jp^K, \end{aligned} \tag{5.7}$$

$$J_4^S = \frac{1}{4}\left(p^0\frac{\partial I_4}{\partial q_0} - q_0\frac{\partial I_4}{\partial p^0} + q_I\eta^{IJ}\frac{\partial I_4}{\partial p^J} - p^I\eta_{IJ}\frac{\partial I_4}{\partial q_J}\right), \tag{5.8}$$

$$\begin{aligned} I_6^S &= \left(\frac{\partial I_4}{\partial p^0}\right)^2 + \left(\frac{\partial I_4}{\partial q_0}\right)^2 + \left(\frac{\partial I_4}{\partial p^I}\right)\eta^{IJ}\left(\frac{\partial I_4}{\partial p^J}\right) \\ &\quad + \left(\frac{\partial I_4}{\partial q_I}\right)\eta_{IJ}\left(\frac{\partial I_4}{\partial q_J}\right) + 4I_4I_2^S, \end{aligned} \tag{5.9}$$

$$\begin{aligned} H_4^S &= 27\eta^{IJ}((p^\#)_I - \sqrt{3}p^0q_I)((p^\#)_J - \sqrt{3}p^0q_J) \\ &\quad + 27\eta_{IJ}((q^\#)^I - \sqrt{3}q_0p^I)((q^\#)^J - \sqrt{3}q_0p^J) \\ &\quad + 54((q^\#)^I - \sqrt{3}q_0p^I)((p^\#)_I - \sqrt{3}p^0q_I) + C_4^S(J), \end{aligned} \tag{5.10}$$

where $(p^\#)_I = C_{IJK}p^Jp^K$ and $(q^\#)^I = C^{IJK}q_Jq_K$. $C_4^S(J)$ is the "correction" term that vanishes when restricted to the subalgebra $\mathrm{SO}(4,4)$ and has a different form for simple Jordan algebras and non-simple ones. For simple split non-Euclidean Jordan algebras J of degree three the quartic correction term $C_4^S(J)$ is given by

$$\begin{aligned} C_4^S(J_3^{\mathbb{A}_S}) &= 81(\mathrm{Tr}[M_0(p)\circ\tilde{M}_0(q)])^2 + \frac{81}{2}\mathrm{Tr}[M_0(p)^2]\mathrm{Tr}[\tilde{M}_0(q)^2] \\ &\quad - 243\mathrm{Tr}[\{M_0(p),\tilde{M}_0(q),M_0(p)\}\circ\tilde{M}_0(q)], \end{aligned} \tag{5.11}$$

where

$$M_0(q) = M(q) - \frac{1}{3}\mathrm{Tr}M(q), \tag{5.12}$$

$$M(q) = e^Iq_I \in J_3^{\mathbb{A}_S} \tag{5.13}$$

and $\tilde{M}(q) = M^*(q)$ where $*$ is the conjugation that replaces the "imaginary" units j_μ^S that square to $+1$ by $-j_\mu^S$ in the underlying split composition algebra $\mathbb{A}^S$. $\{A, B, C\}$ denotes the Jordan triple product

$$\{A, B, C\} = A \circ (B \circ C) + C \circ (B \circ A) - (A \circ C) \circ B. \tag{5.14}$$

For special Jordan algebras with the Jordan product

$$A \circ B = \frac{1}{2}(AB + BA),$$

one finds

$$\{A, B, A\} = ABA. \tag{5.15}$$

Therefore for Jordan algebras $J_3^{\mathbb{C}_S}$ and $J_3^{\mathbb{H}_S}$ the term $C_4^S(J)$ can be written as

$$\begin{aligned} C_4^S(J_3^{\mathbb{A}_S}) = 81(\mathrm{Tr}[M_0(p)\tilde{M}_0(q)])^2 + \frac{81}{2}\mathrm{Tr}[M_0(p)^2]\mathrm{Tr}[\tilde{M}_0(q)^2] \\ - 243\mathrm{Tr}[M_0(p)\tilde{M}_0(q)M_0(p)\tilde{M}_0(q)]. \end{aligned} \tag{5.16}$$

For the generic nonsimple Jordan algebras $(\mathbb{R} \oplus \Gamma_{(n,m)})$ $(m, n \geqslant 1$ of degree three the cubic form is

$$\mathcal{N}(q) = C^{IJK} q_I q_J q_K = \frac{3\sqrt{3}}{2} q_1 [2q_2 q_3 - \eta^{\hat{I}\hat{J}} q_{\hat{I}} q_{\hat{J}}] \tag{5.17}$$

and the quartic correction term C_4^S that appears in H_4^S is given by

$$\begin{aligned} C_4^S(\mathbb{R} \oplus \Gamma_{(n,m)}) = -\frac{81}{2}\{(p^2 - p^3)(q_2 - q_3) + 2p^{\hat{I}} q_{\hat{I}}\}^2 + \frac{81}{2}\{(p^2 - p^3)^2 \\ + 2\eta_{\hat{I}\hat{J}} p^{\hat{I}} p^{\hat{J}}\}\{(q_2 - q_3)^2 + 2\eta^{\hat{I}\hat{J}} q_{\hat{I}} q_{\hat{J}}\}, \end{aligned} \tag{5.18}$$

where $\hat{I}, \hat{J}, \ldots = 4, 5, \ldots, (m + n + 1)$.

We should perhaps recall that the Euclidean Jordan algebra $J_3^{\mathbb{R}}$ is a subalgebra of all simple split Jordan algebras of degree three. An element of $J_3^{\mathbb{R}}$ can be written as

$$M(p) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}p^1 & p^6 & p^5 \\ p^6 & \sqrt{2}p^2 & p^4 \\ p^5 & p^4 & \sqrt{2}p^3 \end{pmatrix} \tag{5.19}$$

and its cubic norm is simply

$$\mathcal{N}(M(p)) = 3\sqrt{3}\mathrm{Det}M = C_{IJK}p^Ip^Jp^K = 3\sqrt{3}\left\{ p^1p^2p^3 \right. \\ \left. - \frac{1}{2}\left[p^1(p^4)^2 + p^2(p^5)^2 + p^3(p^6)^2\right] + \frac{1}{\sqrt{2}}p^4p^5p^6 \right\}. \quad (5.20)$$

5.1 Quasiconformal group $E_{8(8)}$

The quasiconformal group associated with the split exceptional Jordan algebra $J_3^{\mathbb{O}_S}$ is the split exceptional group $E_{8(8)}$. A general element of $J_3^{\mathbb{O}_S}$ can be written as

$$M(p) = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}p^1 & P_S^6 & \bar{P}_S^5 \\ \bar{P}_S^6 & \sqrt{2}p^2 & P^4 \\ P_S^5 & \bar{P}_S^4 & \sqrt{2}p^3 \end{pmatrix}, \quad (5.21)$$

where P_S^4, P_S^5 and P_S^6 are split octonions. The cubic norm of $M(p)$ is normalized to be

$$\mathcal{N}(M(p)) = 3\sqrt{3}\{p^1p^2p^3 - \frac{1}{2}(p^1P_S^4\bar{P}_S^4 + p^2P_S^5\bar{P}_S^5 + p^3P_S^6\bar{P}_S^6) \\ + \frac{1}{\sqrt{2}}\mathrm{Re}(P_S^4P_S^5P_S^6)\}, \quad (5.22)$$

where $\mathrm{Re}(X_S)$ denotes the real part of $X_S \in \mathbb{O}_S$ and $\bar{X}_S$ is the octonion conjugate of X_S. If we expand the elements P_S^4, P_S^5 and P_S^6 in terms of their real components:

$$\begin{aligned} P_S^4 &= p^4 + p^{4+3i}j_i + p^{4+3\mu}j_\mu^{\mathrm{s}}, \\ \bar{P}_S^4 &= p^4 - p^{4+3i}j_i - p^{4+3\mu}j_\mu^{\mathrm{s}}, \\ P_S^5 &= p^5 + p^{5+3i}j_i + p^{5+3\mu}j_\mu^{\mathrm{s}}, \\ \bar{P}_S^5 &= p^5 - p^{5+3i}j_i - p^{5+3\mu}j_\mu^{\mathrm{s}}, \\ P_S^6 &= p^6 + p^{6+3i}j_i + p^{6+3\mu}j_\mu^{\mathrm{s}}, \\ \bar{P}_S^6 &= p^6 - p^{6+3i}j_i - p^{6+3\mu}j_\mu^{\mathrm{s}}, \end{aligned} \quad (5.23)$$

where the indices i and μ are summed over with i running over $1, 2, 3$ and μ running over $4, 5, 6, 7$, we can write the cubic norm as

$$\mathcal{N}(M(p)) = 3\sqrt{3}\left\{ p^1p^2p^3 - \frac{1}{2}p^1[(p^4)^2 + p^{4+3i}p^{4+3i} - p^{4+3\mu}p^{4+3\mu}] \right.$$

$$
\begin{aligned}
&-\frac{1}{2}p^2[(p^5)^2+p^{5+3i}p^{5+3i}-p^{5+3\mu}p^{5+3\mu}]\\
&-\frac{1}{2}p^3[(p^6)^2+p^{6+3i}p^{6+3i}-p^{6+3\mu}p^{6+3\mu}]\\
&+\frac{1}{\sqrt{2}}\Big\{p^4p^5p^6-p^4(p^{(5+3i)}p^{(6+3i)}-p^{(5+3\mu)}p^{(6+3\mu)})\\
&-p^5(p^{(4+3i)}p^{(6+3i)}-p^{(4+3\mu)}p^{(6+3\mu)})\\
&-p^6(p^{(4+3i)}p^{(5+3i)}-p^{(4+3\mu)}p^{(5+3\mu)})\Big\}\\
&-\frac{1}{\sqrt{2}}\epsilon_{ijk}p^{4+3i}p^{5+3j}p^{6+3k}\\
&+\frac{1}{\sqrt{2}}\eta_{i\mu\nu}p^{4+3i}p^{5+3\mu}p^{6+3\nu}+\frac{1}{\sqrt{2}}\eta_{\mu i\nu}p^{4+3\mu}p^{5+3i}p^{6+3\nu}\\
&+\frac{1}{\sqrt{2}}\eta_{\mu\nu i}p^{4+3\mu}p^{5+3\nu}p^{6+3i}\Bigg\}. \qquad (5.24)
\end{aligned}
$$

With the above labelling of the basis elements we have

$$
\begin{aligned}
\eta_{AB}&=\delta_{AB},\\
\eta_{RS}&=-\delta_{RS},\\
\eta_{AR}&=0,
\end{aligned}
\qquad (5.25)
$$

where $A,B,..=1,2,\dots,15$ and $R,S,..=16,17,\dots,27$. The conjugate elements $\tilde{M}(p)$ of $J_3^{\mathbb{O}_S}$ obtained by conjugation $*$ is given by

$$
M(p)=\frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{2}p^1 & (P_S^6)^* & (\bar{P}_S^5)^*\\ (\bar{P}_S^6)^* & \sqrt{2}p^2 & (P^4)^*\\ (P_S^5)^* & (\bar{P}_S^4)^* & \sqrt{2}p^3\end{pmatrix}, \qquad (5.26)
$$

where $*$ is the conjugation under which

$$
\begin{aligned}
(j_i)^*&=j_i,\\
(j_\mu^{\mathrm{s}})^*&=-j_\mu^{\mathrm{s}},
\end{aligned}
$$

where $i=1,2,3$ and $\mu=4,5,6,7$. Thus we find

$$
\mathrm{Tr}\, M(p)\circ\tilde{M}(p)=\sum_{I=1}^{27}p^Ip^I.
$$

The explicit expressions for the generators of quasiconformal group $E_{8(8)}$ and its spherical vector are obtained by substituting the expressions for the cubic form (5.24) and the metric η_{IJ} in (4.21)–(4.31) and in (5.5). The generators of the maximal compact subgroup SO(16) of $E_{8(8)}$ are

$$\begin{aligned}
&(U_A - \tilde{V}^A),\\
&(U_R + \tilde{V}^R),\\
&(V^A + \tilde{U}_A),\\
&(V^R - \tilde{U}_R),\\
&(R_A + \tilde{R}^A),\\
&(R_R - \tilde{R}^R),\\
&(R_A^B - R_B^A),\\
&(R_R^S - R_S^R),\\
&(R_A^S + R_S^A),\\
&(\tilde{U}_0 + V^0),\\
&(-U_0 + \tilde{V}^0),\\
&(K + \tilde{K}).
\end{aligned} \tag{5.27}$$

SU(8) subgroup of SO(16) generated by $(R_A^B - R_B^A)$, $(R_R^S - R_S^R)$, $(R_A^S + R_S^A)$, $(R_A + \tilde{R}^A)$ and $(R_R - \tilde{R}^R)$ act linearly in the quasiconformal action of $E_{8(8)}$ on the 57-dimensional space with coordinates p^0,p^I,q_0,q_I and x. Thus we shall refer to the coset space

$$\mathcal{K}(\mathrm{QConf}(J_3^{\mathbb{O}_S}) = \frac{\mathrm{SO}(16)}{\mathrm{SU}(8)} \tag{5.28}$$

as the "quasiconformal compactification" of this 57-dimensional space [58]. As explained in detail in [19] one can embed the subgroup $E_{7(7)}$ inside $E_{8(8)}$ in essentially three different ways. The $E_{7(7)}$ subgroup generated by the grade zero generators (with respect to $\mathcal{D}$) acts linearly. In addition we have two different subgroups, which we label as $E^q_{7(7)}$ and $E^p_{7(7)}$ that act as the nonlinear conformal group of $J_3^{\mathbb{O}_S}$ on the coordinates and momenta, respectively. The conformal compactification of the corresponding 27-dimensional space is [58]

$$\mathcal{K}(\mathrm{Conf}(J_3^{\mathbb{O}_S}) = \frac{\mathrm{SU}(8)}{U\mathrm{Sp}(8)} \tag{5.29}$$

It is interesting to compare $E_{8(8)}$ as the quasiconformal group associated with the split exceptional Jordan algebra with $E_{8(-24)}$ as the quasiconformal group of the Euclidea exceptional Jordan algebra defined over the division algebra of real octonions. $E_{8(-24)}$ has the maximal compact subgroup $E_7 \times \mathrm{SU}(2)$. The quasiconformal action $E_{8(-24)}$ extends naturally to a holomorphic quasiconformal action on the complex coordinates of the corresponding twistor space [36]

$$\frac{E_{8(-24)} \times \mathrm{SU}(2)}{E_7 \times \mathrm{SU}(2) \times U(1)} = \frac{E_{8(-24)}}{E_7 \times U(1)}$$

The quasiconformal compactification of the 57-dimensional space on which $E_{8(-24)}$ acts is the coset space [58]

$$\mathcal{K}(\mathrm{QConf}(J_3^{\mathbb{O}}) = \frac{E_7 \times \mathrm{SU}(2)}{E_6 \times U(1)}. \tag{5.30}$$

The subgroup $E_{7(-25)}$ has similarly three inequivalent embeddings inside $E_{8(-24)}$. The conformal compactification of the 27-dimensional space on which $E_{7(-25)} = \mathrm{Conf}(J_3^{\mathbb{O}})$ acts nonlinearly is the coset space

$$\mathcal{K}(\mathrm{Conf}(J_3^{\mathbb{O}}) = \frac{E_6}{F_4} \times S^1. \tag{5.31}$$

5.2 Quasiconformal group $E_{7(7)}$

Quasiconformal group associated with the Jordan algebra $J_3^{\mathbb{H}_S}$ over the split quaternions $\mathbb{H}_S$ is $E_{7(7)}$. A general element of $J_3^{\mathbb{H}_S}$ can be written in the form

$$M(p) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}p^1 & P_S^6 & \bar{P}_S^5 \\ \bar{P}_S^6 & \sqrt{2}p^2 & P^4 \\ P_S^5 & \bar{P}_S^4 & \sqrt{2}p^3 \end{pmatrix}, \tag{5.32}$$

where P_S^4, P_S^5 and P_S^6 are now split quaternions. The cubic norm of $M(p)$ is given by

$$\begin{aligned} \mathcal{N}(M(p)) = 3\sqrt{3} \Big\{ & p^1p^2p^3 - \frac{1}{2}(p^1 P_S^4 \bar{P}_S^4 + p^2 P_S^5 \bar{P}_S^5 + p^3 P_S^6 \bar{P}_S^6) \\ & + \frac{1}{\sqrt{2}} \mathrm{Re}(P_S^4 P_S^5 P_S^6) \Big\}, \end{aligned} \tag{5.33}$$

where $\mathrm{Re}(X_S)$ denotes the real part of $X_S \in \mathbb{H}_S$ and $\bar{X}_S$ is the quaternion conjugate of X_S. If we expand the elements P_S^4, P_S^5 and P_S^6 in terms of their components

$$
\begin{aligned}
P_S^4 &= p^4 + p^7 j_1 + p^{4+3m} j_m^{\mathrm{s}}, \\
\bar{P}_S^4 &= p^4 - p^7 j_1 - p^{4+3m} j_m^{\mathrm{s}}, \\
P_S^5 &= p^5 + p^8 j_1 + p^{5+3m} j_m^{\mathrm{s}}, \\
\bar{P}_S^5 &= p^5 - p^8 j_1 - p^{5+3m} j_m^{\mathrm{s}}, \\
P_S^6 &= p^6 + p^9 j_1 + p^{6+3m} j_m^{\mathrm{s}}, \\
\bar{P}_S^6 &= p^6 - p^9 j_1 - p^{6+3m} j_m^{\mathrm{s}},
\end{aligned} \tag{5.34}
$$

where the indices $m, n, \ldots$ are summed over and take values $2, 3$.

The cubic norm $\mathcal{N}(M(p))$ of $M(p)$ can thus be written as

$$
\begin{aligned}
\mathcal{N}(M(p)) = 3\sqrt{3}\Bigg\{ & p^1 p^2 p^3 - \frac{1}{2} p^1 [(p^4)^2 + (p^7)^2 - p^{4+3m} p^{4+3m}] \\
& - \frac{1}{2} p^2 [(p^5)^2 + (p^8)^2 - p^{5+3m} p^{5+3m}] \\
& - \frac{1}{2} p^3 [(p^6)^2 + (p^9)^2 - p^{6+3m} p^{6+3m}] \\
& + \frac{1}{\sqrt{2}} \Big\{ p^4 p^5 p^6 - p^4 (p^8 p^9 - p^{(5+3m)} p^{(6+3m)}) \\
& - p^5 (p^7 p^9 - p^{(4+3m)} p^{(6+3m)}) \\
& - p^6 (p^7 p^8 - p^{(4+3m)} p^{(5+3m)}) \Big\} + \frac{1}{\sqrt{2}} \epsilon_{1mn} p^7 p^{5+3m} p^{6+3n} \\
& + \frac{1}{\sqrt{2}} \epsilon_{m1n} p^{4+3m} p^8 p^{6+3n} + \frac{1}{\sqrt{2}} \epsilon_{mn1} p^{4+3m} p^{5+3n} p^9 \Bigg\}.
\end{aligned} \tag{5.35}
$$

With the above labelling of the basis elements we have

$$
\begin{aligned}
\eta_{AB} &= \delta_{AB}, \\
\eta_{RS} &= -\delta_{RS}, \\
\eta_{AR} &= 0,
\end{aligned} \tag{5.36}
$$

where $A, B, .. = 1, 2, \ldots, 9$ and $R, S, .. = 10, 11, \ldots, 15$. The conjugate elements $\tilde{M}(p)$ of $J_3^{\mathbb{C}_S}$ obtained by conjugation $*$ is given by

$$
M(p) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} p^1 & (P_S^6)^* & (\bar{P}_S^5)^* \\ (\bar{P}_S^6)^* & \sqrt{2} p^2 & (P^4)^* \\ (P_S^5)^* & (\bar{P}_S^4)^* & \sqrt{2} p^3 \end{pmatrix}, \tag{5.37}
$$

where under the conjugation $*$ we have

$$\begin{aligned} j_1^* &= j_1 \\ (j_m^{\rm s})^* &= -j_m^{\rm s}. \end{aligned} \tag{5.38}$$

The maximal compact subgroup of $E_{7(7)}$ is SU(8) and its generators can be obtained by restricting the range of indices in (5.27) such that $A, B, .. = 1, 2, \ldots, 9$ and $R, S, .. = 10, 11, \ldots, 15$. The quasiconformal compactification of the 33-dimensional space on which $E_{7(7)}$ acts is

$$\mathcal{K}(\mathrm{QConf}(J_3^{\mathbb{H}_S}) = \frac{\mathrm{SU}(8)}{\mathrm{SU}(4) \times \mathrm{SU}(4)}. \tag{5.39}$$

The subgroup SO(6, 6) has three different embeddings in $E_{7(7)}$, two of which are conformal acting on 15 coordinates or momenta, respectively. The conformal compactification of the 15-dimensional space on which SO(6, 6) acts is

$$\mathcal{K}(\mathrm{Conf}(J_3^{\mathbb{H}_S}) = \frac{\mathrm{SU}(4) \times \mathrm{SU}(4)}{\mathrm{SU}(4)}. \tag{5.40}$$

We recall that for the Euclidean Jordan algebra $J_3^{\mathbb{H}}$ the associated quasiconformal group is $E_{7(-5)}$ and the quasiconformal compactification of the corresponding 33 dimensional space is [58]

$$\mathcal{K}(\mathrm{QConf}(J_3^{\mathbb{H}}) = \frac{\mathrm{SO}(12) \times \mathrm{SU}(2)}{\mathrm{SU}(6) \times U(1)}. \tag{5.41}$$

The quasiconformal group $E_{7(-5)}$ has a subgroup $\mathrm{SO}^*(12)$ which acts on the underlying Jordan algebra $J_3^{\mathbb{H}}$ as a conformal group. The conformal compactification of the corresponding 15 dimensional space is

$$\mathcal{K}(\mathrm{Conf}(J_3^{\mathbb{H}}) = \frac{\mathrm{SU}(6)}{U\mathrm{Sp}(6)} \times S^1. \tag{5.42}$$

5.3 Quasiconformal group $E_{6(6)}$

Quasiconformal group associated with the Jordan algebra $J_3^{\mathbb{C}_S}$ over the split complex numbers $\mathbb{C}_S$ is $E_{6(6)}$. A general element of $J_3^{\mathbb{C}_S}$ can be written in the form

$$M(p) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}p^1 & P_S^6 & \bar{P}_S^5 \\ \bar{P}_S^6 & \sqrt{2}p^2 & P^4 \\ P_S^5 & \bar{P}_S^4 & \sqrt{2}p^3 \end{pmatrix}, \tag{5.43}$$

where P_S^4, P_S^5 and P_S^6 are now split complex numbers. The cubic norm of $M(p)$ is given by

$$\begin{aligned} \mathcal{N}(M(p)) = 3\sqrt{3}\{p^1p^2p^3 - \frac{1}{2}(p^1P_S^4\bar{P}_S^4 + p^2P_S^5\bar{P}_S^5 + p^3P_S^6\bar{P}_S^6) \\ + \frac{1}{\sqrt{2}}\mathrm{Re}(P_S^4P_S^5P_S^6)\}, \end{aligned} \tag{5.44}$$

where $\mathrm{Re}(X_S)$ denotes the real part of $X_S \in \mathbb{C}_S$ and $\bar{X}_S$ is the complex conjugate of X_S. If we expand the elements P_S^4, P_S^5 and P_S^6 in terms of their components

$$\begin{aligned} P_S^4 &= p^4 + p^7 j^{\mathrm{s}}, \\ \bar{P}_S^4 &= p^4 - p^7 j^{\mathrm{s}}, \\ P_S^5 &= p^5 + p^8 j^{\mathrm{s}}, \\ \bar{P}_S^5 &= p^5 - p^8 j^{\mathrm{s}}, \\ P_S^6 &= p^6 + p^9 j^{\mathrm{s}}, \\ \bar{P}_S^6 &= p^6 - p^9 j^{\mathrm{s}}, \end{aligned} \tag{5.45}$$

where j^{s} is the split "imaginary" unit that squares to +1

$$(j^{\mathrm{s}})^2 = 1. \tag{5.46}$$

The cubic norm $\mathcal{N}(M(p))$ of $M(p)$ can thus be written as

$$\begin{aligned} \mathcal{N}(M(p)) = 3\sqrt{3}\bigg\{ & p^1p^2p^3 - \frac{1}{2}p^1[(p^4)^2 - (p^7)^2] \\ & - \frac{1}{2}p^2[(p^5)^2 - (p^8)^2] - \frac{1}{2}p^3[(p^6)^2 - (p^9)^2] \\ & + \frac{1}{\sqrt{2}}\{p^4p^5p^6 + p^4(p^8p^9) + p^5(p^7p^9) + p^6(p^7p^8)\}\bigg\}. \end{aligned}$$

With the above labelling of the basis elements we have

$$\begin{aligned} \eta_{AB} &= \delta_{AB}, \\ \eta_{RS} &= -\delta_{RS}, \\ \eta_{AR} &= 0, \end{aligned} \tag{5.47}$$

where $A, B, .. = 1, 2, \ldots, 6$ and $R, S, .. = 7, 8, 9$. The conjugate elements $\tilde{M}(p)$ of $J_3^{\mathbb{H}_S}$ obtained by conjugation $*$ is given by

$$M(p) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}p^1 & (P_S^6)^* & (\bar{P}_S^5)^* \\ (\bar{P}_S^6)^* & \sqrt{2}p^2 & (P^4)^* \\ (P_S^5)^* & (\bar{P}_S^4)^* & \sqrt{2}p^3 \end{pmatrix}, \tag{5.48}$$

where under the conjugation $*$ we have

$$(j^{\mathrm{s}})^* = -(j^{\mathrm{s}}). \tag{5.49}$$

The maximal compact subgroup of $E_{6(6)}$ is USp(8) and its generators can be obtained by restricting the range of indices in (5.27) such that $A, B, .. = 1, 2, ..., 6$ and $R, S, .. = 7, 8, 9$. The quasiconformal compactification of the 21-dimensional space on which $E_{6(6)}$ acts is

$$\mathcal{K}(\mathrm{QConf}(J_3^{\mathbb{C}_S}) = \frac{U\mathrm{Sp}(8)}{\mathrm{SU}(4)}. \tag{5.50}$$

The subgroup $\mathrm{SL}(6, \mathbb{R})$ has three different embeddings in $E_{6(6)}$, two of which are conformal acting on 9 coordinates or momenta, respectively. The conformal compactification of the corresponding 9 dimensional space on which $\mathrm{SL}(6, \mathbb{R})$ acts is

$$\mathcal{K}(\mathrm{Conf}(J_3^{\mathbb{C}_S}) = \frac{\mathrm{SO}(6)}{\mathrm{SO}(3) \times \mathrm{SO}(3)}. \tag{5.51}$$

We recall that for the Euclidean Jordan algebra $J_3^{\mathbb{C}}$ the associated quasiconformal group is $E_{6(2)}$ and the quasiconformal compactification of the corresponding 21-dimensional space is [58]

$$\mathcal{K}(\mathrm{QConf}(J_3^{\mathbb{C}}) = \frac{\mathrm{SU}(6) \times \mathrm{SU}(2)}{\mathrm{SU}(3) \times \mathrm{SU}(3) \times U(1)}. \tag{5.52}$$

The quasiconformal group $E_{6(2)}$ has a subgroup $\mathrm{SU}(3, 3)$ which acts on the underlying Jordan algebra $J_3^{\mathbb{C}}$ as a conformal group. The conformal compactification of the corresponding nine-dimensional space is

$$\mathcal{K}(\mathrm{Conf}(J_3^{\mathbb{C}}) = \frac{\mathrm{SU}(3) \times \mathrm{SU}(3)}{\mathrm{SU}(3)} \times S^1. \tag{5.53}$$

5.4 Quasiconformal group $\mathbf{SO(n+3, m+3)}$

For the generic nonsimple Jordan algebras $(\mathbb{R} \oplus \Gamma_{(n,m)})$ of degree three the cubic form is

$$\mathcal{N}(q) = C^{IJK} q_I q_J q_K = \frac{3\sqrt{3}}{2} q_1 [2 q_2 q_3 - \eta^{\hat{I}\hat{J}} q_{\hat{I}} q_{\hat{J}}], \tag{5.54}$$

where $\hat{I}, \hat{J}, \ldots = 4, 5, \ldots, (m+n+1)$ and

$$\begin{aligned} \eta^{\hat{I}\hat{J}} &= \delta^{\hat{I}\hat{J}}, \quad \hat{I}, \hat{J} = 4, 5, \ldots, m+2, \\ \eta^{\hat{I}\hat{J}} &= -\delta^{\hat{I}\hat{J}}, \quad \hat{I}, \hat{J} = m+3, m+4, \ldots, n+m+1 \end{aligned} \tag{5.55}$$

and the associated quasiconformal group is $\mathrm{SO}(n+3, m+3)$. The quartic correction term C_4^S that appears in H_4^S in the spherical vectors is given by

$$\begin{aligned} C_4^S(\mathbb{R} \oplus \Gamma_{(n,m)}) = -\frac{81}{2} \{(p^2 - p^3)(q_2 - q_3) + 2 p^{\hat{I}} q_{\hat{I}}\}^2 + \frac{81}{2} \{(p^2 - p^3)^2 \\ + 2 \eta_{\hat{I}\hat{J}} p^{\hat{I}} p^{\hat{J}}\} \{(q_2 - q_3)^2 + 2 \eta^{\hat{I}\hat{J}} q_{\hat{I}} q_{\hat{J}}\} \end{aligned} \tag{5.56}$$

and the full metric η_{IJ} is given by

$$\begin{aligned} \eta_{AB} &= \delta_{AB}, \quad \text{for } A, B, .. = 1, 2, ..., m+2, \\ \eta_{RS} &= -\delta_{RS}, \quad \text{for } R, S, .. = m+3, m+4, \ldots, m+n+1. \end{aligned} \tag{5.57}$$

The quasiconformal compactification of the $(2n+2m+5)$-dimensional space on which $\mathrm{SO}(n+3, m+3)$ is realized as the quasiconformal group is

$$\mathcal{K}(\mathrm{QConf}(\mathbb{R} \oplus \Gamma_{(n,m)}) = \frac{\mathrm{SO}(n+3) \times \mathrm{SO}(m+3)}{\mathrm{SO}(n+1) \times \mathrm{SO}(m+1) \times \mathrm{SO}(2)}. \tag{5.58}$$

The subgroup $\mathrm{SO}(m+1, n+1) \times \mathrm{SO}(2,1)$ can be embedded in $\mathrm{SO}(n+3, m+3)$ in three inequivalent ways. Two of these embeddings act nonlinearly as conformal groups on the coordinates and momenta, respectively. The conformal compactification of the $(m+n+1)$ dimensional space on which $\mathrm{SO}(m+1,n+1)\times\mathrm{SO}(2,1)$ acts via conformal transformations is

$$\mathcal{K}(\mathrm{Conf}(\mathbb{R}\oplus\Gamma_{(n,m)}))=\frac{\mathrm{SO}(m+1)\times\mathrm{SO}(n+1)}{\mathrm{SO}(m)\times\mathrm{SO}(n)}\times S^1. \tag{5.59}$$

For Euclidean Jordan algebras $(\mathbb{R}\oplus\Gamma_{(1,m)}$ the quasiconformal group is $\mathrm{SO}(4,m+3)$ and the conformal group is $\mathrm{SO}(2,m+1)\times\mathrm{SO}(2,1)$ leading to the following compactified spaces

$$\mathcal{K}(\mathrm{QConf}(\mathbb{R}\oplus\Gamma_{(1,m)})=\frac{\mathrm{SO}(4)\times\mathrm{SO}(m+3)}{\mathrm{SO}(2)\times\mathrm{SO}(m+1)\times\mathrm{SO}(2)}, \tag{5.60}$$

$$\mathcal{K}(\mathrm{Conf}(\mathbb{R}\oplus\Gamma_{(1,m)}))=\frac{\mathrm{SO}(m+1)}{\mathrm{SO}(m)}\times S^1\times S^1. \tag{5.61}$$

6 Exceptional $N=2$ MESGT, $N=8$ supergravity and Jordan algebras of degree four

As explained above the U-duality symmetry groups of the maximal supergravity in $5,4$ and 3 dimensions are simply the Lorentz (reduced structure), conformal and quasiconformal groups of the split exceptional Jordan algebra $J_3^{\mathbb{O}_S}$, which is not Euclidean. On the other hand the corresponding U-duality groups of the exceptional $N=2$ supergravity are the Lorentz, conformal and quasiconformal groups of the real exceptional Jordan algebra $J_3^{\mathbb{O}}$ which is Euclidean. The scalar manifold of the exceptional theory in five dimensions is simply

$$\mathcal{M}_5(J_3^{\mathbb{O}})=\frac{\mathrm{Str}_0(J_3^{\mathbb{O}})}{\mathrm{Aut}(J_3^{\mathbb{O}})}=\frac{E_{6(-26)}}{F_4}.$$

However the scalar manifold of the maximal supergravity is

$$\frac{E_{6(6)}}{U\mathrm{Sp}(8)}$$

so that the maximal compact subgroup is not the automorphism group of the split exceptional Jordan algebra. Remarkably, there is a formulation of the exceptional Jordan algebras in terms of degree four Jordan algebras in which the situation gets reversed. As was shown in [59] the exceptional supergravity can also be formulated in terms of Lorentzian Jordan algebra $J^{\mathbb{H}}_{(1,3)}$ of 4×4 quaternionic matrices that are Hermitian with respect to a Lorentzian metric. This is achieved by the mapping between the traceless elements , which we denote as $J^{\mathbb{H}}_{(1,3)_0}$, of $J^{\mathbb{H}}_{(1,3)}$ and the elements of the exceptional Jordan algebra $J^{\mathbb{O}}_3$:

$$\mathbf{x} \in J^{\mathbb{O}}_3 \Longrightarrow \mathbf{X} \in J^{\mathbb{H}}_{(1,3)_0} \tag{6.1}$$

such that the cubic norm $\mathcal{N}(\mathbf{x})$ of $\mathbf{x}$ is equal to the trace of $\mathbf{X}^3$:

$$\mathcal{N}(\mathbf{x}) = \mathrm{Tr}(\mathbf{X}^3). \tag{6.2}$$

Now the automorphism group of $J^{\mathbb{H}}_{(1,3)_0}$ is $U\mathrm{Sp}(6,2)$ which is the manifest symmetry of the trace form. Hence the trace of $\mathbf{X}^3$ has "hidden" extra symmetries which extend $U\mathrm{Sp}(6,2)$ to $E_{6(-26)}$, which is the Lorentz (reduced structure) group of $J^{\mathbb{O}}_3$. However, the Lorentz (reduced structure) group $\mathrm{SU}^*(8)$ of $J^{\mathbb{H}}_{(1,3)}$ is not a symmetry of the exceptional MESGT in five dimensions.

Similarly there is a mapping between the elements $\mathbf{y}$ of the split exceptional Jordan algebra $J^{\mathbb{O}_S}_3$ and the traceless elements $\mathbf{Y}$ of Euclidean Jordan algebra $J^{\mathbb{H}}_4$ of 4×4 Hermitian matrices over the division algebra of quaternions $\mathbb{H}$ [60]

$$\mathbf{y} \in J^{\mathbb{O}_S}_3 \Longrightarrow \mathbf{Y} \in J^{\mathbb{H}}_{4_0} \tag{6.3}$$

such that one finds

$$\mathcal{N}(\mathbf{y}) = \mathrm{Tr}(\mathbf{Y}^3). \tag{6.4}$$

The automorphism group $U\mathrm{Sp}(8)$ of $J^{\mathbb{H}}_4$ is the manifest symmetry of the trace form. Again extra hidden symmetries of $\mathrm{Tr}(\mathbf{Y}^3)$ extend it to the Lorentz (reduced structure) group $E_{6(6)}$ of $J^{\mathbb{O}_S}_3$. However the scalar manifold of the maximal supergravity in five dimensions is

$$\mathcal{M}_5(N=8) = \frac{\mathrm{Str}_0(J^{\mathbb{O}_S}_3)}{\mathrm{Aut}(J^{\mathbb{H}}_4)} = \frac{E_{6(6)}}{U\mathrm{Sp}(8)}. \tag{6.5}$$

The Lorentz (reduced) structure group $\mathrm{SU}^*(8)$ of $J^{\mathbb{H}}_4$ is not a symmetry of maximal supergravity in five dimensions. In four dimensions the scalar manifold of maximal supergravity is

$$\mathcal{M}_4(N=8) = \frac{\mathrm{Conf}(J^{\mathbb{O}_S}_3)}{\widetilde{\mathrm{Str}}_0(J^{\mathbb{H}}_4)} = \frac{E_{7(7)}}{\mathrm{SU}(8)}, \tag{6.6}$$

where $\widetilde{\mathrm{Str}}_0(J^{\mathbb{H}}_4)$ denotes the compact real form of the Lorentz (reduced structure) group of $J_4^{\mathbb{H}}$). In three dimensions one has the scalar manifold

$$\mathcal{M}_3(N=8) = \frac{\mathrm{QConf}(J_3^{\mathbb{O}_S})}{\widetilde{\mathrm{Conf}}(J_4^{\mathbb{H}})} = \frac{E_{8(8)}}{\mathrm{SO}(16)}, \tag{6.7}$$

where $\widetilde{\mathrm{Conf}}(J)$ refers to the compact real form of the conformal group of J.

One can truncate the above correspondences between the real and split exceptional Jordan algebras and $J_{(1,3)}^{\mathbb{H}}$ and $J_4^{\mathbb{H}}$, respectively, to correspondences between rank three complex and quaternionic Jordan algebras and rank four real and complex Jordan algebras:

$$\begin{aligned} J_3^{\mathbb{C}} &\Longleftrightarrow J_{(1,3)_0}^{\mathbb{R}}, \\ J_3^{\mathbb{H}} &\Longleftrightarrow J_{(1,3)_0}^{\mathbb{C}}, \\ J_3^{\mathbb{C}_S} &\Longleftrightarrow J_{4_0}^{\mathbb{R}}, \\ J_3^{\mathbb{H}_S} &\Longleftrightarrow J_{4_0}^{\mathbb{C}}. \end{aligned} \tag{6.8}$$

Remarkably the $N=2$ MESGT defined by Lorentzian Jordan algebras $J_{(1,3)}^{\mathbb{A}}$ belong to three infinite families of unified MESGT's in five dimensions defined by Lorentzian Jordan algebras of arbitrary rank [59]. Study of quasiconformal groups associated with Lorentzian Jordan algebras will be left to future studies.

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References

[1] M. Günaydin and C. Saclioglu, *Bosonic construction of the Lie algebras of some noncompact groups appearing in supergravity theories and their oscillator-like unitary representations*, Phys. Lett. **B108** (1982), 180.

[2] M. Günaydin and C. Saclioglu, *Oscillator-like unitary representations of noncompact groups with a Jordan structure and the noncompact groups of supergravity*, Commun. Math. Phys. **87** (1982), 159.

[3] M. Günaydin, *Unitary realizations of the noncompact symmetry groups of supergravity*, Presented at Second Europhysics Study Conference on Unification of Fundamental Interactions, Erice, Sicily, 6–14 October 1981.

[4] J. R. Ellis, M. K. Gaillard, L. Maiani and B. Zumino, *Attempts at superunification*, Presented at Europhysics Study Conference on Unification of the Fundamental Interactions, Erice, Italy, 17–24 March 1980.

[5] J. R. Ellis, M. K. Gaillard and B. Zumino, *A grand unified theory obtained from broken supergravity*, Phys. Lett. **B94** (1980), 343.

[6] M. Günaydin, *Present status of the attempts at a realistic GUT in extended supergravity theories*, Presented at 21st International Conference on High Energy Physics, Paris, France, 26–31 July 1982.

[7] M. Günaydin, G. Sierra and P. K. Townsend, *Exceptional supergravity theories and the magic square*, Phys. Lett. **B133** (1983), 72.

[8] M. B. Green and J. H. Schwarz, *Anomaly cancellation in supersymmetric D=10 gauge theory and superstring theory*, Phys. Lett. **B149** (1984), 117–122.

[9] Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, *Manifest ultraviolet behavior for the three-loop four- point amplitude of N=8 Supergravity*, `0808.4112`.

[10] N. E. J. Bjerrum-Bohr and P. Vanhove, *On cancellations of ultraviolet divergences in supergravity amplitudes*, Fortsch. Phys. **56** (2008), 824–832, `0806.1726`.

[11] N. Arkani-Hamed, F. Cachazo and J. Kaplan, *What is the simplest quantum field theory?* `0808.1446`.

[12] G. Chalmers, *On the finiteness of* $N=8$ *quantum supergravity*, `hep-th/0008162`.

[13] M. B. Green, J. G. Russo and P. Vanhove, *Non-renormalisation conditions in type II string theory and maximal supergravity*, JHEP **02** (2007), 099, `hep-th/0610299`.

[14] M. B. Green, J. G. Russo and P. Vanhove, *Ultraviolet properties of maximal supergravity*, Phys. Rev. Lett. **98** (2007), 131602, `hep-th/0611273`.

[15] M. B. Green, H. Ooguri and J. H. Schwarz, *Decoupling supergravity from the superstring*, Phys. Rev. Lett. **99** (2007), 041601, `0704.0777`.

[16] R. Kallosh, C. H. Lee and T. Rube, $N=8$ *supergravity 4-point amplitudes*, `811.3417`.

[17] R. Kallosh, $N=8$ *supergravity on the light cone*, `0903.4630`.

[18] S. Ferrara and M. Günaydin, *Orbits of exceptional groups, duality and BPS states in string theory*, Int. J. Mod. Phys. **A13** (1998), 2075–2088, `hep-th/9708025`.

[19] M. Günaydin, K. Koepsell and H. Nicolai, *Conformal and quasiconformal realizations of exceptional Lie groups*, Commun. Math. Phys. **221** (2001), 57–76, `hep-th/0008063`.

[20] M. Günaydin, *Realizations of exceptional U-duality groups as conformal and quasiconformal groups and their minimal unitary representations*, Comment. Phys. Math. Soc. Sci. Fenn. **166** (2004), 111–125, hep-th/0409263.

[21] M. Günaydin, *Realizations of exceptional U-duality groups as conformal and quasi-conformal groups and their minimal unitary representations*, Prepared for 3rd International Symposium on Quantum Theory and Symmetries (QTS3), Cincinnati, Ohio, 10–14 September 2003.

[22] M. Günaydin, *Unitary realizations of U-duality groups as conformal and quasiconformal groups and extremal black holes of supergravity theories*, AIP Conf. Proc. **767** (2005), 268–287, hep-th/0502235.

[23] M. Günaydin, K. Koepsell and H. Nicolai, *The minimal unitary representation of $E_{8(8)}$*, Adv. Theor. Math. Phys. **5** (2002), 923–946, hep-th/0109005.

[24] M. Günaydin and O. Pavlyk, *Minimal unitary realizations of exceptional U-duality groups and their subgroups as quasiconformal groups*, JHEP **01** (2005), 019, hep-th/0409272.

[25] M. Günaydin and O. Pavlyk, *Generalized spacetimes defined by cubic forms and the minimal unitary realizations of their quasiconformal groups*, JHEP **08** (2005), 101, hep-th/0506010.

[26] M. Günaydin and O. Pavlyk, *A unified approach to the minimal unitary realizations of noncompact groups and supergroups*, JHEP **09** (2006), 050, hep-th/0604077.

[27] D. Gaiotto, A. Strominger and X. Yin, *5d black rings and 4d black holes*, JHEP **02** (2006), 023, hep-th/0504126.

[28] D. Gaiotto, A. Strominger and X. Yin, *New connections between 4d and 5d black holes*, JHEP **02** (2006), 024, hep-th/0503217.

[29] H. Elvang, R. Emparan, D. Mateos and H. S. Reall, *Supersymmetric 4D rotating black holes from 5D black rings*, hep-th/0504125.

[30] B. Pioline, *BPS black hole degeneracies and minimal automorphic representations*, JHEP **0508** (2005), 071, hep-th/0506228.

[31] A. Bouchareb *et al.*, *G_2 generating technique for minimal $D=5$ supergravity and black rings*, Phys. Rev. **D76** (2007), 104032, 0708.2361.

[32] D. V. Gal'tsov and N. G. Scherbluk, *Generating technique for $U(1)^3, 5D$ supergravity*, 0805.3924.

[33] M. Berkooz and B. Pioline, *5D black holes and non-linear sigma models*, JHEP **05** (2008), 045, 0802.1659.

[34] M. Günaydin, A. Neitzke, B. Pioline and A. Waldron, *BPS black holes, quantum attractor flows and automorphic forms*, Phys. Rev. **D73** (2006), 084019, hep-th/0512296.

[35] M. Günaydin, A. Neitzke, B. Pioline and A. Waldron, *Quantum attractor flows*, JHEP **09** (2007), 056, `0707.0267`.

[36] M. Günaydin, A. Neitzke, O. Pavlyk and B. Pioline, *Quasi-conformal actions, quaternionic discrete series and twistors: $SU(2,1)$ and $G_{2(2)}$*, Commun. Math. Phys. **283** (2008), 169–226, `0707.1669`.

[37] S. Ferrara, R. Kallosh and A. Strominger, *$\mathcal{N}=2$ extremal black holes*, Phys. Rev. **D52** (1995), 5412–5416, `hep-th/9508072`.

[38] S. Ferrara and R. Kallosh, *Universality of supersymmetric attractors*, Phys. Rev. **D54** (1996), 1525–1534, `hep-th/9603090`.

[39] P. Breitenlohner, G. W. Gibbons and D. Maison, *Four-dimensional black holes from Kaluza-Klein theories*, Commun. Math. Phys. **120** (1988), 295.

[40] M. Cvetic and D. Youm, *All the static spherically symmetric black holes of heterotic string on a six torus*, Nucl. Phys. **B472** (1996), 249–267, `hep-th/9512127`.

[41] M. Cvetic and D. Youm, *Dyonic bps saturated black holes of heterotic string on a six torus*, Phys. Rev. **D53** (1996), 584–588, `hep-th/9507090`.

[42] D. Gaiotto, W. W. Li and M. Padi, *Non-Supersymmetric attractor flow in symmetric spaces*, JHEP **12** (2007), 093, `0710.1638`.

[43] E. Bergshoeff, W. Chemissany, A. Ploegh, M. Trigiante and T. Van Riet, *Generating geodesic flows and supergravity solutions*, Nucl. Phys. **B812** (2009), 343–401, `0806.2310`.

[44] A. Neitzke, B. Pioline and S. Vandoren, *Twistors and black holes*, JHEP **04** (2007), 038, `hep-th/0701214`.

[45] M. Günaydin, *Harmonic superspace, minimal unitary representations and quasiconformal groups*, JHEP **05** (2007), 049, `hep-th/0702046`.

[46] B. H. Gross and N. R. Wallach, *On quaternionic discrete series representations, and their continuations*, J. Reine Angew. Math. **481** (1996), 73–123.

[47] M. Gunaydin and O. Pavlyk, *Spectrum gnerating conformal and quasiconformal U-duality groups, supergravity and spherical vectors*, JHEP **04** (2010), 070, `0901.1646`.

[48] K. McCrimmon, *A taste of Jordan algebras*, Universitext, Springer-Verlag, New York, 2004.

[49] P. Jordan, J. von Neumann and E. P. Wigner, *On an algebraic generalization of the quantum mechanical formalism*, Ann. Math. **35** (1934), 29–64.

[50] N. Jacobson, *Structure and representations of Jordan algebras*, American Mathematical Society Colloquium Publications, **XXXIX**, American Mathematical Society, Providence, RI, 1968.

[51] M. Günaydin, G. Sierra and P. K. Townsend, *The geometry of $\mathcal{N} = 2$ Maxwell–Einstein supergravity and Jordan algebras*, Nucl. Phys. **B242** (1984), 244.

[52] M. Günaydin and F. Gürsey, *Quark structure and octonions*, J. Math. Phys. **14** (1973), 1651–1667.

[53] M. Günaydin, *Exceptional realizations of Lorentz group: supersymmetries and leptons*, Nuovo Ciment. **A29** (1975), 467.

[54] M. Günaydin, *The exceptional superspace and the quadratic Jordan formulation of quantum mechanics*, in 'Elementary Particles and the Universe: Essays in Honor of Murray Gell-Mann', ed. J. Schwarz, Cambridge University Press, Pasadena, 1989, pp. 99–119.

[55] M. Günaydin, *Generalized conformal and superconformal group actions and Jordan algebras*, Mod. Phys. Lett. **A8** (1993), 1407–1416, `hep-th/9301050`.

[56] I. Kantor and I. Skopets, *Some results on Freudenthal triple systems*, Sel. Math. Sov. **2** (1982), 293.

[57] M. Günaydin and O. Pavlyk, in preparation.

[58] M. Günaydin, in preparation.

[59] M. Gunaydin, S. McReynolds and M. Zagermann, *Unified $N = 2$ Maxwell–Einstein and Yang–Mills–Einstein supergravity theories in four dimensions*, JHEP **09** (2005), 026, `hep-th/0507227`.

[60] B. N. Allison and J. R. Faulkner, *A Cayley-Dickson process for a class of structurable algebras*, Trans. Amer. Math. Soc. **283**(1), (1984), 185–210.