Adv. Theor. Math. Phys. **7** (2003) 619–665

Planar diagrams and Calabi-Yau spaces

Frank Ferrari

Institut de Physique, Université de Neuchâtel
rue A.-L. Bréguet 1, CH-2000 Neuchâtel, Suisse
and
Division de Physique Théorique
CERN, CH-1211 Genève 23, Suisse
frank.ferrari@unine.ch

Abstract

Large N geometric transitions and the Dijkgraaf-Vafa conjecture suggest a deep relationship between the sum over planar diagrams and Calabi-Yau threefolds. We explore this correspondence in details, explaining how to construct the Calabi-Yau for a large class of M-matrix models, and how the geometry encodes the correlators. We engineer in particular two-matrix theories with potentials $W(X,Y)$ that reduce to arbitrary functions in the commutative limit. We apply the method to calculate all correlators $\langle \operatorname{tr} X^p \rangle$ and $\langle \operatorname{tr} Y^p \rangle$ in models of the form $W(X,Y) = V(X) + U(Y) - XY$ and $W(X,Y) = V(X) + YU(Y^2) + XY^2$. The solution of the latter example was not known, but when U is a constant we are able to solve the loop equations, finding a precise match with the geometric approach. We also discuss special geometry in multi-matrix models, and we derive an important property, the entanglement of eigenvalues, governing the expansion around classical vacua for which the matrices do not commute.

e-print archive: http://lanl.arXiv.org/abs/hep-th/0309151

1 Introduction

Ever since the work of 't Hooft on large n QCD [1], finding methods to sum up planar diagrams has remained a central theme in mathematical physics. Yet, even in the best studied and simplest case of zero-dimensional integrals over hermitian $n \times n$ matrices, only very few techniques are available [2]. The usual approach is to solve a set of saddle point equations by making an ansatz for the analytic structure of the solution [3]. The saddle point equations are themselves derived after reducing the number of degrees of freedom from $\sim n^2$ to $\sim n$ by integrating over angular variables, which can necessitate sophisticated methods [4, 5, 6] that apply only in special cases. Another approach is to write down Schwinger-Dyson equations, called loop equations in this context [7]. This method is very general and does not use assumptions on the analytic structure of the solution. However, it is usually very difficult to find a finite set of equations that closes under the correlators one wish to calculate. This scarcity of tools and of solvable examples makes the search for alternative strategies particularly useful.

It is well-known that the solution of the one-matrix model can be expressed in terms of a complex algebraic curve (see for example [8] or the appendix of [9] for reviews). The idea we wish to pursue in the following is that for multi-matrix models it may be useful to consider higher dimensional Calabi-Yau spaces. This is suggested by conjectures by Vafa and collaborators on the strongly coupled dynamics of four dimensional $\mathcal{N} = 1$ supersymmetric gauge theories [10, 11, 12, 13]. We will engineer multi-matrix models with potentials of the form

$$W(X_1, \ldots, X_M) = \frac{1}{2i\pi} \oint_{C_0} z^{-M-1} E\Big(z, \sum_{i=1}^{M} X_i z^{i-1}\Big)\, \mathrm{d}z\,, \tag{1.1}$$

where $E(z,w) = \sum_{i=-\infty}^{+\infty} E_i(w)\, z^i$ can be expanded in terms of entire functions E_i and C_0 is a small contour encircling $z = 0$. This is a large and interesting class of models. In the one- or two-matrix cases, the commutative limit of (1.1) can be an arbitrary function.

The main body of the paper is divided into four parts. In Section 2 and Appendix A, we discuss special geometry relations for multi-matrix integrals. Special geometry is automatically built in the Calabi-Yau geometry, and it is important to have a satisfying understanding from the matrix model perspective as well. Deriving the relations, we uncover new interesting properties of the saddle point equations and their solutions (eigenvalue entanglement) around vacua where the matrices do not commute. These developments in standard matrix model technology are not strictly required to understand the geometric approach, that we describe in details in Sections 3 and 4. The method is applied in particular to compute the resolvents of both matrices X and Y for the standard [14] two-matrix model $W(X,Y) = V(X) + U(Y) - XY$ as well as for the model $W(X,Y) = XY^2 + V(X) + YU(Y^2)$, for arbitrary polynomials V and U. In Section 5, we use the loop equation technique to solve for all correlators $\langle \operatorname{tr} X^p Y^q \rangle$ in the latter example when U is a constant, finding a perfect match with the geometric approach. Finally, we discuss open directions of research in Section 6.

2 The geometry of matrix models

The existence of special geometry relations constraining the sum over planar diagrams is a basic ingredient of the geometric approach to matrix models. This is an essentially unexplored subject in the case of multi-matrix integrals, and thus we provide a rather detailed, albeit incomplete, discussion in the present Section and in Appendix A.

A matrix model with M matrices has several one-matrix model effective descriptions obtained by integrating out all but one matrix, for example

$$e^{-\frac{n^2}{S} V_{\mathrm{eff}}(X)} = \int \prod_{i=2}^{M} \mathrm{d}X_i \, e^{-\frac{n}{S} \operatorname{tr} W(X, X_2, \ldots, X_M)}. \tag{2.1}$$

By analogy with the ordinary one-matrix model with polynomial potential, it is natural to suspect that special geometry relations could be derived for the multi-matrix models by working with the effective descriptions. The difficulty is that the effective potential V_{eff} can have a non-trivial analytic structure. The main subtleties come from the classical solutions for which the matrices do not commute. By studying the properties of the effective potentials, we derive a remarkable property of the eigenvalue distributions governing the expansion around such solutions. Full details are provided in Appendix A for two models that display some generic features.

The various sets of special geometry relations, derived for each well-behaved one-matrix effective description, must be equivalent. In the geometric approach, this non-trivial constraint is automatically satisfied by expressing the solution in terms of integrals over a higher dimensional complex variety equipped with prefered coordinates associated with the various matrices.

2.1 Definitions

We consider a model with M hermitian $n \times n$ matrices X_k and polynomial potential $W(X_1, \dots X_M)$. The classical equations of motion

$$\mathcal{A}: \ \operatorname{tr} \mathrm{d}W(X_1, \dots, X_M) = 0 \tag{2.2}$$

define an algebra $\mathcal{A}$ with a set of D_K-dimensional irreducible representations R_K. In the generic case there are no moduli and the index K is discrete.

A particular solution to (2.2) of the form

$$R = \oplus n_K R_K \,, \quad \sum_K n_K D_K = n \,, \tag{2.3}$$

is called a classical vacuum. The partition function $\mathcal{F}(R)$ is defined by the formula

$$e^{n^2 \mathcal{F}(R)/S^2} = \int_{R-\text{planar}} \mathrm{d}X_1 \cdots \mathrm{d}X_M \, e^{-\frac{n}{S} \operatorname{tr} W(X_1, \dots, X_M)} \,, \tag{2.4}$$

where the subscript "R-planar" in (2.4) means that we sum up only the contributions from the planar diagrams in the perturbative Feynman expansion around the classical solution R. The partition function depends of the integers n_K in (2.3), or more conveniently on the variables

$$S_K = \frac{n_K}{n} S \tag{2.5}$$

that satisfy the identity

$$\sum_K D_K S_K = S \,. \tag{2.6}$$

In general, the perturbative expansion defining $\mathcal{F}$ involves ghost matrices as explained in [15]. The coupling S plays the rôle of $\hbar$. An important result is that the sum over planar diagrams is always absolutely convergent for small enough S [16]. For that reason the convergence properties of the integral $\int \mathrm{d}X_1 \cdots \mathrm{d}X_M \exp(-n \operatorname{tr} W/S)$ are irrelevant. It can always be made convergent without loss of generality by performing an analytic continuation and/or turning on some couplings. If the integral is convergent, then its strict large n limit yields $\mathcal{F}(R_{\min})$ for the representation $R_{\min}$ that minimizes W (assuming that it is unique). A more useful statement is that for general R, $\mathcal{F}(R)$ is the large n limit of the formal perturbative series around the classical solution R. In this limit the variables S_K defined in (2.5) become continuous.

Correlators are defined by

$$\langle\mathcal{O}\rangle_R = e^{-n^2\mathcal{F}(R)/S^2}\int_{R-\text{planar}} \mathrm{d}X_1\cdots\mathrm{d}X_M\,\mathcal{O}\,e^{-\frac{n}{S}\operatorname{tr}W(X_1,\ldots,X_M)}\,. \tag{2.7}$$

We will not systematically indicate the dependence in R. Basic objects that we want to compute are the resolvents of the matrices X_m defined on the complex plane by

$$g^{X_m}(x) = S\left\langle\frac{\operatorname{tr}}{n}\,\frac{1}{x-X_m}\right\rangle. \tag{2.8}$$

The resolvents are the generating functions of the correlators $\langle\operatorname{tr}X_m^p\rangle$ for arbitrary positive integers p. They are analytic functions with square root branch cuts. The physical sheet is defined by the asymptotics

$$g^{X_m}(x) \underset{x\rightarrow\infty}{\sim} \frac{S}{x}\cdot \tag{2.9}$$

The densities of eigenvalues ρ^{X_m} of the matrices X_m are read off from discontinuities of the resolvents across branch cuts on the physical sheet,

$$\rho^{X_m}(x) = \frac{i}{2\pi S}\left(g^{X_m}(x+i\epsilon) - g^{X_m}(x-i\epsilon)\right)\,, \tag{2.10}$$

and we have

$$g^{X_m}(x) = S\int_{-\infty}^{+\infty}\frac{\rho^{X_m}(z)\,\mathrm{d}z}{x-z}\,\cdot \tag{2.11}$$

2.2 Special geometry

2.2.1 Generalities

We work with the one-matrix model whose potential is defined by (2.1). For a general vacuum (2.3), the eigenvalues of X form cuts $I^X_{K,k}$, $1\leq k\leq D_K$, that generically do not overlap. The number of eigenvalues in the cuts, or equivalently the variables S_K defined in (2.5), are expressed in terms of period integrals,

$$S_K = \frac{1}{2i\pi}\oint_{\alpha^X_{K,k}} g^X(x)\,\mathrm{d}x\,, \tag{2.12}$$

where the contours $\alpha^X_{K,k}$ encircle the cuts $I^X_{K,k}$ on the physical sheet for g^X (see Figure 1 for the definition of various contours). The formula (2.12) could be written in many different-looking but equivalent ways, using other matrices X_i of the original multi-matrix model. Taking into account the

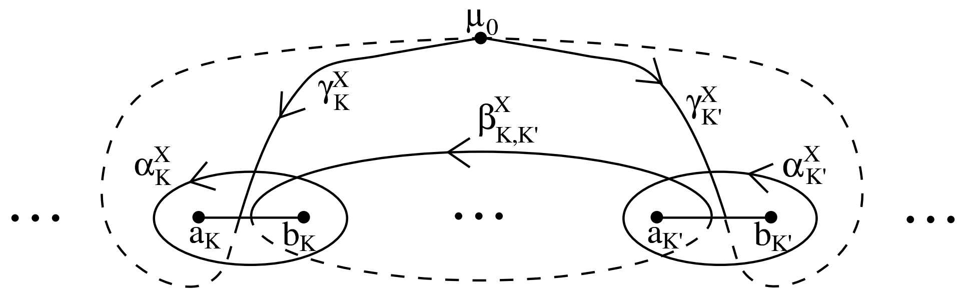

Figure 1: Definition of the contours α_K^X, $\beta_{K,K'}^X$ and γ_K^X used in the main text (similar contours like $\tilde{\alpha}_K^X$ or α_K^Y are also used). The eigenvalues that classically sit in the representations R_K and $R_{K'}$ are spread along the cuts $I_K^X = [a_K, b_K]$ and $I_{K'}^X = [a_{K'}, b_{K'}]$ respectively.

constraints (2.12), the eigenvalues adjust to make the partition function

$$\mathcal{F} = -SV_{\text{eff}}(x_1, \ldots, x_n) + \frac{S^2}{n^2} \sum_{i \neq j} \ln |x_i - x_j| \tag{2.13}$$

extremal. The corresponding saddle point equations

$$-n \frac{\partial V_{\text{eff}}}{\partial x_i} + \frac{2S}{n} \sum_{j \neq i} \frac{1}{x_i - x_j} = 0 \tag{2.14}$$

are interpreted as vanishing force conditions. The total force on a test eigenvalue on the complex plane

$$f(x) = f_{\text{b}}(x) + 2g^X(x) \tag{2.15}$$

is the sum of a background force

$$f_{\text{b}}(x) = -n \frac{\partial V_{\text{eff}}}{\partial x_i}\Big|_{x_i = x}, \tag{2.16}$$

and of the Coulomb term $2g^X(x)$. Since varying S_K amounts to moving eigenvalues from one cut to another [13], we expect that $\partial\mathcal{F}/\partial S_K$ will be expressed in terms of countour integrals of the differential $f\,\mathrm{d}x$. If the term $f_{\text{b}}\mathrm{d}x$ does not contribute to the integrals, as in the ordinary one-matrix model for which it is an exact differential $-\mathrm{d}W$, the special geometry relations associated with (2.12) straightforwardly follow, for example

$$\frac{\partial \mathcal{F}}{\partial S_K} = \oint_{\gamma_K^X} g^X(x)\,\mathrm{d}x + \text{counterterm}\,, \tag{2.17}$$

where the non-compact γ-cycles are defined in Figure 1. However, in general, $f_{\text{b}}\mathrm{d}x$ do contribute, and a more detailed discussion is needed.

2.2.2 The commutative case

There is a class of models for which the classical equations of motion imply that the matrices commute with each other. The non-commutative structure can then, in some sense, be neglected. This is also true more generally for vacua built from one dimensional representations only. A typical example is the much-studied two-matrix model with potential

$$W(X,Y) = V(X) + U(Y) - XY\,, \tag{2.18}$$

where V and U are polynomials of degree $d_X + 1$ and $d_Y + 1$ respectively. This model was used for example in the study of two dimensional gravity coupled to $c < 1$ matter [8]. The classical equations of motion

$$\mathcal{A}:\ X = U'(Y) = Q(Y)\,,\quad Y = V'(X) = P(X)\,, \tag{2.19}$$

imply that $[X,Y] = 0$, and thus the only irreducible representations are the $d_X d_Y$ one dimensional representations given by $X = xI$ and $Y = yI$ with

$$x = Q\circ P(x)\,,\ y = P(x) \quad\Longleftrightarrow\quad y = P\circ Q(y)\,,\ x = Q(y)\,. \tag{2.20}$$

The effective potential for X is the sum of a classical part $(\operatorname{tr} V)/n$ and of a quantum term proportional to the partition function of the one-matrix model in an external field. For cubic U, this partition function is related to the Kontsevich integral for two-dimensional topological gravity [17], and was studied in [18]. In general, a qualitative picture of the analytic structure of the background force can be obtained by assuming that S is very small. In this limit the matrix Y can be integrated out classically and we get

$$f_{\mathrm{b,\,cl}}(x) = -P(x) + Q^{-1}(x)\,. \tag{2.21}$$

The functional inverse Q^{-1} of the polynomial Q is a multi-valued function with d_Y sheets. There are $d_Y - 1$ branching points at $x = Q'^{-1}(0)$. The analytic structure of f_{b} at finite S is a perturbation of the analytic structure of $f_{\mathrm{b,\,cl}}$, with the same number of sheets and of branching points. This can be explicitly checked when U is cubic by using the results of [18], and no other singularities are expected. In particular, f_{b} does not have branch cuts on the support of ρ^X, and thus does not contribute to contour integrals over α-, β- or γ-cycles. The relations (2.17) are thus valid. Moreover, the saddle point equation (2.14) has a simple $n \to \infty$ limit,

$$f_{\mathrm{b}}(x) + g^X(x + i\epsilon) + g^X(x - i\epsilon) = 0 \quad \text{for } x \in \operatorname{Support}[\rho^X]\,, \tag{2.22}$$

showing that g^X is a $(d_Y + 1)$-sheeted analytic function (the physical sheet and the d_Y additional sheets of f_{b}). This is consistent with the fact that g^X is an algebraic function of degree $d_Y + 1$ (see [14] and Section 4). A symmetric discussion applies to the effective description in terms of the matrix Y.

2.2.3 The non-commutative case

Things are much more subtle and interesting when higher dimensional representations are present. The naïve classical limit of a saddle point equation like (2.22) would yield the one-dimensional representations only. The emergence of the higher dimensional representations, which is a classical effect in the description in terms of the M matrices X_i, must come out as a quantum effect in the effective description in terms of the single matrix X. This means that some of the quantum corrections in f_{b} are singular and can have a finite $S \to 0$ limit that modifies the naïve classical limit. A related crucial fact is that, even though the equations (2.14) are expressed in terms of the eigenvalues of X only, they must know about correlations between X and the other matrices that do not commute with X. This implies that in the continuum limit, (2.14) cannot be expressed in terms of ρ^X only. We will argue that the density ρ^X is still unambiguously determined, because there is a constraint, the entanglement of eigenvalues, that relates ρ^X on the various intervals $I^X_{K,k}$ associated with a given representation R_K. Intuitively, this constraint comes from the $\mathrm{U}(n)$ gauge symmetry of the original matrix model.

Those facts are very general but are best illustrated on an example. Let us consider the model with potential

$$W(X,Y) = XY^2 + \alpha Y + V(X) \tag{2.23}$$

for an arbitrary polynomial V of degree $d+1$. It is useful to separate $V'(X)$ into an even and an odd part,

$$-V'(X) = F_1(X^2) + XF_2(X^2)\,. \tag{2.24}$$

The classical equations of motion

$$\mathcal{A}:\ \{X,Y\} = -\alpha\,, \quad Y^2 = F_1(X^2) + XF_2(X^2)\,, \tag{2.25}$$

admit $d+2$ irreducible representations R_K of dimension one, $X = xI$ and $Y = yI$ with

$$2xy = -\alpha\,, \quad y^2 = -V'(x)\,, \tag{2.26}$$

and $[(d-1)/2]$ irreducible representations $\tilde{R}_K$ of dimension two for which, in terms of Pauli matrices, $X = x\sigma^3$ and $Y = y_1\sigma^1 + y_3\sigma^3$ with

$$2xy_3 = -\alpha\,, \quad y^2 = y_1^2 + y_3^2 = F_1(x^2)\,, \quad F_2(x^2) = 0\,. \tag{2.27}$$

If only the one dimensional representations are considered, then the saddle point equations for the matrix X have standard features and can be solved

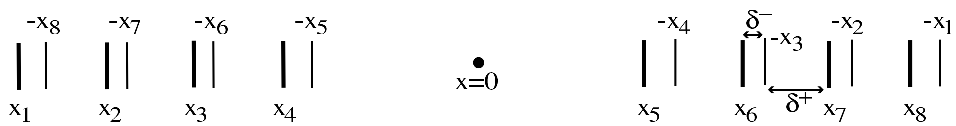

Figure 2: Eigenvalues filling a two dimensional representation $\tilde{R}_K$ for $n_K = 4$ (thick lines) and their images with respect to $x = 0$ (thin lines). When $\delta^+ \neq \delta^-$ the eigenvalues feel a quantum even force that compensates for the non-zero classical force $F_1 - \alpha^2/(4x^2)$ in (2.27).

by the usual tricks. This was done for $\alpha = 0$ and with only one cut that do not intersect the origin in [19]. We are interested in the general case. The effective potential for X, the background force and the saddle point equations can be calculated exactly,

$$V_{\text{eff}}(x_1, \dots x_n) = \frac{1}{n} \sum_i \Bigl(V(x_i) - \frac{\alpha^2}{4x_i} \Bigr) + \frac{S}{2n^2} \sum_{i \neq j} \ln |x_i + x_j| \, , \tag{2.28}$$

$$f_{\text{b}}(x) = -V'(x) - \frac{\alpha^2}{4x^2} - \frac{S}{n} \sum_j \frac{1}{x + x_j} = -V'(x) - \frac{\alpha^2}{4x^2} + g^X(-x) \, , \tag{2.29}$$

$$-V'(x_i) - \frac{\alpha^2}{4x_i^2} - \frac{S}{n} \sum_j \frac{1}{x_i + x_j} + \frac{2S}{n} \sum_{j \neq i} \frac{1}{x_i - x_j} = 0 \, . \tag{2.30}$$

It is important to assume that $\alpha \neq 0$, because otherwise the vacua $X = 0$, $Y = \pm \sqrt{F_1(0)}\, \sigma^1$, for which the $\mathbb{Z}_2$ symmetry $Y \to -Y$ is broken, can no longer be described after the exact integration over Y. If needed, the limit $\alpha \to 0$ can be taken on the final formulas. The singularities in (2.28) and (2.29) can be understood as follows. Classically, integrating out Y amounts to solving the equation

$$\{X, Y\} + \alpha = 0 \tag{2.31}$$

for a given matrix X, and this implies that $[X^2, Y] = 0$. Generically (2.31) has a unique solution $Y = -\alpha X^{-1}/2$, which is singular if X is not invertible, yielding poles at $x_i = 0$. More interestingly, if two eigenvalues of X are such that $x_i = -x_j$, then a different class of solutions to (2.31) is possible, corresponding to the two-dimensional representations. In that case, the $S \to 0$ limit of the term $S \sum_j 1/(x_i + x_j)$ in (2.30) is non-zero and compensates for the non-zero even background force in (2.27).

A typical configuration with a two-dimensional representation $\tilde{R}_K$, corresponding to a solution $x^2 = x_K^2$ of (2.27), is depicted in Figure 2. We put n_K eigenvalues $x_1, \dots, x_{n_K}$ at $x = -x_{(K)}$ and n_K eigenvalues $x_{n_K+1}, \dots, x_{2n_K}$ at $x = x_{(K)}$. Turning on S, the eigenvalues fill intervals $\tilde{I}_K^X = [\tilde{a}_K, \tilde{b}_K]$ and

$\tilde{I}'^X_K = [\tilde{a}'_K, \tilde{b}'_K]$ around $x_{(K)}$ and $-x_{(K)}$ respectively. The attractive force between eigenvalues and their images, described by the term $S\sum_j 1/(x_i + x_j)$, induces strong correlations between the eigenvalues around $+x_K$ and the eigenvalues around $-x_K$. If we choose the labels such that $x_1 < x_2 < \cdots < x_{2n_K}$, classically $x_1 = \cdots = x_{n_K} = -x_{(K)}$ and $x_{n_K+1} = \cdots = x_{2n_K} = x_{(K)}$. Quantum mechanically, the only stable, finite energy configurations are then such that $-x_{2n_K} < x_1$ and $x_i < -x_{2n_K-i} < x_{i+1}$ for $1 \le i \le n_K - 1$, or $x_i < -x_{2n_K-i+1} < x_{i+1}$ for $1 \le i \le n_K - 1$ and $x_{n_K} < -x_{n_K+1}$. This entanglement of eigenvalues can be proven as follows. If more than one image eigenvalue $-x_j$, $j > n_K$, were in a given interval $]x_i, x_{i+1}[$, $i \le n_K - 1$, then, because of the attractive force between eigenvalues and image eigenvalues, the system would collapse to a singular configuration with divergent energy. If there were some intervals $]x_i, x_{i+1}[$ without image eigenvalues then, due to the repulsive force, the eigenvalues that are not entangled would not converge to the equilibrium positions x_K or $-x_K$ for the two-dimensional representations $\tilde{R}_K$ when $S \to 0$.

A direct consequence of the entanglement is that, in the continuum large n limit, the intervals $\tilde{I}_K$ and $\tilde{I}'_K$ are symmetric, $\tilde{b}'_K = -\tilde{a}_K$ and $\tilde{a}'_K = -\tilde{b}_K$, and

$$\rho^X(x) = \rho^X(-x) \quad \text{for } x \in \tilde{I}^X_K\,. \tag{2.32}$$

This shows that the background force (2.29) has branch cuts that coincide with those of g^X, and in particular will contribute to contour integrals. Moreover, the continuum limit of the saddle point equation (2.14, 2.30) must be studied with care. One might want to replace $f_{\mathrm{b}}(x)$ by $(f_{\mathrm{b}}(x+i\epsilon) + f_{\mathrm{b}}(x-i\epsilon))/2$, but this is not correct. As explained in Appendix A, Section 1, if $\delta^+(x)$ and $\delta^-(x)$ are the even functions defined in Figure 2, then the continuum limit is

$$\frac{1}{2}\big(f_{\mathrm{b}}(x+i\epsilon) + f_{\mathrm{b}}(x-i\epsilon)\big) + g^X(x+i\epsilon) + g^X(x-i\epsilon) = \\ S\rho^X(-x)\ln\frac{\delta^-(x)}{\delta^+(x)}\,, \quad \text{for } x \in \mathrm{Support}[\rho^X]\,. \tag{2.33}$$

The even "quantum" force on the right hand side of (2.33) can compensate for the even classical force $F_1(x^2) - \alpha^2/(4x^2)$ in the $S \to 0$ limit. Consequently, there can exist classical equilibrium configurations for which only the odd part $xF_2(x^2)$ of the classical force vanishes. Those configurations correspond precisely to the two dimensional representations (2.27). By analysing (2.32) and (2.33), we further show in Appendix A that g^X is generically three-sheeted. It satisfies a degree three algebraic equation that we compute by two different methods in Sections 4 and 5.

2.2.4 Lesson

For higher dimensional representations, we have seen that the continuum saddle point equation involves additional unknown functions and is supplemented by conditions coming from the entanglement of eigenvalues. Once this is understood, one can proceed and try to derive the special geometry relations. This is done is Appendix A for the model (2.23), both in the description in terms of X discussed above and in the description in terms of Y. The same analysis applies straightforwardly in a large class of models, including the famous $\mathcal{N}=1^*$ theory, in which case an ansatz corresponding to eigenvalue entanglement was first made in [21].

In general, the effective potential, as a function of the eigenvalues of a given matrix X, has singularities when the eigenvalues satisfy some conditions corresponding to the emergence of representations for which X does not commute with all the other matrices. In our case the condition was $x_i=-x_j$, but other type of conditions, for example $x_i-x_j\in\mathbb{Z}$, are possible. The logarithmic singularities that we found in (2.28) are singled out by the fact that V_{eff} is a well-defined functional of ρ^X in the continuum limit (which is necessary for the idea of a master field, and the corresponding factorization of correlation functions, to apply), unlike its derivatives, that know about the correlations between X and the other matrices. The generated force must be attractive in order to stabilize new higher dimensional solutions in the classical limit. These features presumably imply eigenvalue entanglement and special geometry in a large class of models.

2.3 Solving the constraints from special geometry

A natural way to implement special geometry is to express the solution in terms of a Calabi-Yau manifold with nowhere vanishing holomorphic top-form Ω,

$$S_K=\frac{1}{2i\pi}\oint_{A_K}\Omega\,,\quad \frac{\partial\mathcal{F}}{\partial S_K}=\oint_{C_K}\Omega+\text{counterterm}\,. \qquad (2.34)$$

The A_K and C_K are holomorphic spheres in one-to-one correspondence with the $\alpha_K^{X_m}$ and $\gamma_K^{X_m}$ cycles. A very important additional property is that, at least in a large class of examples, there are privileged coordinates associated with the matrices, or more precisely with the generators of the center of the algebra $\mathcal{A}$. Integrating over the coordinates in different orders in (2.34), we can obtain the description in terms of the various matrices. We will argue in Section 4 that this property makes possible the computation of the resolvents.

3 Matrix models and Calabi-Yau threefolds

Let us consider type IIB string theory one some non-compact Calabi-Yau threefold. Let us assume that there is a $\mathbb{P}^1$ with normal bundle $\mathcal{N}$ and quantum volume V in the geometry. By wrapping N D5 branes on the $\mathbb{P}^1$, we engineer a four dimensional $\mathcal{N} = 1$ supersymmetric gauge theory, with bare Yang-Mills coupling constant $g^2_{\mu_{UV}} \sim 1/V$ and gauge group $\mathrm{U}(N)$. The deformation space of the $\mathbb{P}^1$, or equivalently the space of fluctuations of the brane, is generated locally by the holomorphic sections of the normal bundle $\mathcal{N}$. Physically, this means that if $\mathcal{N}$ has h linearly independent holomorphic sections, the gauge theory is coupled to h scalar superfields in the adjoint representation. For a generic geometry, there is an obstruction to the deformation space, which in favorable cases can be interpreted physically in terms of a superpotential W.

If $h \leq 2$, the resulting gauge theory is asymptotically free and dynamically generates a mass gap Λ. The renormalized Yang-Mills coupling g^2_μ at scale μ grows in the IR, and becomes infinite at some scale $\mu_{\rm c} \sim \Lambda$. Beyond that scale the fundamental gauge fields strongly fluctuate and are no longer appropriate degrees of freedom. From the type IIB perspective the RG flow corresponds to the shrinking of the $\mathbb{P}^1$ to zero size. At scale $\mu = \mu_{\rm c}$, the $\mathbb{P}^1$ vanishes and the original smooth Calabi-Yau is replaced by a singular geometry. What happens for $\mu < \mu_{\rm c}$ has not been derived from first principles, but there are remarkable conjectures that partially address the problem. On the gauge theory side, it is believed that good degrees of freedom are given by the so-called glueball superfields S_i, and that the exact quantum superpotential $W_{\rm glueballs}$ for the S_i can be calculated from the partition function of an associated h-matrix model whose potential is equal to the superpotential of the gauge theory [13]. On the string theory side, it is believed that the singular Calabi-Yau is deformed to a smooth Calabi-Yau. The branes and the $\mathbb{P}^1$ disappear and are replaced by three-form flux through three-spheres. The non-zero flux generates a superpotential $W_{\rm flux}$ that can be computed from special geometry [22]. The consistency between the conjectures,

$$W_{\rm glueballs} = W_{\rm flux}\,, \tag{3.1}$$

implies a highly non-trivial relationship between the partition function of the matrix model and the Calabi-Yau geometry, whose form coincide precisely with (2.34). This relationship is indicated by a question mark in Figure 3.

The great advantage of this gauge theory/string theory set-up is that we have a precise recipe, explained below, to construct the relevant Calabi-Yau spaces. On the other hand, it is important to emphasize that there is no

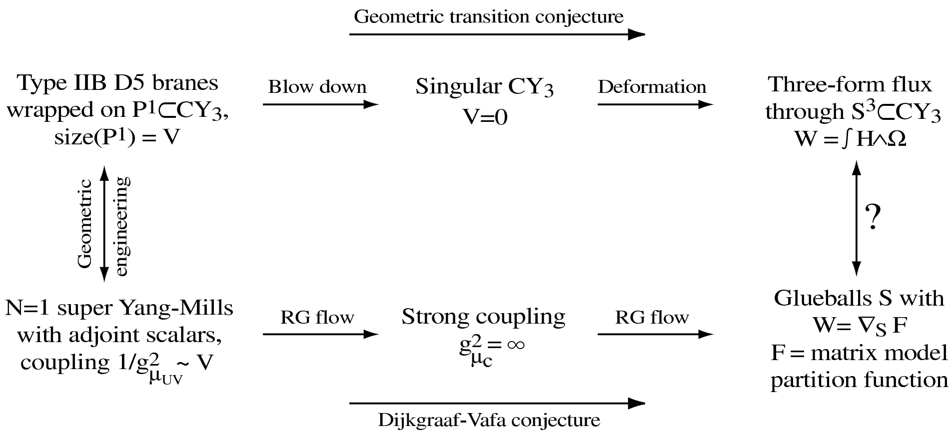

Figure 3: Chain of conjectured dualities suggesting a relation between Calabi-Yau spaces and matrix models. The question mark emphasizes the link that we want to explore. See details in the main text.

general proof of the validity of the approach. For example, the full relationship between $\mathcal{N}=1$ gauge theories and matrix models can be proven only in special cases and under some assumptions [23]. A perturbative argument can also be given [24], but it does not apply to the objects that are relevant for us. In particular, there is no understanding of $\mathcal{N}=1$ special geometry [25] in gauge theory from first principles.

As is clear from Figure 3, there are three geometries that play a rôle, and that we discuss in turn in the following. The first geometry is the original Calabi-Yau space $\hat{\mathcal{M}}$ containing the $\mathbb{P}^1$s with normal bunble $\mathcal{N}$ on which we wrap the D5 branes. This space $\hat{\mathcal{M}}$ is called the resolved Calabi-Yau, because it is a small resolution of the singular Calabi-Yau $\mathcal{M}_0$ obtained from $\hat{\mathcal{M}}$ by shrinking the $\mathbb{P}^1$ (the resolution is small because the singular points are replaced by curves). Precisely, there exists a blow down map $\pi:\hat{\mathcal{M}}\rightarrow\mathcal{M}_0$ that locally induces a birational isomorphism between $\hat{\mathcal{M}}\backslash\{\mathbb{P}^1\}$ and $\mathcal{M}_0\backslash\{p\}$ where p is a singularity and $\pi^{-1}(p)=\mathbb{P}^1$. Finally there is the smooth deformed space $\mathcal{M}$ obtained by perturbing the algebraic equation defining $\mathcal{M}_0$. All we need to know about $\mathcal{M}$ is this algebraic equation, that encodes the complex structure, because the metric datas are irrelevant in integrals like (2.34).

3.1 The resolved geometry

3.1.1 The transition functions

Let us describe $\hat{\mathcal{M}}$ by two coordinate patches (z, w_1, w_2) and (z', w_1', w_2'), where z and $z' = 1/z$ are the stereographic coordinates on the two-spheres, and consider the following transition functions

$$z' = 1/z\,, \quad w_1' = z^{-n} w_1\,, \quad w_2' = z^{-m} w_2\,. \tag{3.2}$$

If $n \geq 0$ and $m < 0$ (other cases could be discussed as well, but are irrelevant for our purposes), the geometry (3.2) has a $(n+1)$-dimensional continuous family of $\mathbb{P}^1$s that sit at

$$w_1(z) = \sum_{i=1}^{n+1} x_i z^{i-1}\,, \quad w_2(z) = 0\,. \tag{3.3}$$

The normal bundle to the $\mathbb{P}^1$s is $\mathcal{N} = \mathcal{O}(n) \oplus \mathcal{O}(m)$ by construction. The functions $w_1(z)$ and $w_2(z)$ in (3.3) are characterized by the fact that they must define globally holomorphic sections of $\mathcal{N}$. This is indeed the case because in the other coordinate patch, (3.2) yields

$$w_1'(z') = \sum_{i=1}^{n+1} x_i z'^{n-i+1}\,, \quad w_2'(z') = 0\,. \tag{3.4}$$

The parameters x_i, $1 \leq i \leq n+1$, span the versal deformation space of the $\mathbb{P}^1$s. A more interesting geometry is obtained by perturbing the transition functions given by (3.2). The perturbation is described in terms of a geometric potential. The potential is a function $E(z, w)$ of two complex variables that can be Laurent expanded in terms of entire functions E_i,

$$E(z, w) = \sum_{i=-\infty}^{+\infty} E_i(w) z^i\,. \tag{3.5}$$

The most general geometry we will consider is given by

$$z' = 1/z\,, \quad w_1' = z^{-n} w_1\,, \quad w_2' = z^{-m} w_2 + \partial_w E(z, w_1)\,. \tag{3.6}$$

Since the relation between w_1 and w_1' is unchanged with respect to (3.2), the most general holomorphic section $(w_1(z), w_2(z))$ of $\mathcal{N}$ is still such that

$$w_1(z) = \sum_{i=1}^{n+1} x_i z^{i-1} \Longleftrightarrow w_1'(z') = \sum_{i=1}^{n+1} x_i z'^{n-i+1}\,. \tag{3.7}$$

On the other hand, $w_2(z)$ must now be adjusted non-trivially to insure that $w_2'(z')$ is holomorphic. Because of the term $z^{-m}w_2 = z'^{-|m|}w_2$ in (3.6), one can always choose $w_2(z)$ to cancel all the poles in z'^{-j} for $j \geq |m|$, but there remains $|m|-1$ singular terms that cancel only if the parameters x_i in (3.7) satisfy $|m|-1$ constraints. Thus we find that the versal deformation space of $\mathbb{P}^1$ is spanned by $n+1$ parameters satisfying $|m|-1$ constraints (the same result could be straightforwardly obtained for the most general perturbation of (3.2), see [26]). Since the tangent bundle to $\mathbb{P}^1$ is $\mathcal{O}(2)$, the Calabi-Yau condition of vanishing first Chern class for the total space is

$$M = n+1 = -m-1 \tag{3.8}$$

and is thus equivalent to the equality of the number of parameters and the number of constraints.

3.1.2 The superpotential

The special form (3.6) of the perturbation is particularly interesting because in this case the constraints are integrable and are equivalent to the extremization $\mathrm{d}W = 0$ of a superpotential $W(x_1, \ldots, x_{n+1})$. This was proven in [27] in the case where $E(z,w)$ is regular at $z=0$, and the same argument could be straightforwardly generalized to (3.5). We can actually derive an explicit formula for W that makes this property manifest. By using (3.6), (3.7) and (3.8) we obtain

$$w_2'(z') = z^{M+1}w_2(z) + \partial_w E\Big(z, \sum_{i=1}^{M} x_i z^{i-1}\Big). \tag{3.9}$$

Let us define the coutour C_z to encircle both the points $u=0$ and $u=z$. If we choose in a manifestly z-holomorphic way w_2 to be

$$w_2(z) = -\frac{1}{2i\pi}\oint_{C_z} \frac{\partial_w E\left(u, \sum_{i=1}^{M} x_i u^{i-1}\right)\mathrm{d}u}{u^{M+1}(u-z)}\,, \tag{3.10}$$

and impose the M integrable constraints

$$\frac{\partial W}{\partial x_i} = \frac{1}{2i\pi}\oint_{C_0} \mathrm{d}z\, z^{i-M-2}\partial_w E\Big(z, \sum_{j=1}^{M} x_j z^{j-1}\Big) = 0\,, \quad 1 \leq i \leq M\,, \tag{3.11}$$

then we get a manifestly z'-holomorphic w_2',

$$w_2'(z') = \frac{1}{2i\pi}\oint_{C_{z'}} \frac{\partial_w E\left(1/u, \sum_{i=1}^{M} x_i u^{1-i}\right)\mathrm{d}u}{u-z'}\,. \tag{3.12}$$

The superpotential is

$$W(x_1,\dots,x_M) = \frac{1}{2i\pi}\oint_{C_0} z^{-M-1} E\Big(z, \sum_{i=1}^{M} x_i z^{i-1}\Big)\, \mathrm{d}z. \tag{3.13}$$

For example, in the case of one variable we obtain $W(x) = E_1(x)$, and in the case of two variables

$$W(x,y) = \sum_{i\geq 0} E_{2-i}^{(i)}(x) y^i / i!\,. \tag{3.14}$$

It might seem that different geometries (3.6) can yield the same W, but this is not the case. For example, in the case $M = 2$, the holomorphic change of variables

$$\begin{aligned} w_2 &\to w_2 - \sum_{i<0} E'_{2-i}(w_1) z^{-i-1}\,, \\ w'_2 &\to w'_2 + \sum_{i\geq 2}\sum_{j=0}^{i-2} E_{2-i}^{(j+1)}(0) w_1'^{j} z'^{i-2-j}/j!\,, \end{aligned} \tag{3.15}$$

put the geometry in the form

$$z' = 1/z\,,\quad w'_1 = z^{-1} w_1\,,\quad w'_2 = z^3 w_2 + \sum_{i\geq 0} E'_{2-i}(w_1) z^{2-i}\,, \tag{3.16}$$

with the conditions

$$E_{2-i}^{(j)}(0) = 0 \quad \text{for} \quad 0 \leq j \leq i-1 \tag{3.17}$$

on the derivatives of the functions E_k. There is thus a unique geometry for a given superpotential.

3.1.3 The non-commutative structure

Equation (3.14) shows that we can engineer an arbitrary superpotential of two variables when only one brane is wrapped on $\mathbb{P}^1$. However, ordering ambiguities arise when $N \geq 2$ D5 branes are considered. Eventually we expect that only special, in some sense integrable, matrix models can be constructed. The rigorous computation of the matrix model potential W as a function of matrices X_i would require an extensive use of the theory of branes in Calabi-Yau threefolds, and this is beyong the scope of the present paper. There is however little doubt as to what the correct answer must be. Our proposal is that the parameters x_i parametrizing the fluctuations of the branes in (3.7) must be replaced by hermitian matrices X_i. Supersymmetry then implies (3.11) with $x_j \to X_j$. Remarkably, with this prescription for the ordering, the constraints are still integrable, and the potential is given by (1.1).

In the two-matrix case, on which we shall mainly focus in the following, the above prescription tells that the non-commutative version of a monomial $x^i y^j$ is simply obtained by expanding $\mathrm{tr}(X+Y)^{i+j}$ and picking up the terms with the right degrees in X and Y. For example, $x^2y^2 \to \mathrm{tr}(2X^2Y^2/3 + XYXY/3)$. The general formula is

$$x^i y^j \longrightarrow \frac{i!j!}{(i+j)!} \oint_{C_0} \frac{\mathrm{d}z}{2i\pi}\, z^{-1-j}\, \mathrm{tr}(X+Yz)^{i+j}\,. \tag{3.18}$$

For polynomials V_X, V_Y and V, the geometric potential

$$E(z,w) = z^2 V_X(w) + z^2 V_Y(w/z) + \frac{z^2 V(w)}{1-1/z} \tag{3.19}$$

yields models of the form

$$W(X,Y) = V_X(X) + V_Y(Y) + V(X+Y)\,. \tag{3.20}$$

The term $(1-1/z)^{-1}$ in (3.19) must be understood as a formal power series in z^{-1}, for which only the first $\deg V + 1$ terms are relevant. Special cases of (3.20) have been obtained by considering Calabi-Yau threefolds that are monodromic fibrations of ALE spaces with ADE singularities [12]. The A-series correspond to $\deg V = 2$, the D-series to $\deg V = \deg V_Y = 3$ and the E-series to $\deg V = 3$, $\deg V_Y = 4$ and $4 \leq \deg V_X \leq 6$. For those models, the deformed geometry $\mathcal{M}$ can in principle be calculated by using the results of [28].

3.2 Blowing down

3.2.1 Theorems

Let us first state an important theorem by Laufer [29]:

THEOREM (Laufer): Let $\mathcal{M}_0$ be an analytic space of dimension $D \geq 3$ with an isolated singularity at p. Suppose that there exists a non-zero holomorphic D-form Ω on $\mathcal{M}_0 \backslash \{p\}$. Let $\pi : \hat{\mathcal{M}} \to \mathcal{M}_0$ be a resolution of $\mathcal{M}_0$. Suppose that the exceptional set $A = \pi^{-1}(p)$ is one-dimensional and irreducible. Then A is isomorphic to $\mathbb{P}^1$ and $D = 3$. Moreover, the normal bundle of

$\mathbb{P}^1$ in $\hat{\mathcal{M}}$ must be either $\mathcal{N} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, or $\mathcal{N} = \mathcal{O}(0) \oplus \mathcal{O}(-2)$, or $\mathcal{N} = \mathcal{O}(1) \oplus \mathcal{O}(-3)$.

It is known in general that $\mathcal{N}$ is a direct sum of line bundles [30], $\mathcal{N} = \mathcal{O}(n) \oplus \mathcal{O}(m)$. The condition on the normal bundle in Theorem 1 is thus equivalent to asymptotic freedom, $M \leq 2$. This is consistent with the fact that asymptotically free theories are the one for which we can expect the $\mathbb{P}^1$ to be exceptional. The situation for $M \geq 3$ is more subtle. An important point is that the normal bundle to the $\mathbb{P}^1$ in the geometry (3.6) is changed when the perturbation E is added. From the gauge theory point of view, turning on the superpotential amounts to giving a mass to the chiral multiplets, and generically the theory flows to the pure gauge theory in the IR. More precisely, the number of massless chiral multiplets in a given vacuum is equal to the corank of the Hessian of W at the corresponding critical point. This translates mathematically in the following

CONJECTURE (RG flow): Consider the geometry (3.6) for $m = -n-2$ and associated superpotential W given by (3.13). Let $\mathcal{N}$ be the normal bundle of a $\mathbb{P}^1$ that sits at a given critical point of W. Let r be the corank of the Hessian of W at the critical point. Then $\mathcal{N} = \mathcal{O}(r-1) \oplus \mathcal{O}(-r-1)$.

We give an elementary proof of this conjecture for $n = 1$ in the Appendix. This result suggests that models with an arbitrary number of multiplets, or integrals over an arbitrary number of matrices, may still be described by a geometric transition as in Figure 3. The difficulty is that Laufer's theorem implies that the blow down map must be singular when the parameters are adjusted in such a way that the corank of the Hessian jumps to a value greater than two. Because of this complication, we restrict ourselves to two-matrix models in the following.

3.2.2 Examples

The blow down map $\pi : \hat{\mathcal{M}} \to \mathcal{M}_0$ is given explicitly by four globally holomorphic functions π_i, $1 \leq i \leq 4$. The mapping π must be a birational isomorphism except on the $\mathbb{P}^1$s that are mapped onto the singular points of $\mathcal{M}_0$. We present the construction for three examples.

EXAMPLE 1: The resolved geometry

$$z' = 1/z\,,\ w_1' = w_1\,,\ w_2' = z^2 w_2 + zP(w_1)\,, \tag{3.21}$$

engineers the one-matrix model with a potential $W(X)$ such that $W' = P$.

The $\mathbb{P}^1$s sit at

$$w_1(z) = x = w_1'(z')\,, \quad w_2(z) = 0 = w_2'(z')\,, \tag{3.22}$$

with $P(x) = 0$. The blow down is straightforwardly found to be

$$\pi_1 = w_1 = w_1'\,, \tag{3.23}$$
$$\pi_2 = 2zw_2 + P(w_1) = 2z'w_2' - P(w_1')\,, \tag{3.24}$$
$$\pi_3 = w_2 - zP(w_1) - z^2 w_2 = -w_2' - z'P(w_1') + z'^2 w_2'\,, \tag{3.25}$$
$$\pi_4 = w_2 + zP(w_1) + z^2 w_2 = w_2' - z'P(w_1') + z'^2 w_2'\,. \tag{3.26}$$

As required, $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ maps the $\mathbb{P}^1$s onto points,

$$\pi_1(\mathbb{P}^1) = x\,, \quad \pi_2(\mathbb{P}^1) = \pi_3(\mathbb{P}^1) = \pi_4(\mathbb{P}^1) = 0\,, \tag{3.27}$$

and is a birational isomorphism outside the $\mathbb{P}^1$s whose inverse is given by

$$w_1 = \pi_1\,, \quad w_2 = \frac{1}{2}\big(\pi_3 + \pi_4\big)\,, \quad z = \frac{\pi_2 - P(\pi_1)}{\pi_3 + \pi_4}\,\cdot \tag{3.28}$$

By taking the difference between (3.25) and (3.26), we obtain the following algebraic constraint

$$\mathcal{M}_0 :\ \pi_4^2 = \pi_3^2 + \pi_2^2 - P^2(\pi_1) \tag{3.29}$$

that defines the singular Calabi-Yau geometry.

Example 2: The resolved geometry for the model (2.18) takes the form

$$z' = 1/z\,,\ w_1' = z^{-1} w_1\,,\ w_2' = z^3 w_2 + zQ(w_1/z) + z^2 P(w_1) - w_1 z\,. \tag{3.30}$$

From (3.7) and (3.10) we see that the $\mathbb{P}^1$s sit at

$$w_1(z) = x + yz\,, \quad w_1'(z') = y + xz'\,, \tag{3.31}$$
$$w_2(z) = \frac{P(x) - P(x + yz)}{z}\,, \quad w_2'(z') = \frac{Q(y + xz') - Q(y)}{z'}\,\cdot \tag{3.32}$$

The blow down map can be constructed by trial and error. For example, starting with the ansatz

$$\pi_1 = -z'w_2' + \cdots = -z^2 w_2 - Q(w_1/z) + w_1 - zP(w_1) + \cdots\,, \tag{3.33}$$

we see that the missing part can be adjusted to cancel the non-holomorphic piece $Q(w_1/z)$ in the right hand side, finally yielding

$$\pi_1 = -z'w_2' + Q(w_1') = w_1 - zP(w_1) - z^2 w_2\,. \tag{3.34}$$

One can construct similarly

$$\pi_2 = zw_2 + P(w_1) = w_1' - z'Q(w_1') + z'^2 w_2' \,. \tag{3.35}$$

The construction of π_3 and π_4 requires further thought. A hint is given by calculating π_1 and π_2 on the spheres (3.31), (3.32),

$$\pi_1(\mathbb{P}^1) = x \,, \quad \pi_2(\mathbb{P}^1) = y \,. \tag{3.36}$$

Consistently with the fact that the $\mathbb{P}^1$s are mapped onto points, the result is independent of z or z'. The result also suggests that $P(\pi_1)$ and $Q(\pi_2)$ could be useful globally holomorphic objects to consider. One then finds that

$$\begin{aligned} \pi_3 &= -z'^3 w_2' + z'^2 Q(w_1') + z' P(\pi_1) - z' w_1' = \frac{P(\pi_1) - P(w_1)}{z} - w_2 \,, & (3.37) \\ \pi_4 &= z^3 w_2 + z^2 P(w_1) + z Q(\pi_2) - z w_1 = \frac{Q(\pi_2) - Q(w_1')}{z'} + w_2' \,, & (3.38) \end{aligned}$$

have all the required properties. In particular

$$\pi_3(\mathbb{P}^1) = \pi_4(\mathbb{P}^1) = 0 \tag{3.39}$$

are z-independent, and the inverse is given by

$$w_1 = \pi_1 + \frac{\pi_2\pi_4}{Q(\pi_2) - \pi_1} \,, \quad w_2 = \frac{(\pi_2 - P(w_1))\pi_3}{P(\pi_1) - \pi_2} \,, \quad z = \frac{\pi_4}{Q(\pi_2) - \pi_1} \,, \tag{3.40}$$

with similar formulas for the primed variables. The singular Calabi-Yau is then found to be

$$\mathcal{M}_0 : \ \pi_3\pi_4 = \big(P(\pi_1) - \pi_2\big)\big(Q(\pi_2) - \pi_1\big) \,. \tag{3.41}$$

EXAMPLE 3: We consider the superpotential

$$W(X,Y) = XY^2 + V(X) + YU(Y^2) \,. \tag{3.42}$$

Note that a term in Y^2 could be generated by a simple shift in X. It is useful to introduce

$$\begin{aligned} V'(x) = P(x) &= -F_1(x^2) - xF_2(x^2) \,, \\ Q(y) = U(y^2) + 2y^2 U'(y^2) &= -G(y^2) \,. \end{aligned} \tag{3.43}$$

The resolved geometry is

$$z' = 1/z \,, \ w_1' = z^{-1} w_1 \,, \ w_2' = z^3 w_2 + zQ(w_1/z) + z^2 P(w_1) + w_1^2 \,, \tag{3.44}$$

and the equations for the spheres are

$$w_1(z) = x + yz\,, \quad w_1'(z') = y + xz'\,, \tag{3.45}$$

$$w_2(z) = \frac{P(x) - P(x+yz)}{z}\,, \quad w_2'(z') = x^2 + \frac{Q(y+xz') - Q(y)}{z'}\,\cdot \tag{3.46}$$

The blow down map is constructed using the same tricks as for Example 1. A special case of the map also appears in [29] and [12]. We find

$$\begin{aligned}
\pi_1 &= w_1^2 - zG(\pi_2) + z^2 P(w_1) + z^3 w_2 = w_2' + \frac{G(w_1'^2) - G(\pi_2)}{z'}\,, && (3.47)\\
\pi_2 &= -zw_2 - P(w_1) = w_1'^2 + z'Q(w_1') - z'^2 w_2'\,, && (3.48)\\
\pi_3 &= z'\pi_2 - w_1' F_2(\pi_1) - z' F_1(\pi_1)\\
&= -w_2 + \frac{F_1(w_1^2) - F_1(\pi_1) + w_1\bigl(F_2(w_1^2) - F_2(\pi_1)\bigr)}{z}\,, && (3.49)\\
\pi_4 &= w_1'\pi_2 - z'\pi_1 F_2(\pi_1) - w_1' F_1(\pi_1)\\
&= -w_1 w_2 + \frac{w_1\bigl(F_1(w_1^2) - F_1(\pi_1)\bigr) + w_1^2 F_2(w_1^2) - \pi_1 F_2(\pi_1))}{z}\,\cdot && (3.50)
\end{aligned}$$

The blow down can be inverted using, for example, the relations

$$\begin{aligned}
z' &= \frac{(\pi_2 - F_1(\pi_1))\pi_3 + F_2(\pi_1)\pi_4}{(\pi_2 - F_1(\pi_1))^2 - \pi_1 F_2^2(\pi_1)}\,, && (3.51)\\
w_1' &= \frac{\pi_1\pi_3 F_2(\pi_1) + (\pi_2 - F_1(\pi_1))\pi_4}{(\pi_2 - F_1(\pi_1))^2 - \pi_1 F_2^2(\pi_1)}\,, && (3.52)\\
w_2' &= \pi_1 + \frac{G(\pi_2) - G(w_1'^2)}{z'}\,\cdot && (3.53)
\end{aligned}$$

The $\mathbb{P}^1$s are mapped onto points,

$$\pi_1(\mathbb{P}^1) = x^2\,,\ \pi_2(\mathbb{P}^1) = y^2\,,\ \pi_3(\mathbb{P}^1) = -yF_2(x^2)\,,\ \pi_4(\mathbb{P}^1) = xyF_2(x^2), \tag{3.54}$$

and the singular geometry is given by

$$\begin{aligned}
\mathcal{M}_0:\ \pi_2\Bigl[\bigl(\pi_2 - F_1(\pi_1)\bigr)^2 - \pi_1 F_2^2(\pi_1)\Bigr] &=\\
\pi_4^2 - \pi_1\pi_3^2 - G(\pi_2)\Bigl[\bigl(\pi_2 - F_1(\pi_1)\bigr)\,\pi_3 + \pi_4 F_2(\pi_1)\Bigr]&. && (3.55)
\end{aligned}$$

3.3 The singular geometry and privileged coordinates

The singular points in a geometry

$$\mathcal{M}_0:\ \mathcal{G}_0(\pi_1,\pi_2,\pi_3,\pi_4) = 0 \tag{3.56}$$

are obtained by solving the equations

$$\mathcal{G}_0 = 0\,, \quad \mathrm{d}\mathcal{G}_0 = 0 \tag{3.57}$$

simultaneously. By construction, the points $\pi_i(\mathbb{P}^1)$, which are the images of the two-spheres, are singular. They are associated with the one dimensional representations of the algebra of classical equations of motion, since the $\mathbb{P}^1$s sit at the extrema of the superpotential W as a function of commuting variables. A deep consistency requirement is that there are also singular points associated with the higher dimensional representations [12], because the geometry encodes the most general solution of the matrix model. The full set of representations must thus be found from the algebraic equations (3.57) over commuting variables. This is possible if there is a correspondence between some combinations of the coordinates and the Casimir operators that characterize the representations. These combinations are related to the privileged coordinates that we have already discussed in Section 2.3. There can also be additional singularities, that are reminiscent of the commonly found branch cuts on unphysical sheets for the resolvents.

EXAMPLE 1: The singular points of the geometry (3.29) are at

$$P(\pi_1) = 0\,, \quad \pi_2 = \pi_3 = \pi_4 = 0\,, \tag{3.58}$$

which yields the obvious identification

$$X \equiv \pi_1 = x\,. \tag{3.59}$$

EXAMPLE 2: For (3.41), equations (3.57) are equivalent to

$$\pi_1 = Q(\pi_2)\,, \quad \pi_2 = P(\pi_1)\,, \quad \pi_3 = \pi_4 = 0\,. \tag{3.60}$$

This is perfectly consistent with (2.20), with the identification

$$X \equiv \pi_1 = x\,, \quad Y \equiv \pi_2 = y\,, \tag{3.61}$$

that is also suggested by (3.36).

EXAMPLE 3: The algebra of classical equations of motion associated with the model (3.42),

$$\mathcal{A}:\ \{X, Y\} = G(Y^2)\,, \quad Y^2 = F_1(X^2) + XF_2(X^2)\,, \tag{3.62}$$

is very similar to the one studied in Section 2.2.3, and provides an example with both one and two dimensional representations. The generic singular points of the associated $\mathcal{M}_0$ (3.55) are found to satisfy either

$$\begin{aligned} &4\pi_1\pi_2 = G^2(\pi_2)\,, \quad \bigl(\pi_2 - F_1(\pi_1)\bigr)^2 = \pi_1 F_2^2(\pi_1)\,, \\ &\pi_3 = \frac{(F_1(\pi_1) - \pi_2)G(\pi_2)}{2\pi_1}\,, \quad \pi_4 = \frac{1}{2}G(\pi_2)F_2(\pi_1)\,, \end{aligned} \tag{3.63}$$

which yield the one dimensional representations, or

$$F_2(\pi_1) = 0\,, \quad \pi_2 = F_1(\pi_1)\,, \quad \pi_3 = \pi_4 = 0\,, \tag{3.64}$$

which yield the two dimensional representations. We obtain the identifications

$$X^2 \equiv \pi_1 = x^2\,, \quad Y^2 \equiv \pi_2 = y^2\,, \tag{3.65}$$

which also make (3.63) and (3.64) perfectly consistent with (3.54).

3.4 The deformed geometry

Singularities of $\mathcal{M}_0$ lie at the classical critical points of the superpotential. When the coupling S is turned on, the eigenvalues spread and the critical points are replaced by cuts. The defining equation (3.56) of $\mathcal{M}_0$ is then deformed in such a way that the singularities are replaced by three-spheres. The equation of the deformed space is of the form

$$\begin{aligned}\mathcal{M}:\ \mathcal{G}(\pi_1,\pi_2,\pi_3,\pi_4) = \\ \mathcal{G}_0(\pi_1,\pi_2,\pi_3,\pi_4) + S \sum_{(a,b,c,d)\in\mathbb{N}^4} c_{abcd}\, \pi_1^a\pi_2^b\pi_3^c\pi_4^d = 0\,.\end{aligned} \tag{3.66}$$

To find the allowed monomials $\pi_1^a\pi_2^b\pi_3^c\pi_4^d$, we must impose that the only divergences that occur in the period integrals (2.34) are either S-independent or linear in S and logarithmic. This constraint comes from the renormalizability of the gauge theory on the brane. A logarithmic divergence proportional to S is absorbed by the standard renormalization of the gauge coupling constant. It is important to realize that divergences proportional to higher powers of S are not allowed, even if they are logarithmic. For example, this subtlety must be taken into account to find the correct deformation of (3.55). Another constraint, from analyticity, is that a monomial that is generically forbidden cannot appear in a special case, for example when a coupling in the matrix model potential is set to zero. After all those constraints have been taken into account, and up to coordinate redefinitions, the number of deformation parameters c_{abcd} must match the number of irreducible representations, because there should be a uniquely defined deformed space for each vacuum (2.3).

It is usually straightforward to guess the general form of the deformations up to coordinate transformations. For example, the deformed version of

(3.29), (3.41) and (3.55) are respectively

$$\mathcal{M}:\ \pi_4^2 = \pi_3^2 + \pi_2^2 - P^2(\pi_1) + S\Delta(\pi_1)\,, \tag{3.67}$$

$$\mathcal{M}:\ \pi_3\pi_4 = \big(P(\pi_1) - \pi_2\big)\big(Q(\pi_2) - \pi_1\big) + S\Delta(\pi_1, \pi_2)\,, \tag{3.68}$$

$$\begin{aligned}\mathcal{M}:\ &\pi_2\Big[\big(\pi_2 - F_1(\pi_1)\big)^2 - \pi_1 F_2^2(\pi_1)\Big] + S\Delta(\pi_1,\pi_2) = \\ &\pi_4^2 - G(\pi_2)F_2(\pi_1)\pi_4 - \pi_1\pi_3^2 - \Big[G(\pi_2)\big(\pi_2 - F_1(\pi_1)\big) + S\hat{\Delta}(\pi_2)\Big]\pi_3\,,\end{aligned} \tag{3.69}$$

where Δ and $\hat{\Delta}$ are polynomials. The constraints on the degrees of the polynomials can be found by explicit calculations. For the one-matrix model (3.67) it is well-known that $\deg\Delta = \deg P - 1$. For the model (3.68), we will find in the next Section that the most general monomial in Δ is $\pi_1^a\pi_2^b$ with $0 \le a \le \deg P - 1 = d_X - 1$ and $0 \le b \le \deg Q - 1 = d_Y - 1$. The number of deformation parameters is thus $d_X d_Y$, consistently with (2.20). In the case of (3.69), three cases must be distinguished. If $d_G = \deg G = 0$, $\hat{\Delta}$ must be a constant and the allowed monomials in Δ are π_1^a for $0 \le a \le d-1$ and $\pi_1^a\pi_2$ for $0 \le a \le [(d-1)/2]$. If $d_G = d = 1$, $\hat{\Delta}$ is a constant and Δ is linear in π_2 and independent of π_1. In all the other cases, $\deg\hat{\Delta} = d_G - 1$ and Δ is a linear combination of terms $\pi_1^a\pi_2^b$ with $0 \le a \le d-1$ for $0 \le b \le d_G - 1$, $0 \le a \le d-2$ for $d_G \le b \le 2d_G - 1$, and $0 \le a \le [(d-1)/2] - 1$ for $b = 2d_G$. It is easily to check that the total number of parameters is given by $\max(d+2, 2dd_G) + [(d-1)/2]$, matching the number of irreducible representations of the algebra (3.62).

4 Solutions from the geometry

The aim of this Section is twofold. First we want to explain how to get geometrically the resolvents for various matrices. Second we provide full calculations for the models (2.18) and (3.42). The solution of this latter example was not known, but we will be able to check the results in a particular case in the next Section. Let us emphasize that any model of the form (1.1), at least for two matrices, should be in principle solvable using the same strategy.

4.1 The resolvents from the geometry

The idea is that, since any multi-matrix model can be formulated as a one-matrix model by integrating out all but one matrix, the Calabi-Yau geometry for a general model should have some common features with the geometry

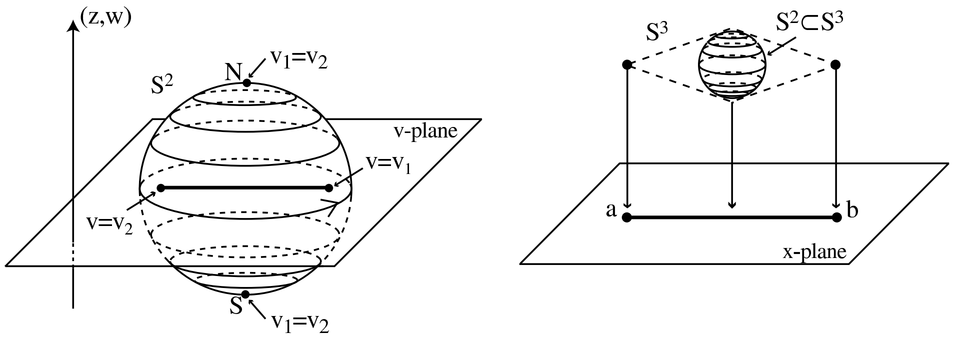

Figure 4: Integrating over S^3. We first integrate over $S^2 \subset S^3$ (left inset) and then over a meridian of S^3 (right inset). From the matrix model perspective, the meridian $[a, b]$ is either an eigenvalue-filled interval corresponding to a branch cut in Figure 1, or to a intervals of the type $[b_K, a_{K'}]$ or $]\mu_0, b_K]$.

(3.67) for the one-matrix model. The relevant property of the latter geometry is that it is a fibration of the deformation of the simplest ALE space $\mathbb{C}^2/\mathbb{Z}_2$,

$$u^2 = v^2 + w^2 + \lambda(z)\,. \tag{4.1}$$

The space $u^2 = v^2 + w^2 + \lambda$ has a single S^2 of holomorphic volume λ. When $\lambda(z) = 0$, the two-sphere shrinks. Classically this is equivalent to the equations of motion. The base coordinate z describes the fluctuations of S^2 and is associated with the matrix X, $z = x$ as in (3.59). We thus expect that the more general spaces we have to deal with are natural fibrations over bases parametrized by the privileged coordinates discussed in Section 3.3. The fiber F_x over a point x is in general much more complicated than a simple deformed $\mathbb{C}^2/\mathbb{Z}_2$ space. For example, the multi-valuedness of the effective potential implies that the fibers must contain several S^2s.

To make those ideas quantitative, we propose that integrating over the two-spheres in the fibers yields the discontinuity of the resolvent across a branch cut. Let us introduce the coordinate x associated with the matrix X, and let us denote by i_X the interior product associated with the vector field $\partial/\partial x$. If g^X is the resolvent for X on the physical sheet, and $\hat{g}^X$ is the analytic continuation of g^X through a branch cut, then

$$\int_{S^2 \subset F_x} \mathrm{i}_X \Omega = g^X(x) - \hat{g}^X(x)\,. \tag{4.2}$$

This is the fundamental equation that unambiguously determines g^X, as can be seen for example by taking x to be in the support of ρ^X and using (2.10) and (2.11).

We can now integrate over a three-sphere in the full three-fold geometry by first integrating over the S^2 in the fiber and then over a meridian $[a, b]$ of S^3 in the base, as depicted in Figure 4. Equation (4.2) implies that

$$\oint_{S^3} \Omega = \int_a^b \int_{S^2 \subset F_x} \Omega = \int_a^b \bigl(g^X(x) - \hat{g}^X(x)\bigr)\, \mathrm{d}x = \oint g^X(x)\, \mathrm{d}x\,, \qquad (4.3)$$

which links the special geometry relations in the Calabi-Yau (2.34) and matrix model (2.12), (2.17) forms.

As we have said, there are in general several $S^2 \subset F_x$ over which to integrate. The relevant S^2s can be easily identified by looking at the classical $S \to 0$ limit of (4.2). Equations (4.2) and (2.14) imply

$$\lim_{S\to 0} \int_{S^2 \subset F_x} \mathrm{i}_X \Omega = f_{\mathrm{b,\,cl}}(x)\,, \qquad (4.4)$$

where the background force was defined in (2.16) (for the $D \geq 2$ dimensional representations, one must take into account the terms discussed in Section 2.2.3 in the classical background force). Note that a rigorous consistency condition that follows from the second equation in (2.34) is that the non-compact integrals of $\lim_{S\to 0} \int_{S^2 \subset F_x} \mathrm{i}_X \Omega$ are related to the critical values of the potential W. We believe that the stronger condition (4.4) holds if and only if the ansatz (4.2) is correct. Equation (4.4) is very handy to fix the overall normalization of Ω. Let us also point out that the classical equations of motion, including the higher dimensional representations, are strictly equivalent to the condition $f_{\mathrm{b,\,cl}}(x) = 0$,

$$\operatorname{tr} \mathrm{d}W = 0 \Longleftrightarrow \lim_{S\to 0} \int_{S^2 \subset F_x} \mathrm{i}_X \Omega = 0\,. \qquad (4.5)$$

The basic requirement of renormalizability is equivalent to the fact that the quantum corrections to $\int_{S^2 \subset F_x} \mathrm{i}_X \Omega$ must vanish faster than $1/x$ when $x \to \infty$, except for terms linear in S that may go as $1/x$. As explained in Section 3.4, this must restrict the number of moduli of the deformed space to be equal to the number of irreducible representations of $\mathcal{A}$.

EXAMPLE: Let us consider a non-compact Calabi-Yau threefold in $\mathbb{C}^4$ defined by the equation

$$\mathcal{M}:\ u^2 = f(z,w)v^2 + g(z,w)v + h(z,w)\,, \qquad (4.6)$$

where f, g and h are polynomials. The nowhere vanishing holomorphic three-form is

$$\Omega = N\, \frac{\mathrm{d}v \wedge \mathrm{d}z \wedge \mathrm{d}w}{2i\pi u}\,, \qquad (4.7)$$

where N is a normalization constant. We assume that the privileged coordinate x is expressed in terms of the variables z and w only, so that

$$\mathrm{i}_X\Omega = -N\,\frac{\mathrm{d}v}{2i\pi u}\wedge \mathrm{i}_X(\mathrm{d}z\wedge\mathrm{d}w)\,. \tag{4.8}$$

As in Figure 4, the two-spheres are generated by closed contours encircling the branch cut $[v_1, v_2]$ in the v-plane. Equation (4.2) then yields

$$\begin{aligned} g^X(x)-\hat{g}^X(x) &= -\frac{N}{2i\pi}\int_{\mathrm{S}}^{\mathrm{N}}\oint\frac{\mathrm{d}v}{\sqrt{(v-v_1)(v-v_2)}}\wedge\frac{\mathrm{i}_X(\mathrm{d}z\wedge\mathrm{d}w)}{\sqrt{f(z,w)}} \\ &= -N\int_{\mathrm{S}}^{\mathrm{N}}\frac{\mathrm{i}_X(\mathrm{d}z\wedge\mathrm{d}w)}{\sqrt{f(z,w)}}\,. \end{aligned} \tag{4.9}$$

The North and South poles N and S lie on the algebraic curve $v_1 = v_2$,

$$\mathcal{C}:\ g^2(z,w)-4f(z,w)h(z,w)=0\,. \tag{4.10}$$

4.2 The two-matrix model with XY interaction

By introducing the privileged coordinates (3.61) $z=\pi_1=x$ and $w=\pi_2=y$ associated with the matrices X and Y respectively, the geometry (3.68) can be put in the form

$$\mathcal{M}:\ u^2=v^2+\mathcal{C}(x,y)=v^2+\bigl(P(x)-y\bigr)\bigl(Q(y)-x\bigr)+S\Delta(x,y)\,. \tag{4.11}$$

Equation (4.9) yields

$$g^X(x)-\hat{g}^X(x)=-N\int_{\mathrm{S}}^{\mathrm{N}}\mathrm{d}y\,. \tag{4.12}$$

Let us denote by $y=y_i(x)$ the d_Y+1 solutions to the equation

$$\mathcal{C}\bigl(x,y(x)\bigr)=0\,, \tag{4.13}$$

with labels chosen in such a way that, classically, $y_{1,\mathrm{cl}}(x) = P(x)$ and $y_{i,\mathrm{cl}}(x)=Q^{-1}(x)$ for $i\geq 2$. Consistency with (4.4) and (2.21) then requires that $N=\pm 1$ and

$$g^X(x)-\hat{g}^X(x)=y_i(x)-y_1(x)\,. \tag{4.14}$$

Let us note that the physical sheet branch cuts of the resolvent correspond to and only to $y_1=y_i$, $i\geq 2$. From (4.14) we thus deduce that g^X and $-y_1$

have exactly the same branch cuts and discontinuity across branch cuts, and thus must be equal up to some entire function. From (2.9) we finally get

$$g^X(x) = P(x) - y_1(x)\,. \tag{4.15}$$

The same analysis could be repeated to compute the resolvent for the matrix Y.

Let us now implement the constraint of renormalizability, which restricts the large x behaviour of $y_i(x) - y_1(x)$. These restrictions actually apply to the S-dependent part of $y_1(x)$ and $y_i(x)$ separately, because generically no cancellation can occur. A straightforward calculation then shows that the allowed monomials $x^a y^b$ in $\Delta(x,y)$ are such that

$$a + bd_X \leq d_X d_Y - 1\,, \quad a + b/d^Y \leq d_X + (d_Y - 1)/d_Y - 1\,. \tag{4.16}$$

This is equivalent to $a \leq d_X - 1$ and $b \leq d_Y - 1$, showing that there are exactly $d_X d_Y$ free parameters in Δ, matching the number of irreducible representations of $\mathcal{A}$.

Summary: If $\mathcal{C}(x,y) = \big(P(x) - y\big)\big(Q(y) - x\big) + S\Delta(x,y)$, with

$$\Delta(x,y) = \sum_{i=0}^{d_X-1}\sum_{j=0}^{d_Y-1} c_{ij} x^i y^j\,, \tag{4.17}$$

then the resolvents g^X and g^Y for the model (2.18) satisfy

$$\mathcal{C}\left(x, P(x) - g^X(x)\right) = \mathcal{C}\left(Q(y) - g^Y(y), y\right) = 0\,, \tag{4.18}$$

with the asymptotics (2.9). The $d_X d_Y$ parameters c_{ij} are determined by a choice of vacuum through the $d_X d_Y$ conditions (2.12).

The above result is in perfect agreement with the known solution from the loop equations [14], which yields

$$\Delta(x,y) = 1 - \Big\langle \frac{\mathrm{tr}}{n}\, \frac{P(x) - P(X)}{x - X}\,\frac{Q(y) - Q(Y)}{y - Y} \Big\rangle\,. \tag{4.19}$$

Note that in the loop equation approach the vacua are naturally characterized by a set of basic correlators, whereas in the geometric approach the natural parameters are the filling fractions (2.5). There is a one-to-one correspondence between those two sets of parameters.

4.3 The generalized Laufer's matrix model

We now solve the model (3.42), with $\deg V' = d$ and $\deg U = d_G$. When $U = 0$ and $V(x) = x^q$, the relevant Calabi-Yau (3.69) reduces to the geometry originally constructed by Laufer in [29]. With an obvious redefinition of coordinates, (3.69) can be cast in the form (4.6),

$$\begin{aligned}\mathcal{M}:\ u^2 = zv^2 &+ \Big[\big(w - F_1(z)\big)\, G(w) + S\hat{\Delta}(w)\Big] v \\ &+ \frac{1}{4} G^2(w) F_2^2(z) + w\Big[\big(w - F_1(z)\big)^2 - zF_2^2(z)\Big] + S\Delta(z,w)\,, \quad (4.20)\end{aligned}$$

and the algebraic curve (4.10) is given by the equation

$$\begin{aligned}\mathcal{C}(z,w) = \Big(zw - \frac{G^2(w)}{4}\Big)&\Big(\big(w - F_1(z)\big)^2 - zF_2^2(z)\Big) \\ + Sz\Delta(z,w) - \frac{1}{2} S\,\big(w - F_1(z)\big)\, &G(w)\hat{\Delta}(w) - \frac{1}{4} S^2 \hat{\Delta}^2(w) = 0\,. \quad (4.21)\end{aligned}$$

4.3.1 The resolvent for X

By using the privileged coordinates (3.65) $z = \pi_1 = x^2$ and (4.9) we get

$$g^X(x) - \hat{g}^X(x) = -2N\Delta w(x)\,, \quad (4.22)$$

where $\Delta w(x)$ denotes the difference between two roots of the equation

$$\mathcal{C}\left(x^2, w(x)\right) = 0\,. \quad (4.23)$$

There are $\max(3, 2 + 2d_G)$ roots, labeled such that classically

$$\begin{aligned}w_{1,\,\mathrm{cl}}(x) = -V'(x)\,, \quad w_{2,\,\mathrm{cl}}(x) &= -V'(-x)\,, \\ 4x^2 w_{i,\,\mathrm{cl}}(x) &= G^2\left(w_{i,\,\mathrm{cl}}(x)\right) \quad \text{for } i \geq 3\,. \quad (4.24)\end{aligned}$$

The classical background force for one dimensional representations is simply $f_{\mathrm{b,\,cl}}(x) = -y^2(x) - V'(x)$ with $4x^2 y^2(x) = G^2(y^2(x))$. Consistency with (4.4) and (4.5) then immediately requires that $N = \pm 1/2$ and

$$g^X(x) - \hat{g}^X(x) = w_1(x) - w_i(x)\,, \quad i \geq 2\,, \quad (4.25)$$

where the cases $i = 2$ and $i \geq 3$ correspond to two and one dimensional representations respectively. Finally the branch cut structure and asymptotics of the resolvent imply

$$g^X(x) = w_1(x) + V'(x)\,. \quad (4.26)$$

Let us now study the normalizability constraints that follow from (4.25). One must be careful that the term in S^2 in (4.21) does not produce any divergences, while terms in S may yield logarithmic divergences. The relevant constraints on the monomials $z^a w^b$ in $\Delta(z,w)$ read

$$\begin{aligned} 2a+bd &\leq \max(d+2, 2dd_G)+2[(d-1)/2]-2\,, && (4.27)\\ (2d_G-1)a+b &\leq (2d_G-1)d-d_G \text{ for } d_G\geq 1 \text{ and } d\geq 2 \text{ if } d_G=1\,, && (4.28)\\ a+b &\leq 1 \text{ for } d=d_G=1\,, && (4.29)\end{aligned}$$

and the relevant constraints on $\hat{\Delta}(w)$ read

$$\begin{aligned} 2d\deg\hat{\Delta} &\leq \max(d+2, 2dd_G)+2[(d-1)/2]-1\,, && (4.30)\\ d\deg\hat{\Delta} &\leq \max(d+2, 2dd_G)+2[(d-1)/2]-d(1+d_G)\,. && (4.31)\end{aligned}$$

The inequalities (4.27), (4.30) and (4.31) are obtained by studying the asymptotics of the roots w_1 or w_2, and (4.28) and (4.29) follow from looking at w_i for $i\geq 3$. It is straightforward to check that the general solution to this set of inequalities yields exactly the deformed geometry described at the end of Section 3.4.

SUMMARY: Let $\mathcal{C}$ be defined by (4.21), with the constraints on Δ and $\hat{\Delta}$ as in Section 3.4. The resolvent g^X for the model (3.42) satisfies the degree $\max(3, 2+2d_G)$ algebraic equation

$$\mathcal{C}\left(x^2, g^X(x)-V'(x)\right)=0\,, \tag{4.32}$$

with the asymptotics (2.9).

4.3.2 The resolvent for Y

Using (3.65), (4.9), and the normalization constant N deduced in the preceding subsection, we obtain

$$g^Y(y)-\hat{g}^Y(y)=2y\,\Delta x(y)\,, \tag{4.33}$$

where $\Delta x(y)$ denotes the difference between two roots of the equation

$$\mathcal{C}\left(x(y)^2, y^2\right)=0\,. \tag{4.34}$$

Let us separate the $2(d+1)$ roots into two sets $\{x_i\}$ and $\{-x_i\}$ characterized by

$$x_{1,\mathrm{cl}}(y)=-\frac{G(y^2)}{2y}\,,\quad V'\left(x_{i,\mathrm{cl}}(y)\right)+y^2=0 \text{ for } i\geq 2\,. \tag{4.35}$$

One and two dimensional representations correspond respectively to $x_1 = -x_i$ and $x_j = -x_i$ for $i, j \geq 2$. From the classical background force in one dimensional representations, $f_{\mathrm{b,\,cl}} = G(y^2) - 2yx(y)$ with $y^2 + V'(x(y)) = 0$, we deduce that

$$g^Y(y) - \hat{g}^Y(y) = -2y\left(x_i(y) + x_j(y)\right), \tag{4.36}$$

where $i \geq 2$ and $j \geq 1$. The resolvent g^Y is the only analytic function satisfying (4.36), with no other branch cuts than those associated with the representations of $\mathcal{A}$, and that goes as S/y at large y on the physical sheet. A naïve guess might have been to identify g^Y with one of the root x_i for $i \geq 2$, but this cannot work since for example there would be unphysical branch cuts corresponding to the permutation of the root x_i with another root x_j. It is easy to eliminate those latter branch cuts by taking a permutation invariant sum, normalized in such a way that (4.36) is satisfied,

$$g^Y(y) = -y\sum_{i=1}^{d+1} x_i(y) + \text{polynomial}. \tag{4.37}$$

The only possible remaining unwanted branch cuts in $\sum_i x_i$ would correspond to the permutation of x_1 with $-x_1$, but it is actually straighforward to show that there are no cuts permuting any root x_i with its opposite. Indeed, the associated $S = 0$ double points do not open up because the curve (4.21) factorizes at $z = 0$ for any S,

$$\mathcal{C}(0, w) = -\frac{1}{4}\Big[\big(w - F_1(0)\big)\, G(w) + S\hat{\Delta}(w)\Big]^2. \tag{4.38}$$

Finally, we fix the polynomial part in (4.37) by looking at the large y asymptotics.

Summary: Suppose that $V'(x) = \sum_{k=0}^{d} t_{k+1}x^k$. The resolvent g^Y for the model (3.42) is given by

$$\begin{aligned} g^Y(y) &= -y\sum_{i=1}^{d+1} x_i(y) - \frac{1}{2}G(y^2) - \frac{y t_d}{t_{d+1}} \quad \text{if } d \geq 2\,, && (4.39)\\ &= -y\left(x_1(y) + x_2(y)\right) - \frac{1}{2}G(y^2) - \frac{y(t_1 + y^2)}{t_2} \quad \text{if } d = 1\,, && (4.40) \end{aligned}$$

where the sum is taken over half of the roots of the equation (4.34) satisfying the conditions (4.35).

The resolvent g^Y, being a sum of algebraic functions, must itself be an algebraic function. It is not difficult to find the algebraic equation satisfied by g^Y. Let us introduce the symmetric polynomials $\sigma_0 = 1$ and $\sigma_k = \sum x_{i_1} \cdots x_{i_k}$ for $1 \leq k \leq d+1$. The resolvent is essentially $-y\sigma_1$. We can write

$$\begin{aligned} \mathcal{C}(x^2, y^2) &= (-1)^d t_{d+1}^2 y^2 \prod_{i=1}^{d+1} \big(x - x_i(y)\big)\big(x + x_i(y)\big) \\ &= (-1)^d t_{d+1}^2 y^2 \sum_{0 \leq k, k' \leq d+1} (-1)^k \sigma_k \sigma_{k'} x^{2d+2-k-k'} . \end{aligned} \tag{4.41}$$

Comparing with (4.21), we get $d+1$ quadratic equations for the $d+1$ unknown σ_k. By looking at the constant term and the classical limit (4.35), we find that $\sigma_{d+1} = (-1)^{d+1}\big(G(y^2)(t_1+y^2) + S\hat{\Delta}(y^2)\big)/(2yt_{d+1})$. There remains d quadratic equations for $\sigma_1, \ldots, \sigma_d$. Geometrically, this means that the resolvent lies at the intersection of d quadrics. By elimination of variables, we can find a degree 2^d algebraic equation satisfied by g^Y. For example, when $d=1$, we recover the hyperelliptic curve of the ordinary one-matrix model obtained by integrating out X.

5 Solution from the loop equations

The general class of models (1.1) ought to have some very special mathematical properties that make possible a solution in terms of an algebraic variety. In some sense, the loop equations should be integrable in the planar limit. A general analysis is far beyong the scope of the present paper, but we have been able to solve the example (2.23), which is a particular case of (3.42) for $G = -\alpha$. We find a precise match with the result of the previous Section, providing a very non-trivial test of the geometric approach, and, most importantly, of the underlying conjectures on which it is based. The test is particularly stringent because the model (2.23) is associated with a genuine $\mathcal{N}=1$ gauge theory that is not a deformation of an underlying $\mathcal{N}=2$ theory. In particular, the solution cannot be found by perturbing a Seiberg-Witten curve, unlike all the examples studied so far in the literature.

5.1 General loop equations

We consider the matrix integral

$$\int \mathrm{d}X\mathrm{d}Y e^{-\frac{n}{S}\operatorname{tr}\left(XY^2+\alpha Y+V(X)\right)} \tag{5.1}$$

for an arbitrary polynomial

$$V(X) = \sum_{k=0}^{d+1} t_k X^k / k\,. \tag{5.2}$$

The basic object we are going to compute is the generating function

$$\Gamma(x,y) = S\left\langle \frac{\mathrm{tr}}{n} \frac{1}{x-X}\frac{1}{y-Y}\right\rangle. \tag{5.3}$$

It can be expanded, either at large y or at large x,

$$\Gamma(x,y) = \sum_{k\geq 0} g_k^X(x) y^{-k-1} = \sum_{k\geq 0} g_k^Y(y) x^{-k-1}\,, \tag{5.4}$$

in terms of generalized resolvents

$$g_k^X(x) = S\left\langle \frac{\mathrm{tr}}{n}\frac{Y^k}{x-X}\right\rangle, \quad g_k^Y(y) = S\left\langle \frac{\mathrm{tr}}{n}\frac{X^k}{y-Y}\right\rangle. \tag{5.5}$$

The ordinary resolvents are $g^X = g_0^X$ and $g^Y = g_0^Y$.

Let us now consider the following variations in (5.1),

$$\delta X = 0\,, \quad \delta Y = \frac{\epsilon}{x-X}\,, \tag{5.6}$$

$$\delta X = \frac{\epsilon}{2}\Big(\frac{1}{x-X}\frac{1}{y-Y} + \frac{1}{y-Y}\frac{1}{x-X}\Big)\,, \quad \delta Y = 0\,, \tag{5.7}$$

$$\delta X = 0\,, \quad \delta Y = \frac{\epsilon}{2}\Big(\frac{1}{x-X}\frac{1}{y-Y}\frac{1}{-x-X} + \frac{1}{-x-X}\frac{1}{y-Y}\frac{1}{x-X}\Big)\,. \tag{5.8}$$

The remarkable property of these changes of variables is that, due to many cancellations of terms, the associated loop equations close in the planar limit under the correlators of the form $\langle \mathrm{tr}\, X^p Y^q\rangle$. After calculating the jacobian of the transformations, taking into account the variation of the classical potential, and using the factorization of multi-trace correlators at large N, we get

$$\begin{aligned}\big(g^X(x) - y^2 - V'(x)\big)\,\Gamma(x,y) = -\Big(y - \frac{\alpha}{2x}\Big)g^X(x) - S\,\frac{\langle \mathrm{tr}\, Y\rangle}{nx} \\ - S\left\langle \frac{\mathrm{tr}}{n}\frac{1}{y-Y}\frac{V'(x)-V'(X)}{x-X}\right\rangle,\end{aligned} \tag{5.9}$$

$$\begin{aligned}\Gamma(x,y)\Gamma(-x,y) = g^X(x) + g^X(-x) - y\,\big(\Gamma(x,y) + \Gamma(-x,y)\big) \\ - \frac{\alpha}{2x}\big(\Gamma(x,y) - \Gamma(-x,y)\big)\,.\end{aligned} \tag{5.10}$$

5.2 The resolvent for X

The functions g_k^X, and in particular the resolvent for X, are obtained by expanding the loop equations at large y. From (5.9) we get the following recursion relations,

$$\begin{aligned} g_1^X(x) &= S\frac{\langle \operatorname{tr} Y\rangle}{nx} - \frac{\alpha}{2x}\, g^X(x)\,, & (5.11)\\ g_{k+2}^X(x) &= \big(g^X(x) - V'(x)\big)\, g_k^X(x) + S\Delta_k(x)\,, & (5.12)\end{aligned}$$

where the $\Delta_k(x)$ are degree $d-1$ polynomials that can be computed from g_k^X and that are defined by

$$\Delta_k(x) = \Big\langle \frac{\operatorname{tr}}{n} Y^k \frac{V'(x) - V'(X)}{x - X}\Big\rangle\,. \tag{5.13}$$

The relations (5.11) and (5.12) determine all the g_k^Xs, and thus $\Gamma(x, y)$, from the single function $g^X(x)$. Moreover, from (5.10) we get

$$g_{q+2}^X(x) + g_{q+2}^X(-x) = -\frac{\alpha}{2x}\big(g_{q+1}^X(x) - g_{q+1}^X(-x)\big) - \sum_{k+k'=q} g_k^X(x) g_{k'}^X(-x)\,. \tag{5.14}$$

By using (5.11) and (5.12) in (5.14) we obtain, for each q, equations that close under $g^X(x)$ and $g^X(-x)$. Any two such independent equations then determine unambiguously $g^X(x)$. It turns out that the equation for $q = 1$ does not yield anything new, so we use $q = 0$ and $q = 2$. Eliminating $g^X(-x)$ we get, after a lengthy but straightforward calculation, a closed equation for $g^X(x)$ only. It is a cubic algebraic equation of the form

$$\mathcal{C}\left(x^2, g^X(x) - V'(x)\right) = 0\,, \tag{5.15}$$

where $\mathcal{C}$ is defined exactly as in (4.21) for $G = -\alpha$. The deformation polynomials are found explicitly in terms of correlators to be

$$\begin{aligned} \hat\Delta(w) &= 2\,\langle \frac{\operatorname{tr}}{n} Y\rangle\,, & (5.16)\\ \Delta(x^2, w) &= \Delta_2(x) + \Delta_2(-x) + xF_2(x^2)\big(\Delta_0(x) - \Delta_0(-x)\big) \\ &\quad - F_1(x^2)\big(\Delta_0(x) + \Delta_0(-x)\big) + \frac{\alpha}{2x}\big(\Delta_1(x) - \Delta_1(-x)\big) \\ &\quad + \big(\Delta_0(x) + \Delta_0(-x)\big)\, w\,. & (5.17)\end{aligned}$$

The form of Δ and $\hat\Delta$ is exactly as discussed at the end of Section 3.4: the term independent of w in Δ is a polynomial of degree $d-1$ in $z = x^2$, the term linear in w is a polynomial of degree $[(d-1)/2]$ in z, and $\hat\Delta$ is a constant. We have thus found a precise match with the result obtained from the geometric approach. Note that similar cubics were found from loop equations in [20] for models that are deformations of $\mathcal{N} = 2$ gauge theories.

5.3 The resolvent for Y

One way to compute the resolvent for Y is to expand the loop equations at large x. From (5.9) we get a set of linear equations

$$\sum_{k=0}^{d} t_{k+1} g^Y_{k+q}(y) = S\,\langle\frac{\mathrm{tr}}{n}Y\rangle\delta_{q,0} - \frac{\alpha S}{2}\,\langle\frac{\mathrm{tr}}{n}X^{q-1}\rangle\,\bigl(1-\delta_{q,0}\bigr) + Sy\,\langle\frac{\mathrm{tr}}{n}X^q\rangle \\ - y^2 g^Y(y) + S\sum_{k+k'=q-1}\langle\frac{\mathrm{tr}}{n}X^k\rangle\, g^Y_{k'}(y)\,, \quad (5.18)$$

and from (5.10) we get a set of quadratic equations

$$y g^Y_{2q-1}(y) = S\,\langle\frac{\mathrm{tr}}{n}X^{2q-1}\rangle - \frac{\alpha}{2}g^Y_{2q-2}(y) + \frac{1}{2}\sum_{k+k'=2q-2}(-1)^k g^Y_k(y) g^Y_{k'}(y)\,. \quad (5.19)$$

The $2d$ equations (5.18) for $0\leq q\leq d-1$ and (5.19) for $1\leq q\leq d$ close under the $2d$ unknown g^Y_k for $0\leq k\leq 2d-1$. We can for example express linearly the g^Y_k for $d\leq k\leq 2d-1$ in terms of the g^Y_k for $0\leq k\leq d-1$ by using (5.18), and then obtain a set of d quadratic equations for the d unknown $g^Y_0,\ldots,g^Y_{d-1}$ from (5.19). We thus discover that the resolvent for Y lies at the intersection of d quadrics, consistently with the result of Section 4.3.2. However, the direct calculation showing that the set of quadratic equations obtained from (5.18) and (5.19) on the one hand and (4.41) and (4.21) on the other hand are equivalent turns out to be extremely tedious.

We are thus going to provide a much simpler proof using the fact that we have already computed the resolvent for X in Section 5.2. Let us consider the equation for the unknown $x(y)$

$$g^X\bigl(x(y)\bigr) - V'(x) = y^2\,. \quad (5.20)$$

We know that $x(y)$ then automatically satisfies (4.34). Not all the solutions to (4.34), though, satisfies (5.20). This is best seen by plugging (5.20) in (5.9), which yields

$$\Bigl(yx(y)-\frac{\alpha}{2}\Bigr)\Bigl(y^2+V'\bigl(x(y)\bigr)\Bigr) + S\langle\frac{\mathrm{tr}}{n}Y\rangle \\ + Sx(y)\Bigl\langle\frac{\mathrm{tr}}{n}\frac{1}{y-Y}\frac{V'\bigl(x(y)\bigr)-V'(X)}{x(y)-X}\Bigr\rangle = 0\,. \quad (5.21)$$

This is a degree $d+1$ algebraic equation for $x(y)$, and thus only half of the roots of (4.34) can actually satisfy (5.20). By taking the classical limit $S\to 0$ of (5.21), we see that those $d+1$ roots are precisely the roots x_i that we have used in Section 4.3.2 and that were characterized by (4.35). By looking at the coefficient of x^{d+1} (for the overall normalization) and of x^d in (5.21), we then immediately get the sum $\sigma_1 = \sum_{i=1}^{d+1} x_i(y)$,

$$\begin{aligned} \sigma_1 &= -\frac{t_d}{t_{d+1}} + \frac{\alpha}{2y} - \frac{g^Y(y)}{y} \quad \text{if} \quad d \geq 2 && (5.22) \\ &= -\frac{t_1 + y^2}{t_2} + \frac{\alpha}{2y} - \frac{g^Y(y)}{y} \quad \text{if} \quad d = 1\,. && (5.23) \end{aligned}$$

Those equations are equivalent to (4.39) and (4.40). Moreover, we see that the higher symmetric polynomials σ_k, $2 \leq k \leq d$, are related to the generalized resolvents g_k^Y for $k \leq d-1$. Since the degree d is arbitrary, we conclude that the Calabi-Yau geometry actually encodes the full generating function Γ.

6 Discussion

The geometric approach to matrix models is singled out by its aesthetic features and its relationship with some of the deepest insights in gauge theory and string theory. It provides an entirely new perspective on the problem of summing planar diagrams, and suggests that a whole new class of models could be solved. The examples that we have studied show that, when appropriate, the geometric approach, which reduces to the calculation of a blow down map, is much more powerful than standard techniques. There remains, however, many open problems, some of which we review below.

MATRIX MODEL TECHNOLOGY: it is now clear that all the irreducible representations of the algebra $\mathcal{A}$ of equations of motion should be taken into account, even though most of the classic matrix model literature considers only one dimensional representations, often with the additional one-cut requirement. We have demonstrated in Section 2 that new qualitative features occur for higher dimensional representations. It would be desirable to develop the analytic techniques to deal with the saddle point equations in those cases, and to have a more general derivation of the property of eigenvalue entanglement. Another basic problem is to understand the general conditions under which special geometry relations can be derived. The geometric approach suggests that Casimir operators play a special rôle in this respect. Another interesting aspect is that the difficulty of a given model seems to be directly related to the complexity of the algebra $\mathcal{A}$ (the representation theory and the structure of the center). It is not clear how this translates in the language of loop equations. An interesting concrete problem would be understand what makes the algebra and the loop equations associated with the models (1.1) special.

ALGEBRAIC GEOMETRY: the infrared slavery of gauge theories suggests that all the Calabi-Yau geometries of the form (3.16), or even (3.6) when the corank of the Hessian of the potential W at critical points is at most two, can be blown down. It is not clear how this works mathematically. In particular, it would be desirable to understand the relationship with the results of [28] on Gorenstein threefold singularities. A naïve guess would have been that blow down can only be found for the cases constructed in [12], but clearly the models (1.1) are much more general, and already the model (3.42) that we have solved provides a counter-example. A startling fact is that *the calculation of the blow down map π is essentially equivalent to finding the solution of the associated matrix model.* It might be possible to devise an algorithm that computes π. Another interesting feature is the interplay between the non-commutative structure and the singularity structure of the blown down geometry, which must reproduce the representation theory of the model. In some sense the blow down map knows about D-branes.

GEOMETRIC APPROACH TECHNOLOGY: we have explained in Section 4 how to calculate the resolvents from the geometry. It is not clear to what extent this approach can be applied in general. The issue is to understand the fibered structure of the geometries, which is probably related to the structure of the center of $\mathcal{A}$. We would also like to understand what are the most general matrix model correlators encoded in the geometry, and also how to extract the non-planar contributions. The answer can probably be found in the topological string set-up [31].

D-BRANES ON CALABI-YAU: an interesting question is to ask what is the most general matrix model that can be engineered by putting branes in a Calabi-Yau. This is the "reverse geometric engineering" problem [32]. When there is no moduli space, which is the generic case we have been considering, we have very little insight into the solution of that problem. We have been able to construct in Section 3 a large class of models, but it might be possible to find additional theories by considering more general perturbations to the geometry (3.2). For example, one would like to know, given a $\mathbb{P}^1$ in a Calabi-Yau, what is the most general geometry for which the obstruction to the versal deformation space of the $\mathbb{P}^1$ is integrable in terms of a potential, taking into account the non-commutativity of the variables. The solution is probably most naturally expressed in terms of some "regularity" conditions on the algebra $\mathcal{A}$. Even if we could understand the reverse geometric engineering, there would remain the fundamental question of why and when the geometric transition conjecture is valid. The phenomenon it describes is very reminiscent, both physically and mathematically, to the "continuation to negative radius" found in two dimensional σ models (see for example [33]). In this latter case, we know that some non-geometrical phases are possible. This suggests that we may presently only catch a glimpse of the full story for what can happen beyond the singular (infinite gauge coupling) point. The matrix models provide in principle a powerful tool to study that question. A full understanding would amounts to describing the space of vacua in string theory.

$\mathcal{N}=1$ GAUGE THEORIES: we have used throughout the full non-perturbative Dijkgraaf-Vafa conjecture. This includes, consistently with special geometry, keeping the Veneziano-Yankielowicz term and the higher powers in the glueball superfields S_i, even when the latter are perturbatively zero in the chiral ring. There is no proof of this conjecture at the moment. When a direct solution of the matrix model exists, as described in Section 5, our result can be interpreted as providing a non-trivial check of the consistency betweeen the Dijkgraaf-Vafa conjecture and the geometric transition picture. The two-matrix models we have studied are particularly interesting because they are not deformation of $\mathcal{N}=2$ theories (which requires either one or three adjoint fields), and thus there is no Seiberg-Witten curve. The matrix model is then the only tool at our disposal. It would be very interesting to work out the quantum space of parameters for these theories, along the lines of [9, 34]. In particular, it is in principle possible to study quantitatively the phase transition between the Higgs and confining phases on parameter space [35].

As a final comment, we would like to note that we have put the emphasis on matrix models and thus on theories with only adjoint fields. However, all the questions we have addressed are also relevant to the more general set-up of quiver theories.

Acknowledgements

I would like to thank D. Berenstein, S. Katz, W. Lerche, C. Vafa and particularly R. Dijkgraaf for very useful discussions, as well as the organizers of the Amsterdam summer workshop "String Theory and Quantum Gravity," June 2003, where some of the results of this article were first presented. This work was supported in part by the Swiss National Science Foundation. The author is on leave of absence from Centre National de la Recherche Scientifique, Laboratoire de Physique Théorique de l'École Normale Supérieure, Paris, France.

A Saddle point equations and special geometry

1 Continuum limit of singular sums

Let us consider the sum

$$\sigma_n(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{x-x_i}\,\cdot \tag{A.1}$$

We assume that when $n\to\infty$, the distribution of eigenvalues $(x_i)_{1\leq i\leq n}$ goes to a smooth function ρ with compact support, and we want to compute

$$\sigma(x) = \lim_{n\to\infty}\sigma_n(x)\,. \tag{A.2}$$

If $x\notin \mathrm{Support}[\rho]$, we have

$$\sigma(x) = g(x)\,, \tag{A.3}$$

where the analytic function $g(x)$ is defined by

$$g(x) = \int_{-\infty}^{+\infty}\frac{\rho(z)\,\mathrm{d}z}{x-z}\,\cdot \tag{A.4}$$

If $x\in\mathrm{Support}[\rho]$, $g(x)$ is ambiguous because g has a branch cut. Let us label the eigenvalues in such a way that $x_1<x_2<\cdots<x_n$, and pick $x_j<x<x_{j+1}$ with $\delta_n^+ = x_{j+1}-x$ and $\delta_n^- = x-x_j$. Let us define

$$\frac{\delta^+(x)}{\delta^-(x)} = \lim_{n\to\infty}\frac{\delta_n^+}{\delta_n^-}\,\cdot \tag{A.5}$$

We can then compute

$$\begin{aligned}\sigma(x) &= \lim_{n\to\infty}\left(\int_{-\infty}^{x-\delta_n^-}+\int_{x+\delta_n^+}^{+\infty}\right)\frac{\rho(z)\,\mathrm{d}z}{x-z}\\ &= \frac{1}{2}\big(g(x+i\epsilon)+g(x-i\epsilon)\big)+\rho(x)\ln\frac{\delta^+(x)}{\delta^-(x)}\,\cdot\end{aligned} \tag{A.6}$$

One can use the above formula to deduce equation (2.33) from (2.30). The same reasoning also yields the standard result

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{j\neq i}\frac{1}{x_i-x_j}\right) = \frac{1}{2}\big(g(x_i+i\epsilon)+g(x_i-i\epsilon)\big) \tag{A.7}$$

because at large n we have in that case $\delta^+ = x_{i+1}-x_i = 1/(n\rho(x_i)) = x_i - x_{i-1} = \delta^-$.

2 Analysis of the saddle point equations

Let us discuss briefly the saddle point equation (2.33) supplemented with the condition (2.32). If the intervals I_K^X are filled with eigenvalues in one dimensional representations, we have

$$g^X(x+i\epsilon)+g^X(x-i\epsilon)+g^X(-x)=V'(x)+\frac{\alpha^2}{4x^2}\quad \text{for } x\in I_K^X\,. \tag{A.8}$$

Moreover, by taking the even and odd parts of (2.33) for two dimensional representations, and using (2.32) and (2.10), we get

$$\begin{aligned} S\rho^X(-x)\ln\frac{\delta^+(x)}{\delta_-(x)} &= \frac{\alpha^2}{4x^2}-F_1(x^2)-3xF_2(x^2) \\ &\qquad -\frac{3}{2}\left(g^X(x+i\epsilon)+g^X(x-i\epsilon)\right), \qquad (\mathrm{A}.9)\\ g^X(x+i\epsilon) &= g^X(-x+i\epsilon)-2xF_2(x^2)\,, \quad \text{for } x\in \tilde{I}_K^X\cup\tilde{I}_K'^X\,. \qquad (\mathrm{A}.10)\end{aligned}$$

Let us denote $g^X_{\text{first}}(x)=g^X(x)$ on the physical sheet. The equation (A.8) shows that the cuts I_K^X glue the physical sheet with a second sheet where the solution is

$$g^X_{\text{second}}(x)=-g^X_{\text{first}}(x)-g^X_{\text{first}}(-x)+V'(x)+\frac{\alpha^2}{4x^2}\,\cdot \tag{A.11}$$

Due to the term $g^X_{\text{first}}(-x)$, this second sheet contains new cuts $I_K'^X$ that are the image with respect to $x=0$ of the physical cuts I_K^X. Using again (A.8), we see that the new cuts glue the second sheet with a third sheet with

$$\begin{aligned} g^X_{\text{third}}(x) &= -g^X_{\text{first}}(x) \\ &\quad -\left(-g^X_{\text{first}}(-x)-g^X_{\text{first}}(x)+V'(-x)+\frac{\alpha^2}{4x^2}\right)+V'(x)+\frac{\alpha^2}{4x^2} \\ &= g^X_{\text{first}}(-x)-2xF_2(x^2)\,. \qquad (\mathrm{A}.12)\end{aligned}$$

Equation (A.10) then shows that this third sheet is glued to the first sheet by the cuts $\tilde{I}_K^X$ and $\tilde{I}_K'^X$. Overall g^X has generically a three-sheeted structure, except when only two dimensional representations are present in which case the second sheet is absent. The general form of the cubic equation satisfied by g^X is derived both in Section 4 and in Section 5. A special form of this cubic in the one-cut assumption and for $\alpha=0$ was found in [19]. Let us also note that models where a similar cubic appears have been studied recently in the literature [20].

3 The effective potential for Y

We have argued in Section 2.2 that the singularity structure found for the background force (2.29) was generic. We want to illustrate this fact by studying the effective potential for Y in the model (2.23), defined by

$$e^{-\frac{n^2}{S}V_{\mathrm{eff}}(Y)} = e^{-\frac{n\alpha}{S}\operatorname{tr}Y}\int \mathrm{d}X e^{-\frac{n}{S}\operatorname{tr}(XY^2+V(X))}\,. \tag{A.13}$$

We know that V_{eff} must have singularities when $\deg V \geq 2$ to make the associated saddle point equations consistent with two dimensional representations. This is a little bit puzzling at first, because the effective potential for Y is essentially the same as the effective potential for the ordinary two-matrix model discussed in Section 2.2.2, except that it is a function of the square of the matrix instead of the matrix itself. A simple way to understand what is going on is to use the Itzykson-Zuber formula [5] to derive

$$e^{-\frac{n^2}{S}V_{\mathrm{eff}}(Y)} = \frac{e^{-\frac{n\alpha}{S}\sum_i y_i}}{\prod_{i<j}\big((y_i-y_j)(y_i+y_j)\big)} \int\prod_i \mathrm{d}x_i\prod_{i<j}(x_i-x_j)\, e^{-\frac{n}{S}\sum_i(x_iy_i^2+V(x_i))}. \tag{A.14}$$

This formula shows that the only singularities may be at $y_i=y_j$ or $y_i=-y_j$. To find which singularities actually occur, we can evaluate (A.14) in the limit $S\to 0$. This yields

$$e^{-\frac{n^2}{S}V_{\mathrm{eff}}(Y)} \underset{S\to 0}{\propto} \frac{\prod_{i<j}\Big(\chi(y_i^2)-\chi(y_j^2)\Big)}{\prod_{i<j}\big((y_i-y_j)(y_i+y_j)\big)}\,, \tag{A.15}$$

where the function $\chi(y^2)=V'^{-1}(-y^2)$ minimizes $\chi y^2+V(\chi)$. When $y_i\to y_j$, the pole in the denominator of (A.15) is cancelled by a zero of the denominator, consistently with the fact that no singularities are expected at $y_i=y_j$. On the other hand, because χ is d-sheeted, if we analytically continue y_i from $y_i=y_j$ to $y_i=-y_j$ we can go to a different sheet $\chi(y_i^2)\to\tilde\chi(y_i^2)$ and thus produce a pole in (A.15). We read from (A.14) that the relevant singular part in $V_{\mathrm{eff}}(Y)$ is exactly the same as that for $V_{\mathrm{eff}}(X)$ in (2.28). In particular the eigenvalues will be entangled for two dimensional representations, implying that

$$\rho^Y(y)=\rho^Y(-y)\quad\text{for } y\in\tilde I_K^Y\,. \tag{A.16}$$

4 Derivation of special geometry relations

An elegant way to derive special geometry relations is to set up a variational formulation of the saddle point equations. Let us first deal with the description in terms of the matrix X. One must be careful because

$$n\frac{\partial V_{\rm eff}}{\partial x_i}\Big|_{x_i=x} \neq \frac{\mathrm{d}}{\mathrm{d}x}\frac{\delta V_{\rm eff}(\rho^X)}{\delta\rho^X(x)} \tag{A.17}$$

due to the logarithmic term in (A.6). Let us consider the functional

$$\begin{aligned}\mathcal{F} = &-SV_{\rm eff}(\rho^X) + S^2\int \mathrm{d}x\mathrm{d}\tilde{x}\,\rho^X(x)\rho^X(\tilde{x})\ln|x-\tilde{x}|\\ &+\sum_{K=1}^{d+2}\ell_K\Big(\int_{I_K^X} S\rho^X(x)\,\mathrm{d}x - S_K\Big) + \sum_{K=1}^{[(d-1)/2]}\tilde{\ell}_K\Big(\int_{\tilde{I}_K^X} S\rho^X(x)\,\mathrm{d}x - \tilde{S}_K\Big)\\ &+\sum_{K=1}^{[(d-1)/2]}\int_{\tilde{I}_K^X} S\left(\rho^X(x)-\rho^X(-x)\right)L_K(x)\,\mathrm{d}x\,.\end{aligned} \tag{A.18}$$

By varying $\mathcal{F}$ with respect to the Lagrange multipliers ℓ_K and $\tilde{\ell}_K$ we obtain

$$\int_{I_K^X} S\rho^X\mathrm{d}x = \frac{1}{2i\pi}\oint_{\alpha_K^X} g^X\mathrm{d}x = S_K\,,\quad \int_{\tilde{I}_K^X} S\rho^X\mathrm{d}x = \frac{1}{2i\pi}\oint_{\tilde{\alpha}_K^X} g^X\mathrm{d}x = \tilde{S}_K\,. \tag{A.19}$$

The notation for the contours is similar to that of Figure 1. The equations (A.19) implement the relations (2.5), with $\tilde{S}_K$ associated with two dimensional representations. By varying $\mathcal{F}$ with respect to the Lagrange multipliers $L_K(x)$, we get the constraint (2.32). Finally, by varying $\mathcal{F}$ with respect to $\rho^X(x)$ we get the saddle point equations

$$0 = -\frac{\delta V_{\rm eff}}{\delta\rho^X(x)} + 2S\int \mathrm{d}\tilde{x}\,\rho^X(\tilde{x})\ln|x-\tilde{x}| + \ell_K \quad \text{for } x\in I_K^X\,, \tag{A.20}$$

$$0 = -\frac{\delta V_{\rm eff}}{\delta\rho^X(x)} + L_K(x) + 2S\int \mathrm{d}\tilde{x}\,\rho^X(\tilde{x})\ln|x-\tilde{x}| + \tilde{\ell}_K \quad \text{for } x\in \tilde{I}_K^X, \tag{A.21}$$

$$0 = -\frac{\delta V_{\rm eff}}{\delta\rho^X(x)} - L_K(-x) + 2S\int \mathrm{d}\tilde{x}\,\rho^X(\tilde{x})\ln|x-\tilde{x}| \quad \text{for } x\in \tilde{I}_K^{\prime X}\,. \tag{A.22}$$

One can check that the derivatives of the above equations are equivalent to (A.8), (A.9) and (A.10), with the identification

$$L_K'(x) = S\rho^X(-x)\ln\frac{\delta^+(x)}{\delta^-(x)} \quad \text{for } x\in\tilde{I}_K^X\,. \tag{A.23}$$

Moreover, the partial derivatives with respect to S_K and $\tilde{S}_K$ take very simple forms,

$$\frac{\partial\mathcal{F}}{\partial S_K} = -\ell_K\,,\quad \frac{\partial\mathcal{F}}{\partial\tilde{S}_K} = -\tilde{\ell}_K\,. \tag{A.24}$$

Using (2.28) and (A.20), it is straightforward to compute

$$\begin{aligned}\oint_{\gamma_K^X} g^X(x)\,\mathrm{d}x &= \int_{\mu_0}^{a_K}\Bigl(2g^X(x) - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\delta V_{\text{eff}}}{\delta\rho^X(x)}\Bigr)\\ &= -\ell_K + \frac{\delta V_{\text{eff}}}{\delta\rho^X(\mu_0)} - 2S\ln\mu_0 + \mathcal{O}(1/\mu_0)\,,\end{aligned}\tag{A.25}$$

which yields explicitly

$$\frac{\partial\mathcal{F}}{\partial S_K} = \lim_{\mu_0\to\infty}\Bigl(\oint_{\gamma_K^X} g^X\mathrm{d}x - V(\mu_0) + S\ln\mu_0\Bigr).\tag{A.26}$$

Similarly, using (A.21), (A.22) and (2.32) we get

$$\oint_{\tilde\gamma_K^X} g^X(x)\,\mathrm{d}x = -\tilde\ell_K + \frac{\delta V_{\text{eff}}}{\delta\rho^X(\mu_0)} + \frac{\delta V_{\text{eff}}}{\delta\rho^X(-\mu_0)} - 4S\ln\mu_0 + \mathcal{O}(1/\mu_0)\,,\tag{A.27}$$

yielding

$$\frac{\partial\mathcal{F}}{\partial\tilde S_K} = \lim_{\mu_0\to\infty}\Bigl(\oint_{\tilde\gamma_K^X} g^X\mathrm{d}x - V(\mu_0) - V(-\mu_0) + 2S\ln\mu_0\Bigr).\tag{A.28}$$

Using the results of Section A.3, we can repeat the discussion above by replacing the matrix X by the matrix Y. There is an additional K-independent term

$$\int\frac{\delta V_{\text{eff}}(\rho^Y)}{\delta\rho^Y(y)}\,\rho^Y(y)\,\mathrm{d}y - \frac{\partial}{\partial S}\bigl(SV_{\text{eff}}\bigr)\tag{A.29}$$

in (A.24) because $V_{\text{eff}}(\rho^Y)$ is not linear in ρ^Y. This term does not affect the relations involving compact cycles,

$$\frac{\partial\mathcal{F}}{\partial S_K} - \frac{\partial\mathcal{F}}{\partial S_{K'}} = \oint_{\beta^Y_{K,K'}} g^Y\mathrm{d}y\,,\quad \frac{\partial\mathcal{F}}{\partial\tilde S_K} - \frac{\partial\mathcal{F}}{\partial\tilde S_{K'}} = \oint_{\tilde\beta^Y_{K,K'}} g^Y\mathrm{d}y\,.\tag{A.30}$$

B RG flow theorem for normal bundles

In this Appendix, we provide a proof of the conjecture in Section 3.2.1 in the special case $n = 1$ relevant to two-matrix models. The proof is a direct consequence of two lemmas.

LEMMA 1: Let $\mathbb{P}^1$ in the geometry (3.16) at a given critical point (x, y) of the superpotential (3.14). Then the transition function characterizing the normal bundle of the $\mathbb{P}^1$ can be cast in the form

$$T(z) = \begin{pmatrix} z^{-1} & 0\\ \partial_y^2 W + z\partial_x\partial_y W + z^2\partial_x^2 W & z^3\end{pmatrix}.\tag{B.1}$$

PROOF: To calculate the transition function, we expand $w_1 = w_1(z) + \delta_1$, $w_2 = w_2(z) + \delta_2$, $w_1' = w_1'(z) + \delta_1'$ and $w_2' = w_2'(z) + \delta_2'$ in (3.16), keeping only the linear terms in the δs. By redefining

$$\begin{aligned} \delta_2 &\to \delta_2 - \delta_1 \sum_{i\geq 3} z^{i-3} \sum_{j\geq 0} E_{i-j}^{(1+j)}(x) y^j/j!\,, \\ \delta_2' &\to \delta_2' + \delta_1' \sum_{i>0} z'^{i-1} \sum_{j\geq 0} E_{-i-j}^{(1+j)}(x) y^j/j!\,, \end{aligned} \tag{B.2}$$

we obtain

$$\begin{pmatrix} \delta_1' \\ \delta_2' \end{pmatrix} = T(z) \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}. \tag{B.3}$$

LEMMA 2: Let $\mathcal{E}$ be a holomorphic vector bundle over $\mathbb{P}^1$ with structure group $\mathrm{GL}(2,\mathbb{C})$ and transition function

$$T(z) = \begin{pmatrix} z^{-1} & 0 \\ a + bz + cz^2 & z^3 \end{pmatrix}. \tag{B.4}$$

Let r be the corank of the quadratic form $Q = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$. Then $\mathcal{E} = \mathcal{O}(r-1) \oplus \mathcal{O}(-r-1)$.

PROOF: We want to construct gauge transformations $P_{\mathrm{N}}(z)$ and $P_{\mathrm{S}}(z' = 1/z)$ such that

$$P_{\mathrm{S}} T P_{\mathrm{N}}^{-1} = \begin{pmatrix} z^{1-r} & 0 \\ 0 & z^{1+r} \end{pmatrix}. \tag{B.5}$$

The important property is that $P_{\mathrm{N}}(z)$, $P_{\mathrm{N}}^{-1}(z)$, $P_{\mathrm{S}}(z')$ and $P_{\mathrm{S}}^{-1}(z')$ must be holomorphic functions. It is not difficult to check that the following matrices have the required properties: for $ac - b^2 \neq 0$ and $b \neq 0$,

$$P_{\mathrm{N}} = \begin{pmatrix} ac/b - b - cz & (a/b - z)z \\ c^2 & cz - b \end{pmatrix}, \quad P_{\mathrm{S}} = \begin{pmatrix} -a^2/b & az'/b - 1 \\ b^2 - ac + abz' & (c - bz')z' \end{pmatrix}, \tag{B.6}$$

for $b = 0$, $a \neq 0$ and $c \neq 0$,

$$\begin{aligned} P_{\mathrm{N}} &= \begin{pmatrix} -c(1+z)/a & 1/c - z(z+1)/a \\ c & z \end{pmatrix}, \\ P_{\mathrm{S}} &= \begin{pmatrix} 1 - az'/c & z'^2/c - (z'+1)/a \\ -a & z' \end{pmatrix}, \end{aligned} \tag{B.7}$$

for $a = \alpha^2$, $b = \alpha\beta$, $c = \beta^2$, $\alpha \neq 0$ and $\beta \neq 0$,

$$P_{\mathrm{N}} = \begin{pmatrix} \beta^2 & z \\ -\beta^3/\alpha & 1 - \beta z/\alpha \end{pmatrix}, \quad P_{\mathrm{S}} = \begin{pmatrix} -\alpha^2 z' - \alpha\beta & z'^2 \\ -\alpha^2 & z' - \beta/\alpha \end{pmatrix}, \tag{B.8}$$

for $a = b = 0$ and $c \neq 0$,

$$P_{\mathrm{N}} = \begin{pmatrix} 0 & -1/c \\ c & z \end{pmatrix}, \ P_{\mathrm{S}} = \begin{pmatrix} 1 & -z'^3/c \\ 0 & 1 \end{pmatrix}, \tag{B.9}$$

for $b = c = 0$ and $a \neq 0$,

$$P_{\mathrm{N}} = \begin{pmatrix} a & z^3 \\ 0 & -1/a \end{pmatrix}, \ P_{\mathrm{S}} = \begin{pmatrix} 0 & 1 \\ 1 & -z'/a \end{pmatrix}, \tag{B.10}$$

and finally for $a = b = c = 0$, $P_{\mathrm{N}} = P_{\mathrm{S}} = I$.

References

[1] G. 't Hooft, *Nucl. Phys.* **B 72** (1974) 461.

[2] V.A. Kazakov, *Solvable Matrix Models,* hep-th/0003064.

[3] É. Brézin, C. Itzykson, G. Parisi and J.-B. Zuber, *Comm. Math. Phys.* **59** (1978) 35.

[4] M.L. Mehta, *Random Matrices,* Academic Press (New-York and London) 1967.

[5] Harish-Chandra, *Amer. J. Math.* **79** (1957) 87,
C. Itzykson and J.-B. Zuber, *J. Math. Phys.* **21** (1980) 411.

[6] P. Di Francesco and C. Itzykson, *Ann. Inst. Henri Poincaré* **59** (1993) 117.

[7] V.A. Kazakov, *Mod. Phys. Lett.* **A 4** (1989) 2125.

[8] P. Di Francesco, P. Ginsparg and J. Zinn-Justin, *Phys. Rep.* **254** (1995) 1.

[9] F. Ferrari, *Phys. Rev.* **D 67** (2003) 85013.

[10] R. Gopakumar and C. Vafa, *Adv. Theor. Math. Phys.* **3** (1999) 1415.

[11] F. Cachazo, K. Intriligator and C. Vafa, *Nucl. Phys.* **B 603** (2001) 3.

[12] F. Cachazo, S. Katz and C. Vafa, *Geometric Transitions and $\mathcal{N} = 1$ Quiver Theories,* HUTP-01-A038, ILL-TH-01-7, OSU-M-2001-4, hep-th/0108120.

[13] R. Dijkgraaf and C. Vafa, *A Perturbative Window into Non-Perturbative Physics,* HUTP-02/A034, ITFA-2002-34, hep-th/0208048.

[14] J. Alfaro, *Phys. Rev.* **D 47** (1993) 4714,
M. Staudacher, *Phys. Lett.* **B 305** (1993) 332,
B. Eynard, *J. High Energy Phys.* **01** (2003) 051.

[15] R. Dijkgraaf, S. Gukov, V.A. Kazakov and C. Vafa, *Phys. Rev.* **D 68** (2003) 45007.

[16] G. 't Hooft, *Comm. Math. Phys.* **86** (1982) 449,
G. 't Hooft, *Comm. Math. Phys.* **88** (1983) 1.

[17] M. Kontsevich, *Comm. Math. Phys.* **147** (1992) 1.

[18] Y. Makeenko and G.W. Semenoff, *Mod. Phys. Lett.* **A 6** (1991) 3455,
D.J. Gross and M.J. Newman, *Phys. Lett.* **B 266** (1991) 291.

[19] B. Eynard and J. Zinn-Justin, *Nucl. Phys.* **B 386** (1992) 558,
B. Eynard and C. Kristjansen, *Nucl. Phys.* **B 455** (1995) 577,
B. Eynard and C. Kristjansen, *Nucl. Phys.* **B 466** (1996) 463.

[20] A. Klemm, K. Landsteiner, C.I. Lazaroiu and I. Runkel, *J. High Energy Phys.* **05** (2003) 066,
S.G. Naculich, H.J. Schnitzer and N. Wyllard, *J. High Energy Phys.* **08** (2003) 021.

[21] N. Dorey, T.J. Hollowood, S.P. Kumar and A. Sinkovics, *J. High Energy Phys.* **11** (2002) 039, *J. High Energy Phys.* **11** (2002) 040.

[22] T.R. Taylor and C. Vafa, *Phys. Lett.* **B 474** (2000) 130,
P. Mayr, *Nucl. Phys.* **B 593** (2001) 99.

[23] F. Ferrari, *Nucl. Phys.* **B 648** (2002) 161.

[24] R. Dijkgraaf, M.T. Grisaru, C.S. Lam, C. Vafa and D. Zanon, *Phys. Lett.* **B 573** (2003) 138,
F. Cachazo, M.R. Douglas, N. Seiberg and E. Witten, *J. High Energy Phys.* **12** (2002) 071.

[25] W. Lerche, P. Mayr and N. Warner, *Holomorphic $\mathcal{N} = 1$ special geometry of open-closed type II strings,* CERN-TH-2002-174, hep-th/0207259,
W. Lerche, P. Mayr and N. Warner, *$\mathcal{N} = 1$ Special Geometry, Mixed Hodge Variations and Toric Geometry,* CERN-TH/2002-175, hep-th/0208039.

[26] M. Namba, *Tôhoku Math. J.* **24** (1972) 581.

[27] S. Katz, *Versal deformations and superpotentials for rational curves in smooth threefolds,* math.AG/0010289.

[28] S. Katz and D. Morrison, *J. Algebraic Geometry* **1** (1992) 449.

[29] H.B. Laufer, *Ann. of Math. Stud.* **100** (1981) 261.

[30] A. Grothendieck, *Amer. J. Math.* **79** (1957) 121.

[31] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Comm. Math. Phys.* **165** (1994) 311,
A. Klemm, M. Mariño and S. Theisen, *J. High Energy Phys.* **03** (2003) 051.

[32] D. Berenstein, *J. High Energy Phys.* **04** (2002) 052.

[33] S. Cecotti and C. Vafa, *Phys. Rev. Lett.* **68** (1992) 903.

[34] F. Cachazo, N. Seiberg and E. Witten, *J. High Energy Phys.* **02** (2003) 042,
F. Ferrari, *Phys. Lett.* **B 557** (2003) 290.

[35] E.H. Fradkin and S.H. Shenker, *Phys. Rev.* **D 19** (1979) 3682.