ASIAN J. MATH.
Vol. 18, No. 3, pp. 439–464, July 2014

ON CERTAIN GENERALIZED HARDY'S INEQUALITIES AND APPLICATIONS*

DEMETRIOS A. PLIAKIS†

Key words. Hardy's inequality, normal crossings, resolvent expansion, singular asymptotics, local growth estimates.

AMS subject classifications. 35B45, 35J10.

Introduction. The classical inequality of Hardy for smooth functions $f \in C_0^\infty(f \in \mathbf{R} \setminus \{0\})$:

$$\int_{\mathbf{R}} x^s f^2 \leq \frac{4}{(s+1)^2} \int_{\mathbf{R}} x^{s+2} (f')^2$$

for $s \neq -1$ can be generalized in various ways and provides a weighted version of Poincaré's inequality. The standard generalizations replace the weight x^s with the radial variable or the boundary defining function of a smooth domain, are reduced to the one-dimensional case and are proved directly by partial integration, as far as the weight stays smooth. Here we we replace the weight by a homogeneous polynomial that is singular also away from the origin, so its zero set is a singular algebraic cone. In this case no direct method of the preceding form is available: the rectilinearization of such a set being non-trivial along its singularities. Specifically, singular algebraic varities are rectilinearized under the process of "resolution of singularities" then, their singularities unfold and appear as "normal crossings". We follow this procedure to the extent of " reduction of multiplicity " of an algebraic set and prove following generalization.

Let $P(x_1, \ldots, x_n)$ be a homogeneous polynomial of degree d in n-real variables belonging to the class $\mathcal{P}^{gH}$ that we define in the next paragraph. Let $V(P) = \{x \in \mathbf{R}^n / P(x) = 0\}$ be the algebraic set that it defines. We introduce the Hardy factors:

$$\mathcal{H}^1(P) = P^{-\frac{2}{d}}, \quad \mathcal{H}^2(P) = \left|\frac{\nabla P}{P}\right|^2.$$

We prove the following generalized Hardy inequalities GHI_i:

$$\int_{\mathbf{R}^n} \mathcal{H}^i(P) f^2 \leq C_i(P) \int_{\mathbf{R}^n} |\nabla f|^2$$

for functions $f \in C_0^\infty(\mathbf{R}^n \setminus V(P))$. This inequality while it is elementary to prove when the algebraic variety $V(P)$ is smooth away from the origin, it is rather cumbersome when the variety is singular. The above inequality may be viewed as direct generalization of Hardy's Here, we will consider the stratification of the algebraic variety $V(P)$ by multiplicity and the inequality will be examined through the resolution of singularities process. This provides a finite covering, in every chart of which the algebraic set is reduced to normal crossings. The inequality is readily reduced to a corresponding one for inhomogeneous polynomials.

*Received November 29, 2010; accepted for publication March 6, 2013.
†TEI CRETE, Dept Electronics, Chania, Crete, Greece (dpliakis@gmail.gr).

THEOREM 0.1. *Let P be a nonhomogeneous polynomial of degree d. Then there is a constant $c_1'(P) > 0, i = 1, 2$ such that*

$$\int_{\mathbf{R}^n} \mathcal{H}^1(P) f^2 \leq c_1'(P) \sum_{i=1}^{h_1} ||\nabla^i f||_2^2.$$

If $f \in \mathcal{P}^{gH}, \mathcal{P}^{gH}$ being a class of polynomials that we define in the sequel, then we have the following:

$$\int_{\mathbf{R}^n} \mathcal{H}^2(P) f^2 \leq c_1'(P) \sum_{i=1}^{h_2} ||\nabla^i f||_2^2.$$

However, it is worthnoticing that we can refine the crude form of the preceding inequality for an inhomogeneous polynomial of degree d, belonging in the class described below; precisely there is a constant $c_3(P) > 0$ such that for functions $f \in C_0^\infty(\mathbf{R}^n \setminus V(P))$ there holds:

$$\int_{\mathbf{R}^n} \mathcal{H}^1(P) f^2 \leq c_3(P) \int_{\mathbf{R}^n} |\nabla f|^2 + (1 + |x|^2) f^2.$$

We present two applications of this inequality:

- We apply the inequality to a particular case of a problem that motivated the study of these inequalities: the existence of an asymptotic expansion in powers and logarithms of the distributional trace of the heat operator corresponding to
$$H_{c,\alpha} = -\Delta + \frac{c}{|P|^\alpha}$$
for $c \in C_0^\infty(\mathbf{R}^n)$ and small $\alpha > 0$. The inequalities provide the required estimates for the domain, closure and the Neumann series of a suitable power of the resolvent of $H_{c,\alpha}$. The existence of the asymptotic expansion follows, in view of the singular asymptotics lemma [C1], from the well known theorem of Bernshtein-Gelfand, [BG], for the meromorphic extension of integrals containing complex powers of polynomials.
- We consider a smooth domain that approaches arbitrarily close a non-smooth one: it is defined by the level sets of a polynomial. In this domain we compare the growth of functions to the growth of the polynomial defining it. This allows us to compare values of functions in the following manner: in a smooth metric in a Euclidean domain represent the domains of given curvature growth by the semialgebraic sets defined by specific polynomials. Then we obtain estimates of the local growth of laplacian eignefunctions in terms of the curvature growth. Here we prove the simple inequality that provides the growth of integrals of functions is such semilagebraic domains.

The article begins with a review of the local reduction of an algebraic set to normal crossings according to [BM1] with certain comments that adjust the process with our purposes. In the sequel we prove the preceding theorem and its further generalizations and then we proceed to the applications.

1. Local reduction of an algebraic set to normal crossings. We commence by reviewing the necessary definitions and results on blowing up and local desingularization of an algebraic set with guide essentially the presentation in [BM1].

1.1. Standard constructions.

1.1.1. Blown up space. In the various steps we will use conical partitions of unity covering the euclidean ball centered at the origin

$$B_{0,\epsilon,n} = \{x \in \mathbf{R}^n / |x|^2 := x_1^2 + \cdots x_n^2 < \epsilon\}$$

subordinate to the cones

$$C_{\alpha;j} = \{x \in \mathbf{R}^n / x_j^2 > \frac{1}{1+\alpha}|x|^2\}$$

for $\alpha > 1$. Let $\mathbf{P}^n$ denote the n-dimensional projective space of lines through the origin in $\mathbf{R}^{n+1}$. Let $B^*_{0,\epsilon,n+1} = B_{0,\epsilon,n+1} \setminus \{0\}$ be the punctured euclidean ball around the origin in $\mathbf{R}^{n+1}$, set

$$\widehat{B}_{0,\epsilon,n+1} = \overline{\{(x,l) \in B^*_{0,\epsilon,n+1} \times \mathbf{P}^n : x \in l\}}$$

and let $\sigma : \widehat{B}_{0,\epsilon,n+1} \to B_{0,\epsilon,n+1}$ $(x,l) \mapsto x$. Then σ is proper, restricts to a homeomorphism over $B^*_{0,\epsilon,n+1}$, and $\sigma^{-1}(0) = \mathbf{P}^n$. This mapping is called the blowing up of $B_{0,\epsilon,n}$ with center $\{0\}$. In a natural way, $B^*_{0,\epsilon,n+1}$ is an algebraic submanifold of $B_{0,\epsilon,n} \times \mathbf{P}^n$:

Coordinates. Let $(x_1, \ldots, x_{n+1})$ denote the affine coordinates in $\mathbf{R}^{n+1}$ and let $t = [t_1 : \ldots : t_{n+1}]$ denote the homogeneous coordinates of $\mathbf{R}^n$. Then

$$\widehat{B}_{0,\epsilon,n+1} = \{(x,t) \in B_{0,\epsilon,n+1} \times \mathbf{P}^n : x \wedge t = 0\}.$$

Furthermore $\widehat{B}_{0,\epsilon,n}$ is covered for $\alpha > 1$ by the conical charts $j = 1, \ldots, n+1$:

$$\widehat{C}_{\alpha:j} = \{(x,t) \in \widehat{B}_{0,\epsilon,n+1} : t_i^2 > \frac{1}{1+\alpha}|t|^2\},$$

with coordinates $(x_{1,i}, \ldots, x_{n+1,i})$, for each i, where

$$x_{ii} = x_i, \qquad x_{ji} = \frac{t_j}{t_i}, i \neq j$$

with respect to these local coordinates, σ is given by

$$x_i = x_{ii} \qquad x_j = x_{ii}x_{ji}, \;\; i \neq j.$$

Let $n > c$ and $B^{n-c}_\epsilon(0)$ then the mapping $\sigma \times id : \widehat{B}_{0,\epsilon,c} \times B_{0,\epsilon,c} \to B^c_\epsilon(0) \times B^{n-c}_\epsilon(0)$ is called the blowing up of $B^c_\epsilon(0) \times B^{n-c}_\epsilon(0)$ with center $C := \{0\} \times B^{n-c}_\epsilon(0) \subset \mathbf{R}^n$ and it is denoted by $\mathrm{Bl}_C(B^c_\epsilon(0) \times B^{n-c}_\epsilon(0)) = \widehat{B}_{0,\epsilon,c} \times B^{n-c}_\epsilon$.

1.1.2. Blown up volumes and vector fields. Let the usual volume in $\mathbf{R}^n$ be denoted by v_n, then under blow up with center of codimension $c > 1$ considered in the i-th chart it pulls back the volume

$$v_n = x_i^{c-1}\widehat{v_n}.$$

It is noteworthy the way that the vector fields that generate dilation transform under blowing up or down. Let then $x = \sigma(y)$ and

$$D_{x_i} = x_i \frac{\partial}{\partial x_i}, \;\; D_{y_i} = y_i \frac{\partial}{\partial y_i},$$

$$E_0 = \sum_i x_i \frac{\partial}{\partial x_i}, \quad E_1 = \sum_i y_i \frac{\partial}{\partial y_i},$$

then in the k-th chart we have the formulas:

$$\begin{aligned} i \neq k: \;\; D_{x_i} &= D_{y_i}, \\ D_{k_1} &= E_0, \\ E_1 &= 2E_0 - D_{x_k}. \end{aligned}$$

Notice theat the Euler vector field E is expressed in the radial variable $r = |x|$ as

$$E = r\partial_r = D_r.$$

We will consider mappings obtained as a finite sequence of local blow-ups ; i.e., $\pi_N = \sigma_1 \circ ... \circ \sigma_N$, where for each $i = 1, ..., r, \sigma_i : \widehat{V}_i \to V_i$ is a local blow up with center $C_i = \{0\} \times B_{0,\epsilon,n-c}$ of the preceding form with $\widehat{V}_i = \widehat{B}_{0,\epsilon,c_i} \times B_{0,\epsilon,n-c_i}$ and $V_i = B_{0,\epsilon,c_i} \times B_{0,\epsilon,n-c_i}$.

The conical atlas partition of unity. We conclude with the partition of unity of the punctured ball $B^*_{0,\epsilon,n+1}$ subordinate to its conical covering. Let $\varphi \in C_0^\infty(\mathbf{R}_+)$ with $\mathrm{supp}(\varphi) \subset [0, 1+\varepsilon), \phi \equiv 1$ in $[0, 1]$ then set

$$\chi_j(x) = \varphi\left(\frac{r}{\sqrt{1+\alpha}|x_j|}\right).$$

We compute its derivatives:

$$|\nabla^\ell \chi_j| \leq \frac{1}{r^\ell} \sum_{i_1+\cdots+i_\ell=\ell} |\varphi^{i_1}| \cdots |\varphi^{i_l}| \leq \frac{C_\ell}{r^\ell}.$$

This formula is very important because in the end of the blow-up process we encounter the derivatives of the localizations functions. These will require the further application of the one dimensional Hardy inequality, i.e. in the radial variable.

1.2. The local desingularization algorithm. Here we 'll follow the proof of the local desingularization theorem in algorithm devised in [BM] and developed in the conventions that we need for the inequalities. It consists of two steps the determination of the center and the reduction of the multiplicity.

THEOREM 1.1. *Let $P : \mathbf{R}^n \to \mathbf{R}$ be a regular function. Then there is a countable collection of regular mappings $\pi_r : W_r \to \mathbf{R}^n$ such that:*

1. *Each π_r is the composition of a finite sequence of local blows up (with smooth centers)*
2. *There is a locally finite covering U_r of $\mathbf{R}^n$ such that $\pi_r(W_r) \subset U_r$ for all r.*
3. *If K is a compact subset of $\mathbf{R}^n$, there are compact subsets $M_r \subset W_r$ such that $K = \bigcup_r \pi_r(M_r)$. The union is finite by (2).*
4. *For each $r, P \circ \pi_r$ is locally normal crossings.*

Determination of the center. Let $a \in V(P), mboxord_a(P) =: m$ and choose coordinates $x = (x_1, \ldots, x_n)$ such that $x(a) = 0$ and, $\mathrm{in}_a(P)(0, ..., 0, x_n) \neq 0$ the lowest degree homogeneous component of the polynomial. Moreover denote by $\tilde{x} = (x_1, ..., x_{n-1})$ and let $h_j = V(x_j)$ be the coordinate hyperplanes. Since $(\partial^m_{x_n} P)(x) \neq 0$ then

$$(\partial^{m-1}_{x_n} P)(x) \sim x_n - H(\tilde{x}) =: x'_n,$$

for some regular function H. Then we perform the division

$$P(x) = Q(x)x_n^m + \sum_{0\le k<m} c_k(\tilde{x})x_n^k.$$

Moreover, possibly after translation, we may assume that $c_{m-1}(\tilde{x}) = 0$ and also observe that each $c_k(\tilde{x}) = \partial_{x_n}^k|_{h_n}P, \tilde{x}_n \sim \partial_{x_n}^{m-1}P$.

Let $a \in h_n$ then for $0 \le k < \mathrm{ord}_a P = \mu_P(a)$ introduce the sets

$$\mathcal{P}(a) := \{(P, \mathrm{ord}_a P)\},\ \ \mathcal{C}_P(a) := \{(c_k, \mu_P - k)\}.$$

The union of stata of multiplicity at least $\mu_P(a)(= m)$ is denoted by

$$S_{\mathcal{P}(a)} := \{x : \mathrm{ord}_x P \ge \mu_P(a)\}$$

as well as that

$$S_{\mathcal{C}_P(a)} := \{x : \mathrm{ord}_x g \ge p,\ \ \text{forall}\ \ g \in \mathcal{C}_P(\alpha)\}.$$

First we will use an induction on n (and on m), to arrive at the particular instance when for all k

$$(*) \qquad c_k(\tilde{x}) = (\tilde{x}^\gamma)^{m-k} c_k^*(\tilde{x})$$

while $\gamma \in \frac{1}{m!}\mathbf{N}^{n-1}$ and for some $k_0, c_{k_0}^*(0) \ne 0$. In order to handle at once the various c_k's we define the auxiliary function

$$A_P(\tilde{x}) :=\ \text{product of all non zero}\ \ c_k^{\frac{m!}{m-k}}\ \ \text{and all their nonzero differences.}$$

The inductive assumption asserts that there is already a uniformization for A_P :

$$A_P \sim x_1^{a_1} \ldots x_{n-1}^{a_{n-1}}.$$

This implies in first place that each nonzero $c_k(\tilde{x}) = (\tilde{x}^{\Omega_k})c_k^*(\tilde{x})$ with $\Omega_k \in \mathbf{N}^{n-1}$ and $c_k(0)^* \ne 0$. Moreover each nonzero $c_k^{\frac{m!}{m-k}} - c_j^{\frac{m!}{m-j}} \sim \tilde{x}^{\Lambda_{ij}}$, with $\Lambda_{ij} \in (\mathbf{N}^{n-1})^*$. The following elementary lemma suggests:

Lemma 1.2. *Let $x = (x_1, \ldots, x_n)$. If $a(x)x^\alpha - b(x)x^\beta = c(x)x^\gamma$ and $a(0)b(0)c(0) \ne 0$ then either $\alpha \in \beta + \mathbf{N}^n$ or $\beta \in \alpha + \mathbf{N}^n$*

The lemma implies that the set $\mathcal{E} := \{\frac{1}{m-k}\Omega_k\} \subset \frac{1}{m!}\mathbf{N}^{n-1}$ is totally ordered with the induced partial ordering from $\mathbf{N}^{n-1}$ and therefore there exist a $\rho = \min(\mathcal{E}) \in \frac{1}{m!}\mathbf{N}^{n-1}$. Therefore we are reduced to the case (*) with $\gamma = \rho \in \frac{1}{m!}\mathbf{N}^{n-1}$. We show that the special case (*) implies reduction in multiplicity by blowing up succesively the components of $S_{\mathcal{P}}(a)$.

$$S_{\mathcal{P}(a)} = S_{\mathcal{C}_P(a)} = \{x : x_n = 0, \quad \mathrm{ord}_x(\tilde{x}^\gamma) \ge 1\} = \bigcup_I Z_I$$

where $Z_I = \bigcap_{i\in I}(h_i \cap h_n)$ and $I \subset \{1, \ldots, n-1\}, \mathrm{card}(I) = \nu_m - 1$ minimal such that $\sum_{i\in I} \gamma_i \ge 1$ or equivalently that $0 \le \sum_{i\in I} \gamma_i - 1 < \gamma_k$ for all $k \in I$. Actually, these serve as centers of the desingularizing blowups $C = Z_I$, for all k we have that

$$\mathrm{ord}_C c_k \ge m - k > 0,$$

where $\mathrm{ord}_C c_k = \inf_{x\in C}\big(\mathrm{ord}_x c_k\big)$.

Reduction of the multiplicity. Let $a \in V(P)$ and choose coordinates such that $x(a) = 0, I = (1, \ldots, \nu_m - 1)$. Let further

$$C_{\alpha;\epsilon;j,n,\nu_m} = (C_{\alpha;j,n} \cap B_{0,\epsilon,\nu_m}) \times B_{0,\epsilon,n-\nu_m}$$
$$\widehat{C}_{\alpha;\epsilon;n,\nu_m} = \left(\widehat{C}_{\alpha;\epsilon;j,n,\nu_m} \cap B_{0,\epsilon,\nu_m}\right) \times B_{0,\epsilon,n-\nu_m}$$

and the blowup

$$\sigma : \widehat{B}_{0,\epsilon,n-\nu_m} \times B_{0,\epsilon,n-\nu_m} \to \widehat{B}_{0,\epsilon,\nu_m} \times B_{0,\epsilon,n-\nu_m}$$

at $\beta \in \sigma^{-1}(0)$. We calculate in these conical charts:

- Let $\beta \in \widehat{C}_{\alpha;\epsilon;n,n,\nu_m}$ then the strict transform $\sigma|C_n$ maps as follows

  $$x_n = y_n, \ \ x_j = y_j y_n, j = 1, \ldots, \nu_m - 1, \ \ x_s = y_s, n > s > \nu_m.$$

  Its effect on the polynomial is

  $$(\sigma^*(\chi_n P))(y) = y_n^m P_n^1(y) = y_n^m[(\sigma^* Q)(y) + \sum_{0 \le k < m} (\sigma^* c_k)(y) y_n^{k-m}]$$

  and we observe that $Q(\sigma(\beta)) \neq 0$ while $\left(\sum_{0 \le k < m} c_k(y) y_n^{k-m}\right)(\sigma(\beta)) = 0$ and hence $P_n^1(\beta) \neq 0$.
- In the conical sector $\widehat{C}_{\alpha;\epsilon;j,n,\nu_m}$, $j \in I$ the strict transform σ maps as follows

  $$x_n = y_n, \ \ y_j, x_j = y_j, \quad x_k = y_k y_j, j \neq k = 1, \ldots, c - 1, \quad x_s = y_s, s > n.$$

  Then the polynomial becomes

  $$(\sigma^*(\chi_j P))(y) = y_j^m P_j^1(y) = y_j^m\big((\sigma^* Q(y)) y_n^m + \sum_{0 \le k < m} (\sigma^* c_k)(\tilde{y}) y_j^{k-m} y_n^k\big)$$

  since in this sector

  $$\tilde{x} = \sigma(\tilde{y}), \tilde{y} = (y_1, ..., y_{n-1}).$$

  Observe that $\partial_{y_n}^{m-1} P_j^1 \sim y_n$ since $Q(\alpha) \neq 0$ while $h_n' = \sigma^{-1}(h_n) = \{y_n = 0\}$ and $c_{m-1} = 0$ identically.

We conclude that for all points on the exceptional divisor the order is not bigger than m, $\beta \in \sigma^{-1}(C), \mathrm{ord}_\beta P^1 \le d$ if P^1 denotes the resulting regular function. Therefore assume that $\mathrm{ord}_\beta P^1 = m$ iff $\beta \in h_n \bigcap \{\mathrm{ord}_y C_k' \ge m - k\}$ where

$$c_k' = y_j^{k-m} (\sigma^* c_k)(\tilde{y}) = (\tilde{y}^{\gamma'})^{m-k}.(c_k^*(\sigma(\tilde{y}))$$

and $\tilde{y}^{\gamma'} := y_j^{-1}.(\tilde{x}^\gamma \circ \sigma)$ and there is $k_0, (c_{k_0}^*)(\sigma(\beta)) \neq 0$ by the particular case (*). Hence, $\gamma_i' = \gamma_i$, if $i \neq j$ and $\gamma_j' = \sum_{i \in I} \gamma_i - 1$ therefore $1 \le |\gamma'| < |\gamma|$ and since $|\gamma|, |\gamma'| \in \frac{1}{m!}\mathbf{N}^{n-1}$ it follows that after no more than $|\gamma| m!$ blows up of this type multiplicity has to decrease.

Remarks.

- Let $P \in \mathbf{R}[x_1, \ldots, x_n]$ and $V(P) := V$ the variety it defines. Set then

  $$P^k(x) = \sum_{j_1 + \cdots + j_n = k} \big(\frac{\partial^k P}{\partial x_1^{j_1} \ldots \partial x_n^{j_n}}(x)\big)^2$$

and consequently $V_k := V(P_k)$ The stratification by multiplicity we 'll use consists of the strata that are semialgebraic sets:

$$\Sigma_k = (V_k \setminus V_{k+1}) \cap V.$$

If P is homogeneous then by Euler's theorem $\Sigma_{k+1} \subset \overline{\Sigma_k}$ while the implicit function theorem asserts that Σ_k are smooth.

- Let $I(V)$ be the sheaf of germs of regular functions on $\mathbf{R}^n$ that vanish on V. Let $x \in \Sigma_k$ then set

$$L_{x,k} = \{\xi \in \mathbf{R}^n : P \in I_x(V), \sum_j \xi_j \frac{\partial P}{\partial x_j} \in I_x(V)\}$$

and also that $E_{x,k} = L_{x,k}^{\perp}$.

- The class $\mathcal{P}^{gH}$ of polynomials consists of those P such that when $codim\Sigma_k = 2$ for some k then

$$\{(x + E_{x,k}) \cap V_x\} \setminus \{x\} \neq \emptyset$$

where V_x is the germ of V at x.

- If $P \geq 0$ everywhere as well as that $P_1, \ldots, P_p \geq 0$ everywhere and it is true that

$$P = \sum_j P_j$$

then $V(P)$ is not a hypersurface. Then the same procedure with more complicated details brings the set to normal crossings, cf [BM]. However, if we assume that each of the P_j's belongs to the class $\mathcal{H}$ then we can proceed without appealing to the desingularization for codimension> 1.

- We would assume that P is irreducible otherwise split it in its factors and use Young's inequality to deal with its factor separately.

The inequalities of Lojasiewicz. Through the paper certain distances from algebraic sets are estimated by the values of the defining polynomials through the fundamental Lojasiewicz inequalities, cf [BM]:

Theorem 1.3. *Let P be a regular function on an open subspace $M \subset \mathbf{R}^n$. Suppose that K is a compact subset of M, on which $V(|\nabla P|^2) \subset V(P)$. Then there exist $c, c' > 0$ and $\mu, 0 < \mu \leq 1, \nu > 1$ such that*

$$|\nabla P(x)| \geq c|f(x)|^{1-\mu}, \quad |P(x)| \geq c' d(x, V(P))^{\nu}$$

in a neighborhood of $K, \sup(\mu), \sup(\nu) \in \mathbf{Q}$.

It is true that $\nu(K) \leq \sup_{a\in K} \mu_a(P)$. Furthermore in the case of a homogeneous polynomial function in $\mathbf{R}^n$ the constant depends on conical neghborhoods of the origin.

In the case of a homogeneous polynomial P of degree m then the algebraic cone decomposes as $V(P) = \mathbf{R}_+ \times K$ where K is the trace of the cone on the sphere. Then Lojasiewicz inequality suggsets that for $C(K) > 0, \mu = \frac{1}{m}$:

$$|\nabla P| \geq C_m |P|^{1-\mu}.$$

This follows from the application of Lojasiewicz on the trace, under stereographic projection in coordinates $(r,\xi)\in\mathbf{R}_+\times\mathbf{R}^{n-1}$, and for applying Young's inequality for $p,q>1,\ pq=p+q$,:

$$|\nabla P(x)|\geq mr^{m-1}|\tilde{P}(\xi)|+r^m|\widehat{\nabla P}(\xi)|\geq$$
$$\geq mr^{m-1}|P|+Cr^m|P|^{1-\mu}\geq C'_m r^{\frac{1}{\mu p}-\frac{m}{q}}|P(x)|^{1-\frac{p}{\mu}}.$$

We choose as $p=\frac{m+\mu}{m}$ scale and arrive at the result $\mu=\frac{1}{m}$.

2. The inequalities. Here we will prove the inequalities $\mathrm{GHI}_1,\mathrm{GHI}_2$ for an inhomogeneous polynomial P of degree d in the class $\mathcal{H}$ and an arbitrary function $f\in C_0^\infty(\mathbf{R}^n\setminus V(P))$:

$$I^i[P](f)=\int_{\mathbf{R}^n}\mathcal{H}^i(P)f^2\leq C\int_{\mathbf{R}^n}\sum_{j=0}^{h_i}|\nabla^j f|^2=||f||_{H^{h_i}(\mathbf{R})},\qquad (\mathrm{GHI}_i)$$

where

$$\mathcal{H}^1(P)=P^{-\frac{2}{d}},\ \ \mathcal{H}^2(P)=\left|\frac{\nabla P}{P}\right|^2.$$

This inequality will be based on the fact that after a suitable number of blow-ups with suitably chosen centers the multiplicity of the polynomial has to decrease. The choice of centers is provided in the proof of the local desingularization theorem.

To fix the ideas assume that we are localised in a tubular neighbourhood of $V(P)$ of width $\epsilon^{\frac{1}{d}}$:

$$N_\epsilon(P)=\{x\in\mathbf{R}^n/|P(x)|<\epsilon\}$$

and also the tubes of width $\epsilon_k^{\frac{1}{d}}$ that enclose the strata of multiplicity Σ_k:

$$N_{\epsilon_k,k}(P)=\{x\in\mathbf{R}^n/Q_k(x)=\sum_{j=0}^{k}P^j(x)<\epsilon_k^2\}.$$

We assume that $\epsilon_1,\dots,\epsilon_m,\epsilon$ are chosen so that this system of tubes $N_1,\dots,N_m$ exhaust N_ϵ and using functions of the form $\chi\left(\frac{Q_k}{\epsilon_k}\right)$, for χ a one-dimensional cut-offs we localize in these sets. Thanks to Lojasiewicz these cut-off functions when differentiated stay away from the variety. The integral then is splitted up as

$$I^i[P](f)=\sum_{j=0}^{m}I^i_j[P](f)$$

where

$$I^i_j[P](f)=\int_{N_j}\mathcal{H}^i(P)f^2.$$

As a matter of fact we have that

$$\mathrm{supp}(f)=\bigcup_{j=0}^{m}(\mathrm{supp}(f)\cap N_j).$$

We are ready to set-up the multiplicity reducing algorithm. Select then points

$$a \in V(P) \cap N_{m,\epsilon}(P): \ \ \mathrm{ord}_a P = m, \ \ \mathrm{codim}(S_{P(a)}) = \nu_m$$

and choose a system of coordinates such that $x(a) = 0$ and also that

$$P(0, \dots, 0, x_n) \sim x_n^m.$$

The initial change of coordnates that rectifies the center we will denote by $K_m : N_m \to \mathbf{R}^{n-\nu_m} \times \mathbf{R}^{\nu_m}$, which consists of algebraic maps localized through φ_{m,ℓ_m} at such points on $a \in S_m$:

$$K_m = \sum_{\ell_m=1}^{N_m} \varphi_{m;\ell_m} K_{m;\ell_m}.$$

These maps have jacobians $0 < c_m(\epsilon_m) \leq J(K_{m;\ell_m}) \leq C_m(\epsilon_m)$

2.1. The inequality GHI_1 for $\mathcal{H}^1 = P^{-\frac{2}{d}}$.

The inequality in the multiplicity reduction step $m \Rightarrow m-1$. Using the conical partition of unity $\{\chi_{m,k}\}$ subordinate to the covering in

$$\bigcup_{k=1}^{\nu_m} K_m(N_{\epsilon,m}(P)) \cap \mathbf{R}^{n-\nu_m} \times C_k$$

we localize and compute that

(1)
$$I_m(P)[f] = \int_{N_{\epsilon_m,m}} \mathcal{H}^1(P) f^2(x) v_n \leq c \sum_{k=1}^{\nu_m} \int_{N_{m;1,k}} \left| D_k(|P_{1,m,k}|^{-\frac{1}{d}} f_{m;1,k}) \right|^2 \left(\mathcal{D}_k^1\right)^2 \hat{v}_n$$

where

$$c = \kappa_m(\nu_m, d) = \left(\frac{16d^2}{d(\nu_m+1)+m)}\right)^2$$
$$N_{m;1,k} = \sigma^{-1}(N_{m;0,k}), \ N_{m,0,k} = K_m(N_{\epsilon_m}) \cap \mathbf{R}^{n-\nu_m} \times C_k$$
$$D_k = x_k \partial_{x_k}, \ \mathcal{D}_k^1 = x_k^{-\frac{m}{d}+\frac{\nu_m-1}{2}}$$
$$f_{m,1,k} = \chi_{m,k} f.$$

The last integral is majorized further by:

$$\int_{N^1_{\epsilon_m,m}} \mathcal{H}^1(P_{m;1,k}) \left[|D_k f_k^1|^2 + \left(Q^m_{1,k,1}(P) f_{m;1,k}\right)^2\right] \left(\mathcal{D}_k^1\right)^2 \hat{v}_n.$$

The singular term is made of the following factors that we are going to keep track in the process:

$$Q^m_{1,k,1}(P) = D_k(\log|P_k^{m,1}|),$$
$$\mathcal{H}^1_{m;1,k} = (P_{m;1,k})^{-\frac{2}{d}}, \ \ P_{m;1,k} = x_k^{-m} \sigma^*(\chi_k P).$$

We perform successive blowups of this type in order to reduce the multiplicity of the algebraic set and every time apply the one dimensional inequality. Therefore, we introduce inductively for the blow up years $j > 1$ and corresponding blow - up chart k_j the following functions that can come out:

$$\begin{aligned} N_{m;j} &= \sigma^{-1}(N_{m;j-1}), \quad N_{m;0} = N_{\epsilon,m} \\ f_{m;0,k,0} = \chi_{m,k} f, \quad f_{m;j,k_j,l} &= \sigma^*(\chi_{k_j} f_{m;j-1,k_{j-1},l}), \\ f_{m;j,k_j,l} &= D_{k_j}\left(f_{m;j,k_{j-1},l-1}\right) \\ \mathcal{H}^1_{m;j,k_j} &= x_{m,k_j}^{\frac{2m}{d}} \sigma^*(\chi_{m,k_j} \mathcal{H}^1_{m;j-1,k_{j-1}}), \\ \mathcal{D}_{m;j,k_j,l} &= D_{k_j}(\mathcal{D}_{m;j,k_{j-1},l-1}), \\ \mathcal{D}_{m;j,k_j,l} &= \sigma^*(\chi_{m,k_j} \mathcal{D}_{m;j-1,k_{j-1},l}) \cdot \mathcal{D}^1_{k_j}, \\ Q^m_{j,k_j,l} &= D_{k_j}(Q^m_{j,k_j,l-1}) \\ Q^m_{j,k_j,l} &= \sigma^*(\chi_{m,k_j} Q^m_{j-1,k_{j-1},l}). \end{aligned}$$

The integral is then majorised by the sum after γ_m generations of blow-ups that are neccessary for the multiplicity to reduce to $m-1$:

$$I_m(P)[f] \leq C(\epsilon_m, m, d, \nu_m) \int_{N_{\gamma_m}} \sum_{k_{\gamma_m}=1}^{c} \mathcal{H}^1_{m;\gamma_m,k_{\gamma_m}} \Phi^2_{m,1;k_{\gamma_m}}$$

for the functions that encode the history of blow ups

$$\left(\Phi_{m,1;k_{\gamma_m}}\right)^2 = \chi^2_{k_{\gamma_m}} \left[\sum_{l_1+l_2+l_3=\gamma_m} \left(f_{m;\gamma_m,k_{\gamma_m},l_1} \mathcal{D}_{m;\gamma_m,k_{\gamma_m},l_2} Q^m_{\gamma_m,k_{\gamma_m},l_3} \right)^2 \right].$$

This sum extends over the set of all ν_m-adic numbers with γ_m digits: $\Lambda_{\gamma_m}(c)$. After these blow ups the polynomial $P_{m;\gamma_m,k_{\gamma_m}}$ has multiplicity $m-1$. In the next γ_{m-1} generations of blow-ups:

$$I_m(P)[f] \leq c \sum_{k_{\gamma_m}=1}^{\nu_m} I_{m-1}[P_{m;\gamma_m,k_{\gamma_m}}](\Phi_{m,1;k_{\gamma_m}}).$$

Therefore we have to work with

$$I_m[P](f) + I_{m-1}(P)[f] \leq c \int_{N_{m;\gamma_m}} \mathcal{H}_1(P_{m;\gamma,k_{\gamma_m}}) \Phi^2_{m,1;k_{\gamma_m}} + \int_{N_{\epsilon_{m-1},m-1}} \mathcal{H}_1(P) f^2$$

and proceed analogously. The desingularization algorithm guarantees that on the set $N_{m;\gamma_m} \cap N_{\epsilon_{m-1},m-1}$ the polynomials $P_{m;\gamma_m,k_{\gamma_m}}, P$ have multiplicity $m-1$. Therefore the choice of blow up centers entails the change of coordinates K_{m-1} that we trace in the summands. Their appearance modifies the constant c. Thus we have m generations of $\gamma_1, \ldots, \gamma_m$ years of blow-ups.

Summing up for $m = 1$ and final step. We arrive at the $m = 1$ stratum with the following sum of integrals:

$$I_1(P)[f] = \sum_{1 \leq i_2 \leq i_1 \leq m} \int_{N_{i_1,i_2}} \mathcal{H}^1(P_{i_1;i_2,k_{i_2}}) \Phi^2_{i_1,i_2;k_{i_2}}$$

where the pair of indices stands keeps track of the origin of the function $\Phi_{i_1,i_2;k_{i_2}}$ while the polynomials $P_{i_1;i_2,k_{i_2}}$ have multiplicity 1. Hence we change coordinates by the map K_1 and conclue with an applcation of Hardy's inequality.

Return to N_ϵ. In order to return back to N_ϵ. We summarize the process that we followed:

Rectilinearization of the centers, blow up till we reduce the multiplicity by 1, again rectilinearization of the new centers and new blow ups etc.

This finally made up a map: $\mathcal{B} : N_\epsilon \to \tilde{N}_\epsilon$, which is a piece-wise algebraic diffeomorphism outside a variety of positive codimension originating from the exceptional divisors of the blow ups. Clearly the process can be reversed and the formula derived above. We will perturb back the centers and blow down. The rectilinearization maps just modify the constants back. The terms that have been produced will be examined then. We examine now the Φ_{i_1,i_2} terms: the Q factors are constants hence they just modify the consnstant coefficients. The $\mathcal{D}$-terms split in two terms: those that consist of the jacobians and the "Hardy divisor factor" $y_k^{-2\frac{i_1}{d}}$. In the course of the blow down process the jacobian terms dissapear while in the conical charts of the $\mathbf{R}^{\nu_m}$ factor of the coordinate system we have that for the radial variable r:

$$y_k^{-2\frac{i_1}{d}} \leq c r^{-2\frac{i_1}{d}},$$

which combine with the blow down formula to give us that

$$|D_{y_k}(f\sigma)|^2 = |E(f)|^2 \leq r^2 |\nabla f|^2.$$

If $r^{-\beta}$ persists then we apply again the usual Hardy inequality. The blowing down process will effect the replacement of the $|D\,(f\mathcal{D}) \circ \sigma)|$ with a term $|\nabla f|$ and finally we get the result stated in the introduction

2.2. The inequality GHI$_2$ for $\mathcal{H}^2(P) = \left|\frac{\nabla P}{P}\right|^2$. Here we separate at each year of blow up the "Hardy divisor term", x_k^{-2} and blow down directly, which could have been done also in the preceding case. We apply the following elementary generalization of the Hardy's inequality refered in the introduction, for $\ell \in \mathbf{N}, f \in C_0^\infty(\mathbf{R} \setminus \{0\})$:

$$\int_{\mathbf{R}} \frac{f^2}{x^{2\ell}} \leq C_\ell \int_{\mathbf{R}} \left(f^{(\ell)}\right)^2, \quad \text{(EGHI)}$$

and the appropriate detemination of the factors $\mathcal{D}, Q$ that appear during the process. Therefore we start with the first blow up in the conical charts and obtain in view of

$$\mathcal{H}^2(P) \leq \frac{m^2}{x_k^2} + \frac{4}{x_k^4} + 4\left(\mathcal{H}^2(P_{m;1,k}\right)^2,$$

that since $m \geq 2$

$$\begin{aligned} I_m(P)[f] &= \int_{N_{\epsilon_m,m}} \mathcal{H}^2(P) f^2(x) v_n \\ &\leq c \sum_{k=1}^{\nu_m} \int_{N_{m;1,k}} \left(\left|D_k f_{m;1,k}\right|^2 + \left|D_k^2 f_{m;1,k}\right|^2\right) (\mathcal{B}_{m;1,k_1})^2 \\ &\quad + c \sum_{k_1=1}^{\nu_m} \int_{N_{m;1,k_1}} \left(\mathcal{H}^2_{m;1,k_1}\right)^2 f^2_{m;1,k_1} (\mathcal{B}_{m;1,k_1})^2 \hat{v}_n \end{aligned}$$

where

$$c = \kappa_m(\nu_m, d) = \frac{16m^2}{(2\nu_m+1)^2}$$

$$N_{m;1,k_1} = \sigma^{-1}(N_{m;0,k_1}),\ N_{m,0,k_1} = K_m(N_{\epsilon_m}) \cap \left(\mathbf{R}^{n-\nu_m} \times C_{k_1}\right)$$

$$f_{m;1,k_1} = \sigma^*(\chi_{m,k_1} f)$$

$$D_{k_1} f_{m;1,k_1} = \frac{1}{x_{k_1}} E(f),\ \mathcal{B}_{m;1,k_1} = |x_{k_1}|^{\frac{\nu_m-1}{2}},$$

$$\mathcal{H}^2_{m;1,k_1} = \left(\nabla \log |P_{m;1,k_1}|\right)^2, \quad P_{m;1,k_1} = x_{k_1}^{-m} \sigma^*(\chi_{k_1} P).$$

The first terms is blown down directly since in the conical chart

$$C_k : |x_k|^2 \geq \frac{\alpha^2}{1+\alpha^2} r^2,$$

and hence implies that

$$|D_k f_{m,k,1}| \leq \frac{\alpha^2}{1+\alpha^2} |\nabla f|^2.$$

Therefore we have by an application of Hardy to the derivative of the jacobian $\mathcal{B}_{m;1,k_1}$ we could comprise all terms in the following inequality

$$I_m[P](f) \leq C \left(||\nabla f||^2_{H_1(N_m)} + I_{1;m;j}[P](f) \right),$$

where we have the usual Sobolev space norm:

$$||\nabla f||_{H^1(N_m)} = \int_{N_m} |\nabla f|^2 + |\nabla^2 f|^2,$$

and

$$I_{1;m;1}[P](f) = \sum_{k=1}^{\nu_m} \int_{N_{m;1,k}} \left(\mathcal{H}^2_{m;1,k}\right)^2 \left(f_{m;1,k} \mathcal{B}_{1,k}\right)^2 \hat{v}_n.$$

Set then

$$f_{m;j,k_j} = \sigma^* \left(\chi_{k_j} f_{m;j-1,k_{j-1}} \mathcal{B}_{m;j-1,k_{j-1}}\right), \quad \mathcal{B}_{m;j,k_j} = \sigma^*(\chi_{m,k_j} \mathcal{B}_{m;j-1,k_{j-1},l}),$$

and

$$I_{\ell;m;\ell}[P](f) = \sum_{k_\ell=1}^{\nu_m} \int_{N_{\ell,k_\ell}} \left(\mathcal{H}^2(P_{m;\ell,k_\ell})\right)^{\ell+1} \left(f_{m;\ell,k_\ell} \mathcal{B}_{\ell,k_\ell}\right)^2.$$

Then we derive the recursive formula for these integrals. and then through the inequality, which is derived thorugh Young's inequality and for $|x_{k_\ell}| < 1$

$$\left(\mathcal{H}^2(P_{m;\ell-1,k_{\ell-1}})\right)^\ell \leq C_\ell m^{2\ell} \left(\frac{1}{x_{k_\ell}^{4\ell(\ell+1)}} + \left(\mathcal{H}^2(P_{m;\ell,k_\ell})\right)^{\ell+1} \right),$$

then

$$I_{\ell-1;\ell-1}[P](f) \leq C_{\ell,m,\epsilon} \left(||\nabla f||_{H^{2\ell(\ell+1)}(N_m)} + I_{\ell;\ell}[P](f) \right).$$

The first term originates from the (EGHI), Leibniz's rule and the application of Hardy's inequality for the term $D_{k_j}\mathcal{B}_{m;j,k_j}$ to transfer the derivative to the f-term. The blow down process in the conical charts gives us the Sobolev space norms. Specifically we have that

$$|D^j_{x_{k_\ell}} f_{m;\ell,k_\ell}| = |\left(\frac{1}{x_{k_\ell}}E\right)^j (\sigma^*\left(\chi_{k_j} f_{m;j-1,k_{j-1}}\mathcal{B}_{m;j-1,k_{j-1}}\right))|.$$

Therefore by expanding the differentiation we arrive at

$$|D^j_{x_{k_\ell}} f_{m;\ell,k_\ell}|^2 \leq C \sum_{i_1+i_2+i_3=j} \frac{1}{r^{2i_1}}|\nabla^{i_2}\sigma^*\left(\chi_{k_j} f_{m;j-1,k_{j-1}}|^2|\nabla^{i_3}\mathcal{B}^2_{m;j-1,k_{j-1}}\right) \leq$$
$$\leq C' \sum_{i_1+i_2=j} \frac{1}{r^{2i_1}}|\nabla^{i_2} f_{m;j-1,k_{j-1}}|^2\mathcal{B}^2_{m;j-1,k_{j-1}}.$$

We iterate this process and finally we end up with the following inequality

$$I_m[P](f) \leq C\left(||\nabla f||_{H^{\beta_m}(N_m)} + I_{m;1}[P](f)\right),$$

where the term

$$I_{m;1}[P](f) = \sum_{k_\gamma=1}^{\nu_m}\int_{N_{m;\gamma_m,k_{\gamma_m}}} \left(\mathcal{H}^2(P_{m;\gamma_m,k_{\gamma_m}})\right)^{\gamma_m+1}(f_{m;\gamma_m,k_{\gamma_m}})^2,$$

contains the polynomial mith mulitplicity $m-1$ after the first generation of γ_m years of blow-ups for the m stratum and $\beta_m = 2\sum_{\ell=1}^{\gamma(m)}(\ell^2+\ell) = 2S_2(\nu_m)+S_1(\nu_m) \sim \nu_m^3$.

Conclusion. Having exhausted the first generation of γ_m-years of blow -ups we reduced the multiplicity to $m-1$ at the cost of bringing in the Hardy - factor in the γ_m+1 power, and we integrate in the tubular neighbourhood of set of lower multiplicity. Again the "center defining maps" enter and are composed and we proceed. Higher order derivatives appear but they are treated due to the formula that we calculated above.

2.3. The inequality for homogeneous polynomials. Let $P \in \mathcal{P}^{gH}$ then the inequality receives the simple form with the first derivatives. We start blowing up the origin which supports the maximal multiplicity of V we obtain inequalities corresponding to the traces of the algebraic set on the balls

$$B_{0,\alpha,n} = V(x_j-1)\cap C_{\alpha;j,n+1}, \quad j=1,\ldots,n+1.$$

Then we use the preceding inequalities for inhomogeneous polynomials and by a scaling transformation we obtain the desired result by trowing the "higher order terms". In the sequel we assume without loss of generality that we are already reduced to the essential variables and proceed through the conical partition of unity and the notation of the preceding paragraph:

$$I_d[P](f) = \int_{\mathbf{R}^{n+1}}\mathcal{H}^1(P)f^2(x)v_{n+1} = \sum_{j=1}^{\nu_d}\int_{\widehat{C}_{\alpha;j,n+1}}(f_{d;1,j})^2 x_j^{\nu_d-3}(P_{d;1,j})^{-\frac{2}{d}}\hat{v}_{n+1}.$$

Now in each cone $\widehat{C}_{\alpha;j,1}$ the function $f_{d,1,j} = \sigma^*(\chi_j f)$ is compactly supported and wehave that the volume form decomposes for the volume form of $\mathbf{R}^n, v_n$

$$\widehat{v}_{n+1} = dx_j v_n$$

and the integral splits as

$$I_d[P](f) = \sum_{j=1}^{\nu_d} \int_{\mathbf{R}} I_{d-1}[P_{d;1,j}](f_{d;1,j})(x_j) x_j^{\nu_d - 3} dx_j \quad \text{(CS)}.$$

In each term we have GHI_1 in the cross section: we obtain for $\tilde{\nabla}_j$ the gradient in all but the x_j-variable:

$$I_{d-1}[P_{d;1,j}](f_{d;1,j}) \leq C \int_{B_{0,\alpha,n}} \sum_{i=0}^{h_{1,j}} |\tilde{\nabla}^i f_{d;1,j}|^2 = \left(R_{d,1,j}[P](f_{d;1,j})(x_j)\right)^2,$$

where the function is compactly supported in $\mathbf{R}$ and smooth with respect to x_j. Hence we return back to (CS) and we have that

$$I_d[P](f) \leq C \sum_{j=1}^{\nu_d} \int_{\mathbf{R}} \left(R_{d,1,j}[P](f_{d;1,j})(x_j)\right)^2 x_j^{\nu_d - 3} dx_j.$$

We apply the Hardy's inequality in each summand and obtain:

$$\int_{\mathbf{R}} \left(R_{d,1,j}[P](f_{d;1,j})(x_j)\right)^2 x_j^{\nu_d - 3} dx_j \leq \frac{16}{(\nu_d - 1)^2} \int_{\mathbf{R}} \left|\partial_{x_j} R_{d,1,j}[P](f_{d;1,j})(x_j)\right|^2 x_j^{\nu_d - 1} dx_j.$$

Cauchy-Schwarz inequality suggests that

$$\left| \partial_{x_j} \left(\sum_{i=1}^{N} G_i^2 \right)^{1/2} \right| = \left| \frac{1}{\sum_{i=1}^{N} G_i^2} \left(\sum_{i=1}^{N} G_i \partial_{x_j} G_i \right) \right| \leq \left| \sum_{i=1} \partial_{x_j} G_i \right|.$$

Then apply this for $G_i(x_j) = \left(\int_{\mathbf{R}^n} f^2(\cdot, x_j)\right)^{1/2}$ and thanks to Cauchy-Schwarz again we pass under the integral to obtain thanks to Leibniz' srule that in gives us the following

$$\int_{\mathbf{R}} \left|\partial_{x_j} R_{d,1,j}[P](f_{d;1,j})(x_j)\right|^2 x_j^{\nu_d - 1} dx_j \leq \int_{\widehat{C}_{\alpha;j,n+1}} \sum_{i=1}^{h_{1,j}} |\nabla^i f|^2 x_j^{\nu_d - 1} \hat{v}_{n+1}.$$

Then we blow down in each conical chart and taking into account the usual estimates (we apply Hardy's inequlaity for the conical atlas partition of unity) and obtain that

$$I_d[P] f \leq C \sum_{j=1}^{n+1} \int_{\widehat{C}_{\alpha;j,n+1}} \left(\sum_{i=1}^{h_{i,j}} |\nabla_i f|^2 \right) v_{n+1}.$$

In the end we scale $x \mapsto \tilde{x} := (\lambda^{-1} x_1, \ldots, \lambda^{-1} x_{n+1})$ Evidently, we let $\lambda \to \infty$ to obtain the inequality. Similarly we obtain the inequality for $\mathcal{H}^2(P)$. □

Remark. Sticking on the inequality for $\mathcal{H}^2 = \left|\frac{\nabla P}{P}\right|^2$ one could also deduce the following inequality, actually by partial integration: for $C_3(P)$ we have

$$\int_{\mathbf{R}^n} \left| \frac{\Delta P}{P} \right| f^2 \leq C_3(P) \int_{\mathbf{R}^n} |\nabla f|^2.$$

2.4. Further inequalities. Let $P \in \mathcal{P}^{gH}$ of degree d and for $s \in \mathbf{R}$ then the following more general inequality is true for $f \in C_0^\infty(\mathbf{R}^n \setminus V(P))$:

$$\int_{\mathbf{R}^n} |P|^s f^2 \leq C(P,s) \int_{\mathbf{R}^n} |P|^{s+\frac{2}{d}} |\nabla f|^2.$$

First we observe that the homogeneity we localize near $V(P)$, the rest being treated by rectilinearization and partial integration. We have for $f \in C_0^\infty(\mathbf{R}^n \setminus V(P)), \beta = s + \frac{2}{d}$: and Young's inequality for $p, q > 1, pq = p + q$:

$$|P|^s f^2 \leq \frac{\varepsilon^p |P|^{p\beta} f^2}{p} + \frac{1}{\varepsilon^q q} |P|^{-\frac{2q}{m}} f^2, \qquad (\mathrm{BI}_1)$$

then we have that

$$\int_{\mathbf{R}^n} |P|^2 f^2 \leq \int_{\mathbf{R}^n} \frac{1}{p} |P|^{p\beta} f^2 + \frac{|P|^{-\frac{2q}{m}} f^2}{q} \leq$$
$$\leq \int_{\mathbf{R}^n} \frac{1}{p} |P|^{p\beta} f^2 + C \int_{\mathbf{R}^n} \left| \nabla \left(P^{-\frac{2(q-1)}{m}} f \right) \right|^2.$$

Then the last term gives us that for $\kappa_1 = \frac{(q-1)^2}{m^2}$

$$\int_{\mathbf{R}^n} |\nabla \left(P^{-\frac{2(q-1)}{m}} f \right) |^2 \leq 2 \left(\kappa_1 I_1 + I_2 \right),$$

where

$$I_2 = \int_{\mathbf{R}^n} |P|^{-\frac{2(q-1)}{m}} |\nabla f|^2 .$$

Now

$$I_1 = \int_{\mathbf{R}^n} P^{-\frac{2(q-1)}{2m}} \mathcal{H}^2(P) f^2 \leq C_1(P) \left[(1+\epsilon)\kappa_1 I_1 + (1 + \frac{1}{\epsilon}) I_2 \right].$$

Now we selct q such that:

$$C_1 \kappa_1 (1+\epsilon) \leq \frac{1}{1+\epsilon} \Rightarrow q \leq 1 + \frac{m}{C_1(1+\epsilon)},$$

and we have that

$$I_1 \leq C_3 I_2,$$

and finally we have that:

$$\int_{\mathbf{R}^n} |P|^s f^2 \leq C \left(\int_{\mathbf{R}^n} |P|^{2\beta p} f^2 + \int_{\mathbf{R}^n} |P|^{-\frac{2(q-1)}{m}} |\nabla f|^2 \right).$$

Now we split $N_\eta(P) = \{x \in \mathbf{R}^n / |P(x)| \leq \eta^m\}, \eta < 1$ in sets

$$N_i = \{x \in \mathbf{R}^n / \eta^{m(i+1)} \leq |P(x)| \leq \eta^{mi}\},$$

and accordingly $F_i = \mathrm{supp}(f) \cap N_i$:

$$\int_{\mathbf{R}^n} |P|^s f^2 = \sum_{i=1}^{\infty} \int_{F_i} |P|^s f^2 + \int_{\mathbf{R}^n \setminus N_\eta} |P|^s f^2.$$

The second term is reduced to the 1-d problem by change of variable. The integral near the zero set is treated using the inequality derived from (BI_1) for $\varepsilon = \eta^{m(i+1}$ or $\varepsilon = \eta^{mi}$ analogously for $\beta > 0, < 0$. Finally we scale and obtain tyhe result. □

The inequality for the homogeneous polynomial allows to improve the inequality for inhomogeneous polynomials, keeping only first derivatives:

PROPOSITION 2.1. *Let $P : \mathbf{R}^n \to \mathbf{R}$ be a polynomial function of degree d from the class $\mathcal{P}^{gH}$ with $V(|\nabla P|^2) \subset V(P)$. Let $f \in C_0^\infty(\mathbf{R}^n \setminus V(P))$ then for $C_3(P)$ it is true that:*

$$\int_{\mathbf{R}^n} P^{-\frac{2}{d}} f^2 v_n \leq C_3(P) \int_{\mathbf{R}^n} f^2 + (1+|x|^2)|\nabla f|^2 v_n.$$

Proof. From the assumption we localize near $V(P)$, the rest being treated by rectilinearization and partial integration. We stratify the algebraic set $V(P)$ by multiplicity:

$$V(P) = \Sigma_1 \bigcup \cdots \bigcup \Sigma_\mu.$$

Since P is inhomogeneous then $d > \mu$. The smooth part being clear, we will study the singular strata. Let $a \in \Sigma_m, x(a) = 0$ and $P_{0,m} = \text{in}_a(P)$ and introduce the tubular neighbourhood of Σ_m of width $\epsilon^{\frac{d}{m}}$

$$N_\epsilon = \{x \in \mathbf{R}^n / P_{0,m}^2(x) + P^2(x) < \epsilon^{2d}\}.$$

We write this as

$$N_\epsilon = C_\epsilon \bigcup S_\epsilon \bigcup R_\epsilon,$$

where for suitable $\delta < 1$:

$$\begin{aligned} C_{\epsilon,\delta} &= \{x \in N_\epsilon : |P| \geq \delta^2 |P_{0,m}|\} \\ S_{\epsilon,\delta} &= \{x \in N_\epsilon : |P| \leq \delta |P_{0,m}|\} \\ R_{\epsilon,\delta} &= \{x \in N_\epsilon : \delta^2|P_{0,m}| \leq |P| < \delta|P_{0,m}|\}. \end{aligned}$$

Notice that the stratum of multiplicity $\Sigma_m \subset S_{\epsilon,\delta}, C_{\epsilon,\delta}, R_{\epsilon,\delta}$.

The integral

$$I_\epsilon = \int_{N_\epsilon} P^{-2/d} f^2$$

splits in three parts , $i = 1, 2, 3$:

$$I_i = \int_{N_\epsilon} P^{-2/d} \psi_i^2 f^2$$

for the functions localizing in the sets $C_{\epsilon,\delta}, S_{\epsilon,\delta}, R_{\epsilon,\delta}$ respectively:

$$\begin{aligned} \psi_1^2 &= \varphi\left(\frac{\delta^2|P_{0,m}|}{|P|}\right) \\ \psi_2^2 &= \varphi\left(\frac{|P|}{\delta|P_{0,m}|}\right) \\ \psi_1^2 &= 1 - \phi_1^2 - \phi_2^2. \end{aligned}$$

We would estimate them seperately:

1. Estimate for I_3. The crucial observation is that $\mathrm{supp}(\psi_3 f) \cap V(P_{0,m}) = \emptyset$ hence we apply the inequality for $P_{0,m}$:

$$I_3 \leq \delta^{-2/d} \int_{R_{\epsilon,\delta}} |\nabla(\phi_3 f)|^2.$$

Now in $R_{\epsilon,\delta}$ we have that for $\kappa_1 < \frac{1}{2}$

$$|\nabla\psi_3| \leq C_1 + \kappa_1 \left|\frac{\nabla P_{0,m}}{P_{0,m}}\right|^2$$

and we apply the inequality for $\mathcal{H}^2(P_{0,m})$ and we are done.

2. The integral I_2 is filtered in the sets, for $j > 1$:

$$S_{\epsilon,\delta,j} = \{x \in N_\epsilon / \delta^{j+1}|P_{0,m}| < |P| < \delta^j |P|\}$$

and we obtain through the cut-off estimate:

$$|\nabla\psi_2| \leq C + \left(\frac{\delta}{2}\right)^j \left|\frac{\nabla P_{0.m}}{P_{0,m}}\right|^2$$

and the fact that $\mathrm{supp}(f) \cap V(P_{0,m})$ therefore we have that

$$I_2 \leq C \int_{\mathbf{R}^n} f^2 + |\nabla f|^2.$$

3. The last is the term near the cone $V(P)$. Then introduce the sets for $\eta < 1$:

$$S_{\epsilon,\eta,j} = \{x \in N_\epsilon / |P| \geq \delta\eta^j \geq \delta|P_{0,m}|\}.$$

Then through the cut-off estamates that allow us to select appropriately ϵ, we conclude that:

$$I_1 \leq \int_{\mathbf{R}^n} f^2 + |\nabla f|^2.$$

Summing up we find

$$I_{1,\epsilon} \leq C \int_{\mathbf{R}^n} |\nabla f|^2 + f^2.$$

□

2.5. Example: cubic hypersurfaces. Here, we treat the inequality for cubic forms $(n+1)$- variables, defining a conic variety with singular part containing a line through the origin. The case $n = 2$ is treated in [P] with elementary methods.

If we choose the x_1- axis such that $V(P)$ is singular along it then the form is written as $P(x) = x_1 Q(\tilde{x}) + C(\tilde{x})$ where $\tilde{x} = (x_2, \ldots, x_{n+1})$ and Q, C are quadratic and cubic forms without common factors since otherwise the inequality is fairly easier. The form P defines a cubic hypersurface in $\mathbf{P}^n$ with a singularity at the point E_1 with homogeneous coordiantes

$$E_1 \equiv [1 : 0 : \cdots : 0]$$

as well as at the points of the variety T where the varieties $V(C), V(Q)$ touch each other or the singular set Σ of $V(C)$ on the hypeplane H_1. We assume also without loss of generality that $\mathrm{rank}(Q) = n$.

For the sake of simplicity we treat first the smooth part of the hypersurface defined for $\epsilon > 0$ by

$$V_\epsilon(P) = \bigcup_j V_{\epsilon,j}(P), \quad V_{\epsilon,j} = \{x \in \mathbf{R}^{n+1} / \left|\frac{\partial P}{\partial x_j}\right| > \epsilon r^2\}.$$

Observe that in $V_\epsilon(P)$ it is true that for $c_i > 0$, depending on the values of P on S^n that $c_1 r^2 \leq |\nabla P| \leq c_2 r^2, c_1 = (n+1)\epsilon$ and also that $c_3 r \leq \left|\frac{\partial^2 P}{\partial x_i \partial_j}\right| \leq c_4 r$. Therefore the integral $I_{1,\epsilon}(P)[f] = I(P)[\chi_{1,\epsilon} f]$ is studied in $V_\epsilon(P)$ by the change of variables for each j by

$$\Psi_j : V_{\epsilon,j} \to \mathbf{R}^{n+1},$$
$$(x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_{j_1}, \xi, x_{j+1}, \ldots, x_{n+1}), \xi = P(x).$$

Then we calculate $v_{j,n+1} = dx_1 \ldots dx_{j-1} d\xi dx_{j+1} \ldots dx_{n+1}$

$$\int_{V_\epsilon(P)} P^{-\frac{2}{3}} f^2 v_{n+1} = \sum_j \int_{\Psi_j(V_{\epsilon,j})} \xi^{-\frac{2}{3}} \tilde{f}^2 \left|\frac{\partial P}{\partial x_j}\right|^2 v_{j,n+1} \leq C \int_{V_\epsilon} |\nabla f|^2 v_{n+1},$$

by applying the preceding inequalities in combination with that of Hardy.

To proceed to $I_{2,\epsilon}(P)[f]$ further we blowup $\sigma_{n+1} : \widehat{\mathbf{R}}^{n+1} \to \mathbf{R}^{n+1}$ through the usual cones for $\alpha > 1$:

$$C_{\alpha;j,n+1} = \{x \in \mathbf{R}^{n+1} / x_j^2 \geq \frac{1}{1+\alpha}|x|^2\}.$$

The procedure will incorporate an induction with respect to n. Explicitly we have the formulas:

Let $j \neq 1$ then after restriction to $C_{\alpha;j,n+1}$, σ_{n+1} maps as

$$\begin{aligned} i \neq j : x_i &= v_i v_j, \\ x_j &= v_j, \\ P(\sigma_{n+1}(v)) &= x_j^3 . P_j^1, \\ P_j^1 &= x_1 Q_j^1(\tilde{v}) + C_j^1(\tilde{v}), \end{aligned}$$

where $\tilde{v} = (v_2, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{n+1})$. At the point $a \in T_j$ or $a \in \Sigma_j$ the order $\mathrm{ord}_a(P_j^1) = 2$ and we choose coordinates such that $u(a) = 0$ and setting $\tilde{u} = (u_2, \ldots, u_n)$ we assume that $P_j^1(u_1, 0, \ldots, 0) \sim u_1^2$ hence

$$P_j^1(u) = (1 + l_j(u))u_1^2 + b_j(\tilde{u})u_1 + R_j(\tilde{u}),$$

where

$$\deg(l_j) = 1, \quad \deg(\mu_j) = 1, \quad \mathrm{ord}_a(l_j) = 0, \mathrm{ord}_a(b_j) = 1,$$

as well as that R_j is of the smae form as P_j^1 but in $(n-1)-$variables. Then we determine inductively the center

$$K_{n,j} = \{x \in B_{0,\alpha,n} / \mathrm{ord}_x(P_j^1) = 2\}$$

of blowups and the years ν_2 of blow-ups for the generation $m = 2$. The formulas for the blowups in the j_1-conical chart, $j_1 \neq 1$ are:

$$u_j = v_j v_{j_1}, j \neq j_1, v_{j_1} = u_{j_1},$$

implying that

$$P_j^1(u) = v_{j_1}^2 P_{jj_1}^2(v), \quad P_{jj_1}^2(v) = (1 + v_{j_1} l_{jj_1}(v))v_1^2 + b_{jj_1}(\tilde{v}))v_1 + R_j(\tilde{v}).$$

These blowups are repeated up to the desingularization of $V(R_j)$ and at each step we apply Hardy's inequality for the factor coming out of the blowup; finally we obtain a smooth chart as precedingly and we apply Hardy's inequality. The $j = 1$ conical chart is carried through analogously.

In conclusion we have that

$$\int_{\mathbf{R}^{n+1}} P^{-\frac{2}{3}} f^2 v_{n+1} \leq C \int_{\mathbf{R}^{n+1}} |\nabla f|^2 v_{n+1} + \sum_{\ell=2}^{\nu_2} \int_{\mathbf{R}^{n+1}} \left|\nabla^\ell f\right|^2 v_{n+1}$$

and scaling we get the desired result.

3. Heat expansion for operators $H_{c,\alpha} = -\Delta + c|P|^{-\alpha}$. In this section we will establish the existence of the small time expansion of the heat trace of the operator

$$H_{c,\alpha} = -\Delta + c|P|^{-\alpha}.$$

Actually we will prove the $\lambda \to \infty$ expansion of the distributional trace $\mathrm{tr}(R_{c,\alpha}(\lambda)^k \chi)$ of the k-th power of

$$R_{c,\alpha}(\lambda) = (H_{c,\alpha} - \lambda)^{-1}.$$

This will achieved in the following steps:

1. Determination of the domain of self-adjointness of $H_{c,\alpha}$
2. Estimates for $R_{c,\alpha}(\lambda) = (H_{c,\alpha} - \lambda)^{-1}$ in various operator norms
3. The preceding allow us to prove the expansion by considering the expansion of each term of the Neumann series for $R_{c,\alpha}(\lambda)$ around $R_0(\lambda) = (-\Delta_n - \lambda)^{-1}$.
4. After identifying the form of these terme we appeal to the usual Mellin transform theorem in view of the Atiyah-Bernshtein-Gelfand theorem on the meromorphic continuation of integrals depending on complex powers

Domain of Self-Adjointness. First we determine the domain of selfadjointness of the operator $H_{c,\alpha} = -\Delta + c|P|^{-\alpha}$ where $c \in C_0^\infty(\mathbf{R}^n)$ and P is a homogeneous polynomial of degree d in the class $\mathcal{P}^{gH}$. We extend $C_0^\infty(\mathbf{R}^n)$ provided that the exponent $\alpha < \frac{n}{d}$.

The Kato-Rellich theorem ([RS]) suggests that the operator $H_{c,\alpha}$ is essentially self adjoint on $C_0^\infty(\mathbf{R}^n)$. The neccessary semiboundedness estimate reads as

PROPOSITION 3.1. *Let $c \geq 0, \phi \in C_0^\infty(\mathbf{R}^n), \alpha < \frac{n}{d}, \kappa < 1$ then it is true that*

$$\| c.|P|^{-\alpha}\phi \|_{L^2} \leq \kappa \| -\Delta\phi \|_{L^2} + \beta \| \phi \|_{L^2} .$$

Furthermore the domain closure of the operator $H_{c,\alpha}|(C_0^\infty(\mathbf{R}^n))$ for small α consists of $H^2(\mathbf{R}^n)$.

Proof. We decompose the integral in two parts, through a suitable partition of unity: one sufficiently close - at distance $\epsilon^{1/d}$- to the algebraic set and one in the complement. Set then

$$N_\epsilon = \{x \in \mathbf{R}^n / |P(x)| < \epsilon\}$$

and

$$K_\epsilon = \mathbf{R}^n \setminus N_\epsilon.$$

Choose $\chi_{1,\epsilon}^2 + \chi_{2,\epsilon}^2 = 1$, $\mathrm{supp}(\chi_{1,\epsilon}) \subset K_\epsilon$ the functions that execute this decomposition (it just suffice to take functions of the form $\chi\left(\frac{P}{\epsilon}\right)$) Split then

$$\phi^2 = \phi_1^2 + \phi_2^2 = (\chi_{1,\epsilon}\phi)^2 + (\chi_{2,\epsilon}\phi)^2.$$

Then for the part localized in $K_{2,\epsilon}$, we use the inequality:

$$|P|^{-\alpha} \leq \varepsilon P^{-2/d} + B_\varepsilon.$$

Hence, we compute and obtain

$$\begin{gathered} \| c|P|^{-\alpha}\phi \|_{L^2} = \| c.|P|^{-\alpha}\phi_1 \|_{L^2} + \| c.|P|^{-\alpha}\phi_2 \|_{L^2} \\ \leq M. \| \phi \|_{L^\infty} \| \chi_{1,\epsilon} \|_{L^2} + \nu \| (-\Delta)\phi_2 \|_{L^2} + \beta \| \phi \|_{L^2} \end{gathered}$$

having applied twice the inequality $P^{-2/d} \leq C(-\Delta)$ for the second term where the first term is due to the fact for $\alpha < \frac{n}{d}$ then $|P|^{-\alpha} \in L^1_{\text{loc}}(\mathbf{R}^n)$. Finally, we let $\epsilon \to 0$ and obtain the desired inequality. The Kato - Rellich theorem gives the essential self-adjointness we have been looking for; the operator $H_{c,\alpha}$ is bounded below by $-\beta$. q.e.d

Further we will examine the closure of the operator, initially considered on $C_0^\infty(\mathbf{R}^n)$. For this we write:

$$\begin{gathered} \| H_{c,\alpha}\phi \|_{L^2}^2 = \| -\Delta\phi \|_{L^2}^2 + \| c|P|^{-\alpha}\phi \|_{L^2} + \\ +(\phi, [(-\Delta) \cdot c|P|^{-\alpha}|P|^{-\alpha} + c|P|^{-\alpha} \cdot (-\Delta)] \cdot \phi). \end{gathered}$$

Therefore combining the obvious inequality for the inner product with the inequality obtained above, we get the desired estimate.

Operator estimates. For the Neumann series we will need the following estimates relative to the resolvent $R_0(\lambda) = ((-\Delta) - \lambda)^{-1}$:

PROPOSITION 3.2. *Let λ be sufficiently large and outside a cone shaped region enclosing the positive real axis then the operator norm is,*

$$\begin{aligned} |||P|^{-\alpha} R_0(\lambda)||_{L^2} &= 0(|\lambda|^{-1+\frac{d\alpha}{2}}), \\ |||P|^{-\alpha} R_0(\lambda)\partial_i||_{L^2} &= 0(|\lambda|^{-\frac{1}{2}+\frac{d\alpha}{2}}) \\ ||[|P|^{-\alpha}, R_0(\lambda)]||_{L^2} &= 0(|\lambda|^{-1+\frac{d\alpha}{2}}), \\ ||[...[|P|^{-\alpha}, R_0(\lambda)], ...], R_0(\lambda)]||_{L^2} &= 0(|\lambda|^{-k+\frac{d\alpha}{2}}). \end{aligned}$$

Proof. Compute

$$(|P|^{-\alpha} R_0(\lambda)\phi, \quad |P|^{-\alpha} R_0(\lambda)\phi)$$

with $\phi \in L^2(\mathbf{R}^n)$. For this use a suitable partition of unity as in Proposition (2), splitting $\psi = R_0(\lambda)\phi \in C_0^\infty(\mathbf{R}^n)$ as before, far and close to the algebraic set $V(P)$. For the term involving $\psi_1 = \chi_1\psi$ apply Cauchy-Schwarz, while for $\psi_2 = \chi_2\psi$ use the inequality as follows:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \qquad \frac{1}{p} + \frac{1}{q} = 1,$$
$$p = \frac{2}{d\alpha}, a = \delta^{\frac{1}{p}} P^{-2\alpha}, b = \delta^{-\frac{1}{p}},$$

and for the two terms we have that

$$\| P^{-\alpha}\psi_2 \|_{L^2} \leq C_\epsilon \| \phi_2 \|,$$

and also that

$$\| \psi_2 \|_{L^2} \leq C_\epsilon |\lambda|^{-\frac{1}{2}} \| \phi_2 \| .$$

Choosing then

$$\delta = \delta^{-\frac{\alpha d}{2-\alpha d}} |\lambda|^{-1}$$

we get the desired estimate.The other estimates follow in the same fashion as the first one. □

The Neumann series. We will prove the existence of an asymptotic expansion for a suitable power

$$R^k_{c,\alpha}(\lambda) = (\lambda - H_{c,\alpha})^{-k}$$

and proving the existence for each term there. Actually, $R^k_{c,\alpha}(\lambda)$ is represented by an integral kernel since as we intend to show, it is in the Hilbert-Schmidt ideal.

The resolvent expansion for $R_{c,\alpha}(\lambda)$ is

$$R_{c,\alpha}(\lambda) = \sum_{j=0}^{N-1} (R_0(\lambda)|P|^{-\alpha})^j.R_0(\lambda) + O(|\lambda|^{-(1-\frac{d\alpha}{2})N-1})$$

the remainder is considered in the L^2-norm. Use the fact that

$$R^k_{c,\alpha}(\lambda) = \frac{1}{k!}\partial^k_\lambda (R_{c,\alpha}(\lambda))$$

to calculate the form of the Neumann series for $R^k_{c,\alpha}(\lambda)$. We set then

$$\Pi^j(\lambda) = (R_0(\lambda)|P|^{-\alpha})^j,$$

so that

$$R_{c,\alpha}(\lambda) = \sum_{i=0}^{N-1} \Pi^i R_0(\lambda) + +0(|\lambda|^{-(1-\frac{d\alpha}{2})N-1}).$$

In order to obtain the Neumann series of $R_{c,\alpha}(\lambda)^k$ to study the beaviour of the terms in this series. Therefore we set us

$$I^{i,j} = \partial^j_\lambda \left(\Pi^i R_0(\lambda)\right) = \sum_{\ell=0}^{j} (-)^{j-\ell+1} \left(\partial^\ell_\lambda \Pi^i\right) R_0(\lambda)^{j-\ell+1}.$$

Therefore we need

$$\partial_\lambda^\ell \Pi^i = \sum_{m_1+\cdots+m_{\ell+1}=i, m_1\geq 1} \Pi^{m_1} R_0(\lambda)\Pi^{m_2} R_0(\lambda)\cdots R_0(\lambda)\Pi^{m_\ell+1}.$$

Here observe that the number of $R_0(\lambda)$-factors is ℓ Conclusively we have that for the operator product factor

$$C_{i;\ell;m_1,\ldots,m_{\ell+1}} = \Pi^{m_1} R_0(\lambda)\Pi^{m_2} R_0(\lambda)\cdots R_0(\lambda)\Pi^{m_\ell+1},$$

$$I^{i,j} = (-1)^{j+1} \sum_{0\leq\ell\leq j, m_1\geq 1, m_1+\cdots+m_{\ell+1}=i} C_{i;\ell;m_1,\ldots,m_{\ell+1}} \cdot R_0(\lambda)^{j-\ell+1}.$$

Now we see that

$$||C_{i;\ell;m_1,\ldots,m_{\ell+1}}||_{L^2} \leq |\lambda|^{-(1-\frac{d\alpha}{2})i-\ell}$$
$$||I^{i,j}||_{L^2} \leq |\lambda|^{-(1-\frac{d\alpha}{2})i-j-1}.$$

The trace class norm estimates are derived through the following estimate from [C1]:

Proposition 3.3. *Suppose* $k > \frac{n(ql-l+2)}{4lq}, l \geq 2, \chi \in L_0^{2p}(\mathbf{R}^n), \frac{1}{p}+\frac{1}{q}=1$. *Then* $\chi.R_0(\lambda)$ *is in* $C_l(\mathbf{R}^n)$ *for* $\lambda \in \mathbf{C}\setminus\bar{\mathbf{R}}_+$ *and*

$$||\chi R_0(\lambda)^k||_l \leq C_{l,\epsilon}||\chi||_{2p}|\lambda|^{-k+\frac{n(ql-l+2)}{4lq}}$$

for $supp(\chi) \subset K \subset\subset \mathbf{R}^n$ *for all* λ *in* $|Im\lambda| > \epsilon Re\lambda + \epsilon$ *for given* $\epsilon > 0$.

Combining the preceding estimates (Proposition 3.1, 3.2) we can conclude that $I^{i,j} \in C_l(\mathbf{R}^n)$ and furthermore that

$$\| \chi I^{i,j}(\lambda) \|_l \leq C_{l,\epsilon,j} \| \chi \|_{2p} |\lambda|^{-j+\frac{n(ql-l+2)}{4lq}-i(1-\frac{d\alpha}{2})}$$

The existence of the Asymptotic Expansion for $tr(\chi R^k_{c,\alpha}(\lambda))$. We conclude this section with the proof of the existence of the $\lambda \to \infty$ asymptotic expansion of the trace $tr(\chi \cdot R^k_{c,\alpha}(\lambda))$. The latter for k chosen conveniently is a sum of integrals of the form

$$\mathcal{T}^{j,k}(\lambda,\alpha_1,\ldots,\alpha_j) := \mathrm{tr}(\chi\mathcal{I}^{j,k})(\lambda,\alpha_1,...,\alpha_j) =$$
$$= \int_{\mathbf{R}^{nj}} \Pi_{i=1}^j (R_0^{k_i}(\lambda)(x_i,x_{i+1}))\cdot\Pi_{i=1}^j |P|^{-\alpha_i}(x_i)\chi(x)\omega_1\cdots\omega_j$$

with the convention $x_{j+1}\equiv x_1, X=(x_1,\ldots,x_j)\in\mathbf{R}^{nj}$. The resolvent is given by

$$R_0(\lambda)(x_1,x_2) = \frac{(\sqrt{-\lambda})^{\frac{n}{2}+2}}{2^{\frac{n}{2}}|x_1-x_2|^{\frac{n}{2}-1}} K_{n-\frac{1}{2}}(\sqrt{-\lambda}|x_1-x_2|) = \frac{1}{(4\pi)^{\frac{n}{2}}}\int_0^\infty e^{\lambda y - \frac{(x_1-x_2)^2}{4y}}\frac{dy}{y^{\frac{n}{2}}}.$$

The existence of the asymptotic expansion of the trace $\mathcal{T}^{j,k}(\lambda,\alpha_1,\ldots,\alpha_j)$ is proved using the Mellin transform suggested by the following classical result - stated in the terminology of [C2]:

Proposition 3.4. *Let* $f \in C_0^\infty(\mathbf{R}_+), m\in\mathbf{R}$ *such that* $x^m f \in L^1_{loc}(\mathbf{R}_+)$. *Suppose that the Mellin transform* $\widehat{f}$ *has a meromorphic extension from the half plane* $\{z \in$

$\mathbf{C}/\Re z > -m\}$ *with poles and multiplicity function* $S, d = deg(S)$ *and further that* $lim_{\Im s\to\infty}(|s|^d\widehat{f}(s)) = 0$. *Then* $f \in \Gamma^\infty(\overline{\mathbf{R}_+})$.

Using the integral representations for the resolvent and changing variables we deduce that the Mellin transform of $\mathcal{T}_{j,k}$ with respect to λ takes the form for $k = k_1 + \cdots + k_j$:

$$\widehat{\mathcal{T}}^{j,k}(s,\alpha_1,\ldots,\alpha_j) = c_{n,j,k}(s)\int_{\mathbf{R}^{nj}\times\overline{\mathbf{R}_+}^j}\mathcal{P}_{(s,\alpha_1,\ldots,\alpha_j)}(x,y)\chi(x,y)dxdy$$

where we have set above

$$c_{n,j,k}(s) := 4^{s-\frac{(2k-n)j}{2}-j}\Gamma(s)\Gamma(-s-\frac{(2k-n)j}{2}-j)$$

$y := (y_1,\ldots,y_j)$ and the integrand consists of:

$$\mathcal{P}_{(s,\alpha_1,\ldots,\alpha_j)}(x,y) = \frac{\prod_{i=1}^j P^{\alpha_i}(x_i)(\prod_{i=2}^j y_i^{s-j(k_i-\frac{n}{2})-1}(\tilde{\mathcal{Q}}(x,y))^{-s+(\frac{n}{2}-k)j+1}}{(1+y_2+\cdots+y_j)^{-s}}$$

and $\tilde{\mathcal{Q}} = i^*\mathcal{Q}$ is the restriction of the homogeneous polynomial function $\mathcal{Q}:\mathbf{R}^{nj}\times\mathbf{R}^j\to\mathbf{R}$ on the $y_1 = 1$ hyperplane

$$\mathcal{Q}(x,y) = \sum_{k=1}^j(y_k+y_{k-1})\pi_k(y)x_k^2 - \sum_{i=1}^j\pi^j_{(i,i+1)}(y)(x_{i-1}y_i + x_{i+1}y_{i-1})\cdot x_i$$

where $\cdot$ denotes the Euclidean inner product and we have set as well

$$\pi_i^j(y) = y_1\ldots y_{i-1}y_{i+1}\ldots y_j, \qquad \pi^j_{(i,i+1)}(y) = y_1\ldots y_{i-1}y_{i+2}\ldots y_j.$$

The existence of the asymptotic expansion is a direct consequence of the following theorem on the meromorphic continuation of integrals depending on complex parameters, [BG].

THEOREM 3.5. *Let* $P_1,\ldots,P_k$ *be regular functions on* $\mathbf{R}^n$ *and* $\phi\in C_0^\infty(\mathbf{R}^n)$, *then the integral*

$$I(\lambda_1,\ldots,\lambda_k) = \int_{\mathbf{R}^n}|P_1|^{\lambda_1}\ldots|P_k|^{\lambda_k}\phi\omega_n$$

can be continued as a meromorphic function on the whole space of the complex variables $\lambda_1,\ldots,\lambda_k$*; at the same time the poles can be situated on a finite number of series of hyperplanes of the form* $a_1\lambda_1+\ldots+a_k\lambda_k+b+s=0$ *where* $a_1,\ldots,a_k,b$ *are fixed nonnegative integers and* s *runs through all the odd natural numbers*

4. Growth of inegrals in semialgebraic sets. Let $P\in\mathcal{P}^{gH}$ be a homogeneous polynomial and consider the following semialgebraic sets:

$$N_P(\eta) = \{x\in\mathbf{R}^n/\epsilon_0\le P(x)\le\eta\}.$$

Then we consider the function $\zeta, \mathrm{supp}(\zeta)\subset B_{0,R}$ that satisfies the gradient estimate for $\gamma,\delta>0$:

$$|\nabla\zeta|\le\gamma|\zeta|+\delta.$$

If $\varphi \in C_0^\infty(\mathbf{R}_+)$ with $\mathrm{supp}(\varphi) \subset [0, 1+\varepsilon), \phi \equiv 1$ in $[0,1]$ then $\varphi\left(\frac{P}{\eta}\right)$ localizes in the space $P < \eta$. We will follow the classical identities for the monotonicity formulas, cf. [HL]. Then we have the following elementary identity:

$$\eta \partial_\eta \phi = -\frac{P}{|\nabla P|^2} \nabla P \cdot \nabla \varphi$$

we can differntiate the integral replacing ζ by $\varphi\left(\frac{\epsilon_0}{P}\right)\zeta$ in order to localize in $N_P(\eta)$:

$$I_\zeta(\eta) = \int_{\mathbf{R}^n} \phi \zeta^2$$

and get:

$$\eta \frac{dI_\zeta}{d\eta} = -\int_{\mathbf{R}^n} \frac{P}{|\nabla P|^2} \zeta^2 \nabla P \cdot \nabla \varphi.$$

Then integrate by parts to obtain for $Q = |\nabla P|^2$:

$$\zeta \frac{dI_\zeta}{d\eta} \leq \int_{\mathbf{R}^n} \left(\frac{P^2}{Q^2} \left| \frac{\Delta P}{P} \right| + \delta + 2\gamma \frac{P^2}{Q} + \frac{|P|}{|\nabla P|} \frac{|\nabla Q|}{Q} \right) \varphi \zeta^2 + \mathrm{vol}(N_\eta).$$

Then applying the conical Lojasiewicz inequality we obtain:

$$\frac{|P|}{|\nabla P|} \leq c|P|^{1/m} \leq c\eta^{1/m}.$$

If we assume that $Q \in \mathcal{P}^{gH}$ -it is not neccessary, since with partial integration we can avoid the term involving ∇Q - then applying the generalized Hardy's inequality we arrive at the differential inequality

$$\frac{dI_\zeta}{d\eta} \leq C\left(\gamma^2 \eta^{\frac{2}{m}-1} I_\zeta + \eta^{\frac{1}{m}}\right).$$

Then we arrive at the conclusion:

$$I_\zeta(\eta) \leq c_1 e^{c_2 \gamma^2 \eta^{2/m}} I_\zeta(\eta/2).$$

Actually this estimate combined with Harnack inequalities for semialgebraic sets (proved in [P1]) provide growth estimates for functions on semialgebraic sets.

REFERENCES

[A] M. F. Atiyah, *Resolution of singularities and division of distributions*, CPAM, 2 (1970), pp. 229–234.

[BG] I. N. Bernshtein and S. I. Gelfand, *On the meromorphic property of the functions P^λ*, Funct. Anal. Applic., 3 (1969), pp. 68–69.

[BM1] E. Bierstone and P. Milman, *Semianalytic and Subanalytic Sets*, Publ. Math. I.H.E.S., 2 (1988), pp. 45–98.

[BM2] E. Bierstone and P. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Inv. Math., 128 (1997), pp. 207–303.

[C1] C. J. Callias, *The heat kernel as a function of the coefficients of the potential*, CWRU Preprint.

[C2] C. J. Callias, *On the existence of small time heat expansions*, MRL, 2 (1995), pp. 1–17.

[G] M. Gromov, *Spectral geometry of semialgebraic sets*, Ann. Inst. Fourier (Grenoble), 42 (1992), fasc. 1-2, pp. 249–274.

[HL] Q. Han and F. H. Lin, *Nodal sets of solutions of elliptic differential equations*, (2007).

[H] L. Hörmander, *The analysis of Linear Partial Differential Operators I*, Springer Verlag, GMW 256, 1990.

[P] S. Papadopoulos, *Notes On A Generalized Hardy Inequality*, Manuscript, Univ. of Crete, (1992).

[P1] D. A. Pliakis, *Nodal volume estimates*, (Preprint).

[RS] M. Reed and B. Simon, *Modern Math. Physics*, vol. II, Self Adjointness, Academic Press, 1975.