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FOCAL LOCI IN G(1,N)*

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Abstract. We introduce the different focal loci (focal points, planes and hyperplanes) of (n-1)-dimensional families (congruences) of lines in $\mathbb{P}^n$ and study their invariants, geometry and the relation among them. We also study some particular congruences whose focal loci have special behavior, namely $(n-1)$-secant lines to an $(n-2)$-fold and $(n-1)$-tangent lines to a hypersurface. In case $n = 4$ we also give, under some smoothness assumptions, a classification result for these congruences.

Key words. Focal locus, congruence

AMS subject classifications. 14N15, 14N10, 14M07

1. Introduction. In a previous work (see [4]), we have done a thorough study of the focal locus of a two-dimensional family of lines in $\mathbb{P}^3$, gathering old and new results, both local and global, about the subject. For other families of linear spaces, few is known about their focal locus (see for instance [6] for families of planes in $\mathbb{P}^4$).

In this paper we want to reconstruct most of the results in [4] for the focal locus of $(n-1)$-dimensional families of lines in $\mathbb{P}^n$ (i.e. line congruences). The main difference when considering arbitrary dimension n is that we do not have the self-duality of the case $n = 3$, in which the dual of a line in $\mathbb{P}^3$ was still a line in $\mathbb{P}^{3^*}$. This self-duality implied a duality for the focal surface, in which the dual of the set of focal points was the set of focal planes. So a first question is whether the natural generalization to arbitrary n of focal planes is the notion of focal hyperplanes or the one of focal planes. As we will see, in order to have some duality result, the notion of focal hyperplanes is more natural, while in order to get easy (and natural) computations the notion of focal planes works usually much better. One of the contributions of the paper is to show that fortunately both notions are essentially equivalent, in the sense that from each of them we can derive the other. This allows us to work at each moment with the most convenient focal locus.

It is worth mentioning that our generalization to arbitrary dimension allows to understand some behavior that looked strange for $n = 3$. There each line had two pairs consisting of a focal point and a focal plane, but the tangent plane to the focal locus at each point was the focal plane of the other pair, which a priori seems quite unnatural. But we will see that, for arbitrary n, each line has $n-1$ pairs consisting of a focal point and a focal plane. It is now natural that, if we want to get the tangent hyperplane to the focal locus at each of the $n - 1$ focal points we use, instead of its own plane, the span of the $n - 2$ others (see Theorem 3.1).

Let us give a sample comparing the advantages of using focal planes or hyperplanes. It is clear that working with incident varieties is always very useful (see Section 2). From this point of view it is thus necessary to work with focal planes, since there is not a natural way of finding an incidence variety of focal points and

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hyperplanes. But once we know that the tangent hyperplane to the focal locus is a focal hyperplane or equivalently the span of $n-2$ planes, in order to compute the class of the focal locus is of course better to consider focal hyperplanes rather than working with a variety parametrizing finite unions of planes (whose intersection ring is very complicated). In this context, the fact that we do not have good incidence varieties for focal hyperplanes will be the reason why we cannot compute for instance the class of hyperplane sections of the focal locus, while for $n=3$ it was not difficult.

Although we will work in arbitrary dimension n, in order to show how our methods allow to obtain global formulae for the focal locus, we will often restrict ourselves to the case $n=4$ to obtain the precise formulae in this case. When the situation is not essentially new with respect to the case $n=3$, we sometimes omit details and refer to the corresponding result in [4]. This is for example what happens when analyzing possible pathologies of the focal hypersurface: these pathologies do not depend essentially on n when considering $(n-1)$-secant lines to a codimension-two subvariety of $\mathbb{P}^n$ or $(n-1)$-tangent lines to a hypersurface in $\mathbb{P}^n$ (we even omit a study of n-th order inflectional lines to a hypersurface, since these congruences would not eventually exist when imposing smoothness).

The paper is distributed as follows: first we start with a section (Section 2) of preliminaries, in which we introduce the different notions of focal points, planes and hyperplanes, stating their main properties and their degrees (sometimes only for $n=4$). In Section 3 we get the first relations between the different focal loci that we defined in the previous section. This is done in the general case, although we also devote a subsection to study what happens in special situations. To complete the study of this relation, we need local coordinates. We thus devote Section 4 to locally study the focal loci, re-obtaining in local coordinates what we did globally, and proving with this local analysis that the set of focal hyperplanes corresponds in general to the dual of the focal hypersurface. Finally, Sections 5 and 6 are devoted respectively to study two particular types of congruences: the congruence of $(n-1)$-secant lines to a codimension-two subvariety of $\mathbb{P}^n$ and the congruence of $(n-1)$-tangent lines to a hypersurface of $\mathbb{P}^n$. In both cases we concentrate in the smooth case and obtain classification results for $n=4$.

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2. General Results and Preliminaries.

2.1. Preliminaries. The notation used in this work is mostly standard from Algebraic Geometry, for instance the one in [8]. In particular, the functor $\mathbb{P}$ will correspond to quotients of rank one. The ground field is always the field $\mathbb{C}$ of complex numbers.

As usual we will view $G=G(1,n)$ as embedded in a projective space under the Plücker embedding. By a *congruence* X we shall mean an $(n-1)$-dimensional (irreducible) subvariety of G. The (arithmetic) *sectional genus* of a congruence is the genus of its hyperplane section. We will usually denote it by g. We will write

- $\Omega(I,J)$ the Schubert variety of lines in G meeting I and contained in J, where I and J are linear subspaces of $\mathbb{P}^n$ such that $I \subset J$
- $\Omega(i,j)$ the class in the Chow ring of G of $\Omega(I,J)$, where $i=$dim$I$ and $j=$dim$J$
- $(a_0,\ldots,a_m)$ for $m=[\frac{n-1}{2}]$, the multidegree of X, where the class of X in $A^*(G)$ is $\sum_i a_i\Omega(i,n-i)$ for $0 \le i \le [\frac{n-1}{2}]$
- $\mathcal{S}$, $\mathcal{Q}$ the universal subbundle of rank $n-1$ and quotient bundle of rank two of G appearing in the exact sequence

$$0 \to \check{\mathcal{S}} \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_G \to \mathcal{Q} \to 0$$

(in general, for $G(k,n)$ with $k \neq 1$ we will write $\mathcal{S}_k$ and $\mathcal{Q}_k$ for the universal bundles, in order to avoid confusion). Observe that with this notation $c_1(\mathcal{Q}) = \Omega(n-2,n)$ (which is the hyperplane section of G), $c_2(\mathcal{Q}) = \Omega(n-2,n-1)$ and $c_i(\mathcal{S}) = \Omega(n-i-1,n)$ for $i=1,\ldots,n-1$.

Convention: In order to distinguish if we refer to a subspace as a subset of $\mathbb{P}^n$ or to an element of the Grassmannian, we will use the following convention: small letters l will represent points of $G(k,n)$, while the corresponding capital letters L will represent the subspace $L \subset \mathbb{P}^n$. If l is a line, we shall also write P_l for the set of planes containing L and P_l^* for the set of hyperplanes containing L. Observe that both sets can be regarded as $\mathbb{P}^{n-2}$, linearly embedded in the Plücker embedding of $G(2,n)$ or in $\mathbb{P}^{n^*}$ respectively.

Following [3] we will say that a point x of $\mathbb{P}^n$ is a *k-fundamental point* of X if there is a k-dimensional family of lines of X through x. A point x of $\mathbb{P}^n$ is a *fundamental point* of X if it is k-fundamental for some $k \ge 1$. A curve C is a *k-fundamental curve* for X if all its points are k-fundamental. Given a congruence X, we will denote

- $a = a_0$ the number of lines of X passing through a general point in $\mathbb{P}^n$
- $b = a_1$ the number of lines of X contained in a general hyperplane H and meeting a general line of H

Equivalently, using the formulas for the Chern classes of the universal bundles, the above numbers can be written as $a = c_{n-1}(\mathcal{S}_{|X})$, $b = c_2(Q_{|X})c_{n-3}(\mathcal{S}_{|X})$.

2.2. Focal points. Intuitively, a focal point of a congruence X is a point in $\mathbb{P}^n$ through which there pass two infinitely close lines. A rigorous way of defining the focal locus is the following.

Definition 1. Let $I_X^0 \subset \mathbb{P}^n \times X$ be the incidence variety consisting of the pairs (x,L) for which l is a line of X and x is a point in the line L. And let $q_0 : I_X^0 \to \mathbb{P}^n$ be the first projection, which has degree a. The *focal locus* of X will be the branch locus F of q_0, and the elements of F will be called *focal points*. If (x,L) is in the ramification locus of q_0 we will say that x is a *focal point for the line* L.

Since $\dim I_X^0 = \dim \mathbb{P}^n = n$, the ramification locus R of q_0 has a scheme structure of pure dimension $n-1$. One should expect the focal locus to be a hypersurface of $\mathbb{P}^n$. Of course there are examples in which it has smaller dimension (see Section 5 or, for very degenerate cases, Remark 3.4); for instance, if X is the set of lines passing through a point $x \in \mathbb{P}^n$, then F is just the point x. When F has the expected dimension, their components of dimension $n-1$ have a natural structure of scheme, precisely the one given by $q_{0*}(R)$. We will refer to it as the *scheme structure of* F.

Proposition 2.1. *Let $X \subset G(1,n)$ be a congruence of multidegree $(a,b,\ldots)$ and let g' be the genus of the curve of X consisting of all the lines meeting a general line of $\mathbb{P}^n$. Then* *(i) If $a > 0$, the degree of the focal locus is, as a scheme, $2a + 2g' - 2$.*

(ii) For any line l of X, the set of focal points for L are the points for which a natural map $\varphi_l : \mathcal{O}_L^{n-1} \to \mathcal{O}_L(1)^{n-1}$ has not maximal rank. In particular, either all the points of L are focal for the line or the number of focal points for L (counted with multiplicities) is $n-1$.

Proof. For (i), if we restrict q_0 to the pull-back of a line L, we first observe that $q_0^{-1}(L)$ is the set of pairs (x, L') such that l' is a line of X and x lies in both L and L'. Therefore, for a general L, the set $q_0^{-1}(L)$ is isomorphic to the curve of X consisting of all the lines meeting L. A direct application of Hurwitz theorem to the restriction of q_0 to $q_0^{-1}(L)$ (which is surjective since $a > 0$) gives (i).

If instead we take now l to be a line of X, $q_0^{-1}(L)$ decomposes into two components: the set $\bar{L}$ of pairs (x, L), with x a point in L and the (closure of the) set of pairs (x, L') with $L' \neq L$ a line of X and x a point in both L and L'. The restriction of the ramification locus of q_0 to $\bar{L}$ will give the pairs (x, L) such that x is a focal point for L. We thus consider the following natural commutative diagram of exact sequences defining φ_l:

$$\begin{array}{ccccccccc}
0 & \to & T_{\bar{L}} & \to & T_{I_X^0|\bar{L}} & \to & N_{\bar{L}, I_X^0} & \to & 0 \\
 & & \downarrow & & \downarrow dq_{0|\bar{L}} & & \downarrow \varphi_l & & \\
0 & \to & q_0^* T_L & \to & q_0^* T_{\mathbb{P}^n|L} & \to & q_0^* N_{L,\mathbb{P}^n} & \to & 0
\end{array}$$

The first vertical arrow is an isomorphism because q_0 defines an isomorphism between $\bar{L}$ and L. Therefore the degeneracy locus of $dq_{0|\bar{L}}$ (which gives the focal points for L) coincides with the degeneration locus of φ_l. But now observe that $\bar{L}$ is, inside the incidence variety $I_X^0 \subset \mathbb{P}^n \times X$, the fiber of a point in X under the second projection (which endows X with a projective bundle structure). Therefore its normal bundle is trivial (of rank $n-1$). On the other hand, the normal bundle of L in $\mathbb{P}^n$ is $\mathcal{O}_L(1)^{n-1}$. Hence φ_l can be interpreted as in the statement of (ii), and so the lemma is proved. □

Remark 2.1. Observe that, if X smooth (in fact it would be enough to assume it is smooth in codimension two) most of the previous lemma could have been proved by computing the class of the ramification locus of $q_0 : I_X^0 \to \mathbb{P}^n$ (which is also interesting by itself). We remark first, as observed in the previous proof, that I_X^0 has a natural structure of projective bundle over X. More precisely $I_X^0 = \mathbb{P}(\mathcal{Q}_{|X})$, and the relative hyperplane section is the pull-back h of the hyperplane section of $\mathbb{P}^n$. This easily implies that the tangent bundle of I_X^0 has first Chern class $2h - H - K$, where H and K are the pull-back of the hyperplane and canonical classes of X (recall that $c_1(\mathcal{Q}_X) = H$). This yields that the class of the ramification locus of q_0 is $(n-1)h + H + K$. Hence, intersecting with the pull-back h^{n-1} of the class of a line in $\mathbb{P}^n$ we thus get intersection product $n - 1 + (H + K) \cdot h^{n-1} = 2a + 2g' - 2$, thus getting (i). On the other hand, intersecting with the pull-back of the class of a point in X we will get the coefficient $n-1$, which is thus the number of focal points for a line. Observe however that this will not give the determinantal description of the focal points on the line (when speaking about focal planes this kind of description will be crucial). Also, it is not easy a priori to figure out that the intersection product $(H + K) \cdot h^{n-1}$ involves g'.

2.3. Focal planes. Again intuitively, if x is a focal point for a line L of a congruence X, this means that through x there pass two infinitely close lines of X.

Since these should span a plane, this naturally yields to the notion of focal plane as a plane containing two infinitely close lines of the congruence. Again we give a precise definition.

Definition 2. Let $I_X^2 \subset X \times G(2,n)$ be the incidence variety consisting of the pairs (l,π) for which l is a line of X and Π is a plane containing the line L. And let $q_2 : I_X^2 \to G(2,n)$ be the second projection, which this time is not surjective if $n \geq 4$. The *locus of focal planes* of X will be the branch locus F_2 of q_2, and if (l,π) is in the ramification locus of q_2 we will say that Π is a *focal plane for the line* L.

We have that the degeneracy locus of dq_2 has expected codimension $n-2$ in I_X^2 (which in turn has dimension $2n-3$), hence we again expect F_2 to have dimension $n-1$. This coincides with the intuition that for a focal point there is a focal plane. This also means that a general line of the congruence should contain a finite number of focal planes for it (precisely $n-1$, according to our philosophy), as we will prove right away. However since F_2 is now a subscheme of $G(2,n)$ it has a multidegree (the number of whose coefficients will depend on n) instead of just a degree. To compute all of its coefficients it will be necessary to make a particular computation in each case by knowing the class of the ramification locus of q_2 in I_X^2 (see Remark 2.1). As a sample, we will do it for $n=4$ (the case $n=3$ is classical and can also be obtained by duality), which shows how complicated the formula becomes in general. We collect all these results in the following.

Proposition 2.2. *Let X be a congruence in $G(1,n)$ of multidegree $(a,b,\ldots)$. Then*

(i) If $n=4$, and F_2 has dimension three, then its bidegree in $G(2,4)$ (as a three-dimensional scheme) $(a',b') = (2b+4g''-4, 8b-2a+6g'+6g''+2K_H^2-12\chi(\mathcal{O}_H)-12)$, where g'' is the genus of the curve of X consisting of all the lines contained in a general hyperplane of $\mathbb{P}^4$ and H is a general hyperplane section of X. Here a' is the number of planes of F_2 contained in a general hyperplane of $\mathbb{P}^4$ and b' is the number of planes passing through a point and meeting along a line a general fixed plane through that point.

(ii) If l is a line of the congruence and P_l is the set of planes of $\mathbb{P}^n$ containing L, identifying P_l with $\mathbb{P}^{n-2}$, then the planes of P_l focal for X are those for which a natural map $\psi_l : \mathcal{O}_{P_l}^{n-1} \to T_{P_l}(-1)^2$ is not injective.

(iii) If the set of focal planes for L has dimension zero (the expected one), then they form a scheme of length $n-1$ whose span is a hyperplane of $\mathbb{P}^n$.

Proof. For (i), we use the same technique as in Remark 2.1, in this case with the map $q_2 : I_X^2 \to G(2,4)$. Now $I_X^2 = \mathbb{P}(\check{\mathcal{S}}_{|X})$, or equivalently it is the Grassmann bundle of rank-two quotients of $\mathcal{S}_{|X}$. As a Grassmann bundle, its universal rank-two quotient bundle $\mathcal{E}$ (which is $T_{I_X^2/X}(-1)$ if we see it as the projective bundle $\mathbb{P}(\check{\mathcal{S}}_{|X})$) is naturally isomorphic to $q_2^*\mathcal{Q}_2$. One next checks that the class of the ramification locus of q_2 is $K^2+2KH+2Kc_1(\mathcal{E})+H^2+3c_1(\mathcal{E})H+c_1(\mathcal{E})^2-c_2(T_X)+2c_2(\mathcal{E})+2C''$, where C'' is the (pull-back from X of the) curve of X consisting of the lines contained in a general hyperplane of $\mathbb{P}^4$. Intersecting this class with $c_3(\mathcal{O}_{I_X^2}^5 - \check{\mathcal{E}})$ and $c_1(\mathcal{E})c_2(\mathcal{E})$ we get respectively a' and b'.

For (ii) we proceed as in Proposition 2.1 by considering the following commutative diagram of exact sequences (defining ψ_l) in which we also identified P_l with the subset of I_X^2 of pairs (l,π), with $\pi \in P_l$ (i.e. with the first factor fixed)

$$\begin{array}{ccccccccc}
0 & \to & T_{P_l} & \to & T_{I_X^2|P_l} & \to & N_{P_l,I_X^2} & \to & 0 \\
 & & || & & \downarrow dq_{2|P_l} & & \downarrow \psi_l & & \\
0 & \to & T_{P_l} & \to & T_{G(2,n)|P_l} & \to & N_{P_l,G(2,n)} & \to & 0
\end{array}$$

It is clear now that the degeneracy locus of $dq_{2|P_l}$ is the degeneracy locus of ψ_l. We observe immediately that the normal bundle $N_{P_l,I_X{}^2}$ is a trivial bundle (whose fiber is canonically isomorphic to the tangent space of X at the point represented by L). In order to identify $N_{P_l,G(2,n)}$, we observe that P_l, as a subvariety of $G(2,n)$, is defined as the zero locus of two sections of the universal bundle $\mathcal{S}_2$, so that $N_{P_l,G(2,n)} \simeq \mathcal{S}_{2|P_l}^2$. On the other hand, the restriction of the universal quotient bundle $\mathcal{Q}_2$ of $G(2,n)$ to P_l is the rank 3 bundle $\mathcal{O}_{P_l}^2 \oplus \mathcal{O}_{P_l}(1)$. This shows that $\mathcal{S}_{2|P_l} \simeq T_{P_l}(-1)$ and hence we are interested in the degeneracy locus of a map $\psi_l : \mathcal{O}_{P_l}^{n-1} \to T_{P_l}(-1)^2$, which proves (ii).

We see then that the expected degeneracy locus of ψ_l is a zero-dimensional scheme Z of length $n-1$. If the degeneracy locus has the expected dimension, then it is easy to see from this description that the scheme Z is not degenerate in P_l, i.e. the focal planes are in general position. A fast way of seeing this is to observe that, if we consider the Euler sequence on P_l summed with itself

$$0 \to \mathcal{O}_{P_l}(-1)^2 \to \mathcal{O}_{P_l}^{2n-2} \xrightarrow{\psi} T_{P_l}(-1)^2 \to 0$$

then the map ψ_l necessarily factorizes through ψ. The induced map $\mathcal{O}_{P_l}(-1)^{n-1} \to \mathcal{O}_{P_l}^{2n-2}$ is necessarily injective since φ is. The degeneracy locus of ψ_l is thus the degeneracy locus of $\mathcal{O}_{P_l}(-1)^2 \to \mathcal{O}_{P_l}^{2n-2}/\mathcal{O}_{P_l}^{n-1} \simeq \mathcal{O}_{P_l}^{n-1}$. But this morphism provides a resolution for the ideal sheaf of Z that immediately shows that Z is not contained in any hyperplane of P_l (for instance, consider P_l as a hyperplane in $\mathbb{P}^{n-1}$, and represent the map $\mathcal{O}_{P_l}(-1)^2 \to \mathcal{O}_{P_l}^{n-1}$ to $\mathbb{P}^{n-1}$ by a suitable matrix of linear forms; then Z is a hyperplane section of a rational normal curve, hence nondegenerate). □

2.4. Focal hyperplanes. Finally, if we want to find the duality that occurs for $n=3$ (in which the focal planes are precisely the tangent planes to the focal surface) we will need to introduce the notion of focal hyperplane. The only possible definition does not seem a priori much natural, since it has not a nice geometric interpretation.

Definition 3. Let X be a congruence of lines in $\mathbb{P}^n$ and consider the incidence variety $I_X^{n-1} \subset X \times \mathbb{P}^{n^*}$ consisting of pairs (l,h) such that the hyperplane H contains the line L. Let $q_{n-1} : I_X^{n-1} \to \mathbb{P}^{n^*}$ be the second projection, which has no finite fibers for $n \geq 4$. The *locus of focal hyperplanes* F_{n-1} is the branch locus of q_{n-1}. If (L,H) is in the ramification locus of q_{n-1} then we will say that H is a *focal hyperplane for the line* L.

Observe that now the ramification locus is the scheme in which the differential map dq_{n-1} is not surjective, but anyway it still has expected codimension $n-2$ in I_X^{n-1}, i.e. expected dimension $n-1$. As for the other focal loci we can prove immediately some numerical results. However, even if this time F_{n-1} is a hypersurface in $\mathbb{P}^{n^*}$, it is not easy to give a general formula for its degree, so that again we will do it only for $n=4$.

Proposition 2.3. *Let X be a congruence in $G(1,n)$ of multidegree $(a,b,\ldots)$. Then* *(i) If $n=4$ and F_3 has dimension three, then its degree as a subscheme of $\mathbb{P}^{4^*}$ is $4g''-4+12\chi(\mathcal{O}_H)-K_H^2$, where H is a hyperplane section of X.*

(ii) Let l be a line of the congruence and let P_l^* be the set of hyperplanes containing L, which we will identify with $\mathbb{P}^{n-2}$. Then the focal hyperplanes for L are those for which a natural map $\xi_l : \mathcal{O}_{P_l^*}^{n-1} \to \mathcal{O}_{P_l^*}(1)^2$ is not surjective.

(iii) If the set of focal hyperplanes for a line l of X has (the expected) dimension zero, then it is a scheme of length $n-1$ which is nondegenerate in P_l^*.

Proof. For (i) we proceed in the same way as in the cases of points and planes, now using the map $q_3 : I_X^3 \to G(3,4) \cong \mathbb{P}^{4^*}$. We can regard I_X^3 as the projective bundle $\mathbb{P}(\mathcal{S}_{|X})$, and in this case the tautological hyperplane section is the pull-back of the hyperplane section h^* of $\mathbb{P}^{4^*}$. We check thus that the degeneracy locus of dq_3 has class $H^2+3Hh^*+KH+3h^{*2}+2h^*K-C''+c_2(T_X)$. Intersecting now with h^{*3} we get the wanted degree.

For (ii) and (iii) we proceed as in Propositions 2.1 and 2.2. Identify P_l^* with the subset of I_X^{n-1} consisting of the pairs of the form (l,h) with $H \in P_l^*$. Then we have the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \to & T_{P_l^*} & \to & T_{I_X^{n-1}|P_l^*} & \to & N_{P_l^*,I_X^{n-1}} & \to & 0 \\
 & & || & & \downarrow dq_{n-1|P_l^*} & & \downarrow \xi_l & & \\
0 & \to & T_{P_l^*} & \to & T_{\mathbb{P}^{n^*}|P_l^*} & \to & N_{P_l^*,\mathbb{P}^{n^*}} & \to & 0
\end{array}$$

As in the previous lemmas, the degeneracy locus of $dq_{n-1|P_l^*}$ is the set of pairs (l,h) for which H is focal for L, and it coincides with the degeneracy locus of ξ_l. And also $N_{P_l^*,I_X^{n-1}}$ is trivial of rank $n-1$ and $N_{P_l^*,\mathbb{P}^{n^*}}$ is isomorphic to $\mathcal{O}_{P_l^*}(1)^2$. This proves (ii), and (iii) is then an immediate consequence of (ii). □

3. Relation among the different focal loci.

3.1. General results. We consider now the incidence variety $I_X^{0,2} \subset \mathbb{P}^n \times X \times G(2,n)$ of triples (x,l,π) such that $x \in L \subset \Pi$ and consider the projection $q_{0,2} : I_X^{0,2} \to I_{0,2}$, where $I_{0,2} \subset \mathbb{P}^n \times G(2,n)$ is the flag variety of pairs (x,π) with $x \in \Pi$. It is natural to consider now the branch locus of $q_{0,2}$ and compare with F and F_2. The first remark is that it has expected dimension $n-1$. We have the following.

PROPOSITION 3.1. *Let X be a congruence of lines in $\mathbb{P}^n$ and let $F_{0,2}$ be the branch locus of $q_{0,2}$.*

(i) For any line L of the congruence, let P_l be the set of planes containing L. Then the set of pairs $(x,\pi) \in L \times P_l$ for which (x,l,π) is in the ramification locus of $q_{0,2}$ is the degeneracy locus of a natural map $\alpha_l : \mathcal{O}_{L\times P_l}^{n-1} \to q_{02}^* N_{L\times P_l, I_{0,2}}$ induced by φ_l and ψ_l.

(ii) If there are finitely many pairs as above, then they form a zero-dimensional scheme of length $n-1$.

(iii) If there are infinitely many such pairs, then the projection from them to L is surjective, and therefore all the points of L are focal for L.

Proof. The first part (i) is completely analogous to Propositions 2.1, 2.2 and 2.3, the only –but not significant– difference being that the projection from $I_X^{0,2}$ to X does not give a structure of projective bundle, but of ($\mathbb{P}^1 \times \mathbb{P}^{n-2}$)-bundle (the fibers being $L \times P_l$). In particular, the ramification locus of $q_{0,2}$ is the degeneracy locus of a natural map $\alpha_l : N_{L\times P_l, I_X^{0,2}} \to q_{02}^* N_{L\times P_l, I_{0,2}}$ and it holds $N_{L\times P_l, I_X^{0,2}} \cong \mathcal{O}_{L\times P_l}^{n-1}$.

To relate α_l with φ_l and ψ_l, observe now that we have an exact sequence of normal bundles

$$0 \to N_{L\times P_l, I_{0,2}} \xrightarrow{\alpha} N_{L\times P_l, \mathbb{P}^n\times G(2,n)} \xrightarrow{\beta} N_{I_{0,2},\mathbb{P}^n\times G(2,n)|L\times P_l} \to 0.$$

We have natural isomorphisms

$$N_{L\times P_l,\mathbb{P}^n\times G(2,n)} \cong pr_1^* N_{L,\mathbb{P}^n} \oplus pr_2^* N_{P_l,G(2,4)} \;\cong\; \mathcal{O}_{L\times P_l}(1,0)^{n-1} \oplus T_{P_l}(0,-1)^2$$

(where $\mathcal{O}_{L\times P_l}(a,b)$ stands for $pr_1^*\mathcal{O}_L(a) \oplus pr_2^*\mathcal{O}_{P_l}(b)$, pr_1 and pr_2 being the projections from $L\times P_l$ to L and P_l respectively, and we identify T_{P_l} with its pull-back by pr_2), and

$$N_{I_{0,2},\mathbb{P}^n\times G(2,n)|L\times P_l} \cong T_{P_l}(1,-1)$$

(for the latter observe that $I_{0,2}$ is the zero locus in $\mathbb{P}^n \times G(2,n)$ of a natural section of the tensor product of the pull-backs of $\mathcal{O}_{\mathbb{P}^n}(1)$ by the first projection and the rank-$(n-2)$ bundle $\mathcal{S}_2$ by the second projection).

With these identifications, a diagram chase shows that the composition of α_l with $q_{02}^*\alpha$ is nothing but (φ_l, ψ_l), which completes the proof of (i).

We have also a natural decomposition of the exact sequence of normal bundles into the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathcal{O}_{L\times P_l}(1,-1) & \to & \mathcal{O}_{L\times P_l}(1,0)^{n-1} & \to & T_{P_l}(1,-1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & N_{L\times P_l,I_{0,2}} & \xrightarrow{\alpha} & \mathcal{O}_{L\times P_l}(1,0)^{n-1} \oplus T_{P_l}(0,-1)^2 & \xrightarrow{\beta} & T_{P_l}(1,-1) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & T_{P_l}(0,-1)^2 & = & T_{P_l}(0,-1)^2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The expected codimension of the dependency locus of $n-1$ sections of $N_{L\times P_l,I_{0,2}}$ is $n-1$, hence the expected dimension is zero. And since the degree of the $(n-1)$-th Chern class of this normal bundle is $n-1$, we get (ii).

If instead $n-1$ sections of $N_{L\times P_l,I_{0,2}}$ are dependent along a set of positive dimension, assume for contradiction that the image of that set in L is not the whole line. This means that this dependency locus contains a subset of the form $\{x\}\times Y$ for some $Y \subset P_l$ of positive dimension. These $n-1$ sections of $N_{L\times P_l,I_{0,2}}$ induce $n-1$ sections of $T_{P_l}(-1)^2$, whose dependency locus is precisely the set of focal planes for L. We are assuming that there is a non-trivial linear combination of the $n-1$ sections producing a section of $T_{P_l}(-1)^2$ vanishing at Y. Thus the first column of the previous diagram indicates that this provides on $\{x\}\times Y$ a section of the restriction of $\mathcal{O}_{L\times P_l}(1,-1)$, in other words, a section of $\mathcal{O}_Y(-1)$, which is absurd. This proves that the dependency locus of the $n-1$ sections of $N_{L\times P_l,I_{0,2}}$ dominates L, just proving (iii). □

We have the following equivalent description of focal points and planes in terms of the tangent space of the congruence (see [7], Lemma 4.4, for $n=3$).

Lemma 3.1. *Let X be a line congruence in $\mathbb{P}^n$. A point $x \in \mathbb{P}^n$ is focal if and only if there exists a line l of the congruence, with $x \in L$, such that the embedded tangent*

$(n-1)$-plane of X at l meets the Schubert variety $\Omega(x,\mathbb{P}^n)$ in at least a line (i.e. a pencil of lines of $\mathbb{P}^n$). Similarly, a plane $\Pi \subset \mathbb{P}^n$ is focal if and only if there exists a line l of the congruence, with $L \subset \Pi$, such that the embedded tangent $(n-1)$-plane of X at l meets the Schubert variety $\Omega(L,\Pi)$ in at least a line.

Proof. This just comes from the observation that a pair (x,π) belongs to $F_{0,2}$ if and only if the pencil $\Omega(x,\Pi)$ contains L with multiplicity at least two, i.e. it is contained in the embedded tangent space of X at l. □

This implies the following (see also [2] or a proof in coordinates in Subsection 4.1).

THEOREM 3.1. *Let X be a line congruence in $\mathbb{P}^n$ and let l be a line of X with exactly $n-1$ pairs $(x_1,\pi_1),\dots,(x_{n-1},\pi_{n-1})$ such that (x_i,L,π_i) is in the ramification locus of $q_{0,2}$. Then L is tangent to the focal locus F at $x_1,\dots,x_{n-1}$ (provided they are smooth for the focal locus, as a scheme) and moreover each Π_i is tangent to F at the points $x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n-1}$. In particular, the tangent hyperplane to F at x_i is generated by $\Pi_1,\dots,\Pi_{i-1},\Pi_{i+1},\dots,\Pi_{n-1}$.*

Proof. We have by assumption that in $\Omega(x_i,\Pi_i)$ there is an infinitesimally close line to l in X. On the other hand, this infinitesimally close line will contain a focal point infinitesimally close to each of $x_1,\dots,x_{n-1}$ (at least if x_i is a smooth point of the focal locus). But it is clear that the direction in which x_i is approached by its infinitesimally close focal point is necessarily the direction of L, so that L is indeed tangent to the focal locus at x_i. Instead, each other x_j is approached in a direction different from L (but contained in Π_i). Since as we just proved L is tangent to F at x_j, we find at x_j two different directions tangent to F and contained in Π_i. Hence Π_i is contained in the tangent space to F at x_j. By Proposition 2.2, the planes $\Pi_1,\dots,\Pi_{i-1},\Pi_{i+1},\dots,\Pi_{n-1}$ span a hyperplane, so that this must be necessarily the tangent hyperplane to F at x_i. □

REMARK 3.2. The hypothesis that the points are smooth for the focal locus means in particular that we are assuming the focal locus to have the expected dimension $n-1$, otherwise the points on the corresponding scheme structure are not smooth. For instance, the congruence of bisecants to a twisted cubic in $C \subset \mathbb{P}^3$ has as focal locus exactly C, but of course the previous lemma does not hold.

Of course the next natural step would be to show that the tangent hyperplanes we have found correspond exactly to the $n-1$ focal hyperplanes containing L. Even if this is true, it is not possible to prove it now (we will need the local coordinates introduced in the next section). The reason is that, due to the different nature of the definition of focal planes and hyperplanes, taking an incidence diagram putting them together does not work.

3.2. Special cases. Most of what we have done so far does not work if all the points of a line are focal to it. So we want to analyze this special case.

DEFINITION 4. A *focal line* of a congruence X is a line L whose points are all focal.

From Lemma 3.1, a line L is focal if and only if the embedded tangent space of X at l meets in a pencil all the varieties $\Omega(x,\mathbb{P}^n)$ with $x \in L$.

The following proposition is a characterization of the focal lines.

Proposition 3.2. *A line L of a congruence X in $\mathbb{P}^n$ is focal if and only if the focal planes for L are not in general position, i.e. they are all contained in a common hyperplane of $\mathbb{P}^n$.*

Proof. The *if* part is an easy consequence of Propositions 2.2 and 3.1. If the focal planes for L are not in general position, then they cannot be a finite number (Proposition 2.2). But then the focal points corresponding to the infinitely many focal planes fill up the whole L (Proposition 3.1).

We prove now the *only if* part. So assume L is a focal line, and hence for each $x \in L$ we have a focal plane $\Pi \supset L$ such that the pencil $\Omega(x, \Pi)$ is contained in the tangent space to X at l. This provides a map $\varphi : L \to P_l$, where again P_l is the $(n-2)$-projective space of all the planes in $\mathbb{P}^n$ containing L, and we want to prove that its image is degenerate.

Consider the projective space $\mathbb{P}^{2n-3}$ of all the tangent directions of $G(1, n)$ at the point represented by L. The set of those tangent directions corresponding to pencils are thus identified with the Segre embedding of $L \times P_l$. The above set of pencils is therefore given by the image of $L \xrightarrow{id \times \varphi} L \times P_l \subset \mathbb{P}^{2n-3}$, and we know that this image is contained in a linear space of dimension $n-2$, namely the set of tangent directions of X at the point represented by L.

Choosing homogeneous coordinates t_0, t_1 for L, the map φ will be defined by $(t_0 : t_1) \mapsto (f_0 : \ldots : f_{n-2})$, where $f_0, \ldots, f_{n-2}$ are homogeneous polynomials in t_0, t_1 of the same degree. Assume by contradiction that the image of φ is nondegenerate. This means that $f_0, \ldots, f_{n-2}$ are linearly independent forms. But then it is easy to see that the homogeneous polynomials $t_0 f_0, \ldots, t_0 f_{n-2}, t_1 f_0, \ldots, t_1 f_{n-2}$ span a linear space of dimension at least n. But this is absurd, since these polynomials define the map $L \to \mathbb{P}^{2n-3}$, and, as we remarked, its image is contained in a linear space of dimension $n-2$. □

Let X_0 be the open set of non-focal lines of a congruence X and consider the map $q_0 : I_X^0 = p_2^{-1}(X) \to \mathbb{P}^n$, where p_2 is the second projection of $I_X \subset \mathbb{P}^n \times X$. Then the restriction of the second projection $p_2^{-1}(X_0) \to X_0$ to the ramification locus of q_0 is finite (typically of degree $n-1$, but it could happen a priori that any line contains less focal points counted with multiplicity). Hence, the branch locus of this restriction has at most $n-1$ components.

Definition 5. We will call the *strict focal locus* of a congruence X the closure F_0 of the reduced structure of the branch locus of $p_2^{-1}(X_0) \to \mathbb{P}^n$. To distinguish from this, we usually refer to the focal locus F (as a scheme) as the *total focal locus*.

We will now analyze some special behaviors of X_0 and F_0.

First of all, X_0 could be empty and this means that all the lines of X are focal lines. This case is characterized in the following theorem.

Theorem 3.3. *For a given congruence X in $\mathbb{P}^n$, all the lines of X are focal if and only if the union of all the lines of X is a proper subvariety of $\mathbb{P}^n$ (i.e. if $a = 0$).*

Proof. If the union of all the lines of X is the whole $\mathbb{P}^n$, this means that the morphism $q_0 : I_X^0 \to \mathbb{P}^n$ is generically finite, so that a general point $\mathbb{P}^n$ (which is a general point in a general line of X) is not in the branch locus of q_0, i.e. it is not focal. Reciprocally, if the union of all the lines of X is a proper subvariety $V \subset \mathbb{P}^n$, then for any point in a line of X (i.e. a point in V) the fiber of q_0 is not finite, and therefore these points are fundamental and hence focal. □

Remark 3.4. For a general congruence, F_0 is a hypersurface.

The cases in which F_0 has a very low dimension are easy to describe.

If F_0 is a point x, then X is $\Omega(x, \mathbb{P}^n)$.

If F_0 is a curve, then it is an $(n-1)$-*fundamental curve*, (i.e. all lines of X intersect F_0). Smooth congruences with an $(n-1)$-fundamental curve are classified in [3].

Remark 3.5. It is not superfluous to take the reduced structure in the definition of F_0. As we have seen for $n = 3$ in [4] (and we will see in the sequel), the focal locus can appear with high multiplicity, for instance for congruences of $(n-1)-$tangents to an $(n-1)$-fold in $\mathbb{P}^n$ (see Section 6).

Remark 3.6. Also the total focal locus could have more components different from F_0, for example for the congruences of $(n-1)-$secants to a $(n-2)$-fold in $\mathbb{P}^n$ (see [4] for the case $n = 3$ or Section 5).

4. Local description of the focal loci and duality. The goal of this section is to show that the focal hyperplanes of a congruence of lines in $\mathbb{P}^n$ are tangent to the focal hypersurface. To this purpose, we will need to use local coordinates for the focal loci, which we will also allow to recover some of the results we have already proved.

4.1. Local coordinates. We first want to give local coordinates for the incidence variety $I_X^{0,2}$ of triples (x, l, π) such that $l \in X$. We fix first a triple $(\overline{x}, \overline{l}, \overline{\pi})$ in $I_X^{0,2}$. We first choose homogeneous coordinates $(x_0 : x_1 : \ldots : x_n)$ in $\mathbb{P}^n$ such that the point $\overline{x}$ has coordinates $(1 : 0 : \ldots : 0)$, the line $\overline{L}$ has equations $x_2 = x_3 = \ldots = x_n = 0$ and the plane $\overline{\pi}$ has equations $x_3 = \ldots = x_n = 0$. We can thus take $u_1, \ldots, u_{n-1}$ to be a system of parameters of X at $\overline{l}$ and assume that near $\overline{l}$ the lines of the congruence are given by the span of the rows of the matrix

$$\begin{pmatrix} 1 & 0 & f_2 & f_3 & \ldots & f_n \\ 0 & 1 & g_2 & g_3 & \ldots & g_n \end{pmatrix} \tag{1}$$

where f_i and g_j are regular functions on the variables $u_1, \ldots, u_{n-1}$ in a neighborhood of $\overline{l}$ such that $f_2(0, \ldots, 0) = \ldots = f_n(0, \ldots, 0) = g_2(0, \ldots, 0) = \ldots = g_n(0, \ldots, 0) = 0$. Hence our choice of coordinates of $I_X^{0,2}$ near $(\overline{x}, \overline{l}, \overline{\pi})$ will be the following: the coordinates $(\lambda, u_1, \ldots, u_{n-1}, v_3, \ldots, v_n)$ will represent the triple (x, l, π) in which l is the line corresponding to $u_1, \ldots, u_{n-1}$ (i.e. the one spanned by the rows of the above matrix), x is the point $x = (1 : \lambda : f_2 + \lambda g_2 : \ldots : \ldots : f_n + \lambda g_n)$ (i.e. the first row plus λ times the second row of the matrix) and Π is the plane given by the following equations:

$$\Pi : \begin{cases} (x_3 - f_3 x_0 - g_3 x_1) + v_3(x_2 - f_2 x_0 - g_2 x_1) = 0 \\ \vdots \\ (x_n - f_n x_0 - g_n x_1) + v_n(x_2 - f_2 x_0 - g_2 x_1) = 0 \end{cases} \tag{2}$$

Observe that if we take only coordinates $(\lambda, u_1, \ldots, u_{n-1})$ we have a local parametrization of I_X^0 near $(\overline{x}, \overline{l})$, while if we take coordinates $(u_1, \ldots, u_{n-1}, v_3, \ldots, v_n)$ we get a parametrization of I_X^2 near $(\overline{l}, \overline{\pi})$.

Now we want to study the injectivity of the differential map of $q_{0,2} : I_X^{0,2} \to I_{0,2}$. It will be easier (as we did in fact in Proposition 3.1) to consider the map as a map from $I_X^{0,2}$ to $\mathbb{P}^n \times G(2, n)$, i.e. to study its two components $q'_0 : I_X^{0,2} \to \mathbb{P}^n$ and $q'_2 : I_X^{0,2} \to G(2, n)$. We first choose affine coordinates for $\mathbb{P}^n$ and $G(2, n)$ around

$\overline{x}$ and $\overline{\pi}$. Of course for $\mathbb{P}^n$ we just consider the open set $x_0 \neq 0$ and take affine coordinates $x_1, \ldots, x_n$.

A local expression for q_0' is given by

$$(\lambda, u_1, \ldots, u_{n-1}, v_3, \ldots, v_n) \mapsto (\lambda, f_2 + \lambda g_2, \ldots, f_n + \lambda g_n).$$

About $G(2, n)$, we consider the affine set of the planes not meeting the space $x_0 = x_1 = x_2 = 0$. Then the coordinates $a_3, \ldots, a_n, b_3, \ldots, b_n, c_3, \ldots, c_n$ will represent the plane of equations

$$\begin{cases} x_3 = a_3 x_0 + b_3 x_1 - c_3 x_2 \\ \vdots \\ x_n = a_n x_0 + b_n x_1 - c_n x_2 \end{cases}.$$

With this choice, a local expression for q_2' is given by

$$(\lambda, u_1, \ldots, u_{n-1}, v_3, \ldots, v_n) \mapsto$$
$$(f_3 + v_3 f_2, \ldots, f_n + v_n f_2, g_3 + v_3 g_2, \ldots, g_n + v_n g_2, v_3, \ldots, v_n).$$

The Jacobian matrix A of the map (q_0', q_2') without considering the first row and the first column is of the form:

$$A = \left(\begin{array}{c|c|c|c|c} 1 & g & 0 & 0 & 0 \\ \hline 0 & B & C_1 & C_2 & 0 \\ \hline 0 & 0 & f_2 I_{n-2} & g_2 I_{n-2} & I_{n-2} \end{array} \right)$$

where the blocks B, C_1 and C_2 are respectively the following (where f_{ij} stands for $\frac{\partial f_i}{\partial u_j}$ and similarlyg_{ij} stands for $\frac{\partial g_i}{\partial u_j}$):

$$B = \begin{pmatrix} f_{21} + \lambda g_{21} & f_{31} + \lambda g_{31} & \ldots & f_{n1} + \lambda g_{n1} \\ f_{22} + \lambda g_{22} & f_{32} + \lambda g_{32} & \ldots & f_{n2} + \lambda g_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ f_{2,n-1} + \lambda g_{2,n-1} & f_{2,n-1} + \lambda g_{2,n-1} & \ldots & f_{n,n-1} + \lambda g_{n,n-1} \end{pmatrix}$$

$$C_1 = \begin{pmatrix} f_{31} + v_3 f_{21} & \ldots & f_{n1} + v_n f_{21} \\ f_{32} + v_3 f_{22} & \ldots & f_{n2} + v_n f_{22} \\ \vdots & \ddots & \vdots \\ f_{3,n-1} + v_3 f_{2,n-1} & \ldots & f_{n,n-1} + v_n f_{2,n-1} \end{pmatrix}$$

$$C_2 = \begin{pmatrix} g_{31} + v_3 g_{21} & \ldots & g_{n1} + v_n g_{21} \\ g_{32} + v_3 g_{22} & \ldots & g_{n2} + v_n g_{22} \\ \vdots & \ddots & \vdots \\ g_{3,n-1} + v_3 g_{2,n-1}, & \ldots & g_{n,n-1} + v_n g_{2,n-1} \end{pmatrix}.$$

Notice that, with these notations and coordinates, the Jacobian matrix of $q_0 : I_X^0 \to \mathbb{P}^n$ is

$$\left(\begin{array}{c|c} 1 & g \\ \hline 0 & B \end{array}\right).$$

Hence, for fixed values of the $(n-1)$-tuple $(u_1, \dots, u_{n-1})$ (i.e. for a fixed line L of the congruence), the determinant of B vanishes at the values of λ that give the focal points for L. Observe that this determinant is expected to be a polynomial of degree $n-1$ in λ, hence we just verified locally Proposition 2.1 (in fact, B can be regarded as the matrix of φ_l).

Similarly, the Jacobian matrix of $q_2 : I_X^2 \to G(2, n)$ is

$$\left(\begin{array}{c|c|c} C_1 & C_2 & 0 \\ \hline f_2 I_{n-2} & g_2 I_{n-2} & I_{n-2} \end{array}\right).$$

Hence, again for fixed values of $u_1, \dots, u_{n-1}$, the values of the $(n-2)$-tuple $(v_3, \dots, v_n)$ for which the rank of the $(n-1) \times 2(n-2)$ matrix $C = (C_1|C_2)$ is less than $n-1$ correspond to the focal planes for the given line. Now a priori it is not clear that we expect $n-1$ solutions, but again C can be regarded as the matrix of ψ_l.

Putting both parts together, a focal point and a focal plane correspond to each other if and only if they are in the ramification locus of the map (q_0, q_2), hence if and only if $(B|C)$ (the matrix of $\alpha \circ \alpha_l$), has rank less than $n-1$ at the pair.

Now we want to use these coordinates to re-prove Theorem 3.1, i.e. to describe the tangent hyperplane at $\overline{x}$ to F in terms of the focal planes at the other focal points for $\overline{l}$. For simplicity we assume that $\overline{l}$ contains exactly $n-1$ distinct focal points $\overline{x}, y_1, \dots, y_{n-2}$ and is contained in $n-1$ focal planes $\overline{\pi}, \pi_1, \dots, \pi_{n-2}$. We want to verify that each of the π_j (but not in general $\overline{\pi}$) is tangent to F at $\overline{x}$. Observe that a local parametrization of F near $\overline{x}$ is given by

$$(u_1, \dots, u_{n-1}) \mapsto (\lambda, f_2 + \lambda g_2, \dots, f_n + \lambda g_n) \tag{3}$$

where λ is implicitly parametrized in terms of $u_1, \dots, u_{n-1}$ from the equation $\det B = 0$ and we take the branch for which $\lambda(0, \dots, 0) = 0$.

Substituting these parametrization in equations (2), it follows that the plane defined by these equations is tangent at $\overline{x}$ if and only if the matrix D given by the first $n-2$ columns of the matrix C has rank less than $n-2$. We know that C has rank less than $n-1$ at the values of the $(n-2)$-tuple $(v_3, \dots, v_n)$ that correspond to the focal planes $\overline{\pi}, \pi_1, \dots, \pi_{n-2}$. We have now two possibilities for each of these values. One of them is that the rank of D is less than $n-2$ and therefore the corresponding focal plane is tangent at $\overline{x}$, as wanted. If instead D has rank $n-2$, then the columns of D are independent and any other column of C depends linearly on them. Let λ be the value that gives the corresponding focal point to the focal plane. For this λ the whole matrix $(B|C)$ is not of maximal rank. In particular each column of B depends on the columns of D. This gives the only solution $\lambda = 0$, hence the corresponding focal point is exactly $\overline{x}$, i.e. the focal plane for this solution was $\overline{\Pi}$, which does not need to be tangent to F. We thus recovered Theorem 3.1.

4.2. Focal points and focal hyperplanes: a duality result. In this section we will parametrize locally F_{n-1} at a smooth hyperplane $\overline{H}$. This will allow us to prove a duality result, i.e. to show that, for general congruences, the focal variety F_{n-1} is dual to the focal variety F_0. The precise result we will prove in this section is the following:

THEOREM 4.1. *Let X be a congruence of lines in $\mathbb{P}^n$. Let $\bar{l}$ be a line of X with exactly $n-1$ focal points for it. Then each tangent hyperplane to each of these points is focal. In particular, if $\bar{l}$ is contained in exactly $n-1$ focal hyperplanes for it, they are exactly the $n-1$ tangent hyperplanes to F_0 at the focal points for $\bar{l}$.*

Proof. We choose coordinates as in Section 4.1, but now assuming also that $\overline{H} : X_n = 0$ is a focal hyperplane for $\bar{l} : x_2 = \ldots = x_n = 0$. Following the notations of Section 2.4, we need to study the rank of the Jacobian matrix at $(\bar{l}, \overline{H})$ of the map $q_{n-1} : I_X^{n-1} \to G(n-1, n)$. For this we choose local parameters $u_1, \ldots, u_{n-1}, \delta_2, \ldots, \delta_{n-1}$ for I_X^{n-1} near $(\bar{l}, \overline{H})$ to represent the pair (l, H), where l is the line generated by the rows of the matrix (1) of Section 4.1 and H is the hyperplane of equation

$$x_n - f_n x_0 - g_n x_1 + \delta_2(x_2 - f_2 x_0 - g_2 x_1) + \ldots + \delta_{n-1}(x_{n-1} - f_{n-1}x_0 - g_{n-1}x_1) = 0.$$

On the other hand, we can take affine coordinates $d_0, \ldots, d_{n-1}$ in $\mathbb{P}^{n^*}$ to represent the hyperplane $x_n = d_0 x_0 + d_1 x_1 - d_2 x_2 - \ldots - d_{n-1}x_{n-1}$

With these coordinates, a local expression for q_{n-1} is given by

$$(u_1, \ldots, u_{n-1}, \delta_2, \ldots, \delta_{n-1}) \mapsto$$
$$(f_n + \delta_2 f_2 + \ldots + \delta_{n-1}f_{n-1}, g_n + \delta_2 g_2 + \ldots + \delta_{n-1}g_{n-1}, \delta_2, \ldots, \delta_{n-1}).$$

Hence the Jacobian matrix A' of q_{n-1}, is:

$$A' = \left(\begin{array}{c|c} B' & 0 \\ \hline C' & I_{n-2} \end{array}\right)$$

where the blocks B' and C' are respectively the following:

$$B' = \begin{pmatrix} f_{n1} + \delta_2 f_{21} + \ldots + \delta_{n-1}f_{n-1,1} & g_{n1} + \delta_2 g_{21} + \ldots + \delta_{n-1}g_{n-1,1} \\ f_{n2} + \delta_2 f_{22} + \ldots + \delta_{n-1}f_{n-1,2} & g_{n2} + \delta_2 g_{22} + \ldots + \delta_{n-1}g_{n-1,2} \\ \vdots & \vdots \\ f_{nn} + \delta_2 f_{2n} + \ldots + \delta_{n-1}f_{n-1,n} & g_{nn} + \delta_2 g_{2n} + \ldots + \delta_{n-1}g_{n-1,n} \end{pmatrix}$$

$$C' = \begin{pmatrix} f_2 & g_2 \\ f_3 & g_3 \\ \vdots & \vdots \\ f_{n-1} & g_{n-1} \end{pmatrix}.$$

It is now immediate to see that the points of the focal variety F_{n-1} are the hyperplanes given by the coordinates $\delta_2, \dots, \delta_{n-1}$ for which the two first columns of A' are dependent.

In order to prove the theorem, we now consider a hyperplane H' of F_{n-1} containing $\overline{l}$ and want to see when it is tangent to F_0 at the point $\overline{x}$. The equation for H' has the form

$$\delta_2 x_2 + \ldots + \delta_{n-1} x_{n-1} + x_n = 0.$$

Substituting in the equation of the hyperplane H' the parametrization (3) of F_0 near $\overline{x}$ given in Section 4.1 we obtain

$$\delta_2(f_2 + \lambda g_2) + \ldots + \delta_{n-1}(f_{n-1} + \lambda g_{n-1}) + f_n + \lambda g_n = 0.$$

The partial derivatives of the above expression with respect to the coordinates $(u_1, u_2, \dots, u_{n-1})$ evaluated at $(u_1, u_2, \dots, u_{n-1}) = (0, \dots, 0)$ are

$$\delta_2 f_{2i} + \ldots + \delta_{n-1} f_{n-1,i} + f_{ni}, \qquad i = 1, \dots, n-1.$$

Hence H' is tangent to F at $\overline{x}$ if and only if all these partial derivatives vanish, i.e. the first column of A' vanish. This proves that any tangent hyperplane to F at each of the $n-1$ focal points for $\overline{l}$ is a focal hyperplane.

We also deduce from Theorem 3.1 and Proposition 2.2 that the $n-1$ tangent hyperplanes at the $n-1$ focal points for $\overline{l}$ are different (the span of two of them must be the span of the focal planes for the line, which is $\mathbb{P}^n$). Hence, if there are exactly $n-1$ focal hyperplanes for $\overline{l}$, then they are the tangent hyperplanes. □

Remark 4.2. A classical example confirming the above duality is the following. Consider in $G(2,4)$ the family Y^* of trisecant planes to a rational normal curve Γ. Since Y^* is parametrized by the set of effective divisors of degree three of Γ, it follows that it is isomorphic to $\mathbb{P}^3$. A precise isomorphism can be obtained any time we fix a point $P \in \Gamma$, if we identify $\mathbb{P}^3$ with the set of hyperplanes passing through P: to any plane in Y^* defined by the divisor D on Γ we associate the hyperplane spanned by the divisor $D + P$. So if we fix two points $P, Q \in \Gamma$, we get in this way an isomorphism between the set of hyperplanes through P and the set of hyperplanes through Q. With this setting, each plane of Y^* is obtained as the intersection of a hyperplane through P and its corresponding hyperplane through Q. Dualizing, we get two hyperplanes in $\mathbb{P}^4$, an isomorphism between them and a congruence $Y \subset G(1,4)$ consisting of the lines joining pairs of corresponding points of both hyperplanes. This is nothing that the embedding of $\mathbb{P}^3$ in $G(1,4)$ by the vector bundle $\mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(1)$, which in fact comes projected from $G(1,7)$ (see [1]), i.e. a Veronese variety after the Plücker embedding. We thus get $a = 4, b = 2, g' = 0, g'' = 0, \chi(\mathcal{O}_H) = 1, K_H^2 = 8$.

We then get from our formulas that F has degree 6, while F_2 has bidegree $(0,0)$ and F_3 has degree 0. The geometric explanation is easy. A focal point for Y corresponds in the dual space to a hyperplane that is tangent to Γ. In other words, F is the dual hypersurface of Γ, hence of degree 6 and of course with class 0. Similarly, a focal plane to Y corresponds by duality to a bisecant line to Γ. Therefore F_2 has dimension two, hence bidegree $(0,0)$ as a threefold. And finally a focal hyperplane to Y corresponds by duality to a point in Γ, hence F_3 has dimension one, therefore its degree as a hypersurface is 0.

5. Congruences of $(n-1)$-secants to an $(n-2)$-fold in $\mathbb{P}^n$. When the expected focal locus in not a hypersurface, any $(n-1)$-dimensional component of the focal locus is special. This is what happens for a congruence of $(n-1)$-secants to an $(n-2)$-fold, in which one expects the $(n-2)$-fold to be the only focal locus. We will devote this section to show that, as it happens in the case $n=3$, one should also expect some $(n-1)$-dimensional part of the focal locus, which we will analyze, at least in the smooth case. We will also see that, imposing that both the $(n-2)$-fold and the congruence are smooth implies for $n=4$ that there is only a finite number of families.

5.1. General study. Throughout this section Σ will be a smooth irreducible $(n-2)$-fold in $\mathbb{P}^n$, not contained in a hyperplane, and we will denote by $X \subset G(1,n)$ the set of the $(n-1)$-secants to Σ. We will also assume that X has the expected dimension and it is irreducible, i.e. it is a congruence. We will study the tangent space of X at a sufficiently general line, so that from Theorem 3.1 we can understand the focal locus of X.

LEMMA 5.1. *Let Σ be as above and let $\mathcal{Z} \subset G(1,n)$ be the Chow complex of lines intersecting Σ. Let x be a point in Σ, consider a line l passing through x, and denote by $t_x \subset \mathbb{P}^n$ the tangent $(n-2)$-plane to Σ at x. Then*

- *(i) The branch of $\mathcal{Z}$ corresponding to x is smooth at l if and only if l is not contained in t_x.*
- *(ii) In the situation of (i), the embedded tangent space of this branch of $\mathcal{Z}$ at l is spanned by the Schubert cycle $\Omega(x,\mathbb{P}^n)$ (of lines passing through x) and by the tangent space at l to the Schubert variety $\Omega(n-2,T_x)$ (of lines contained in the hyperplane $T_x \subset \mathbb{P}^n$ generated by L and t_x).*
- *(iii) The intersection of $G(1,n)$ with the embedded tangent space of the branch of $\mathcal{Z}$ at l is exactly the union of $\Omega(x,\mathbb{P}^n)$ and the Schubert cycle $\Omega(L,T_x)$ of lines in T_x meeting L (which coincides with the tangent space at l to $\Omega(n-2,T_x)$).*

Proof. This is a local computation. Choose coordinates $z_0, z_1, \ldots, z_n$ in $\mathbb{P}^n$ so that the point x becomes $(1:0:\ldots:0)$ and the line L has equations $z_2=\ldots=z_n=0$. Choose local parameters $\alpha_1,\ldots,\alpha_{n-2}$ of Σ at x such that we have a local parametrization of Σ at x

$$(\alpha_1,\ldots,\alpha_{n-2}) \mapsto (1:f_1:\ldots:f_n) \tag{4}$$

where $f_1,\ldots,f_n$ vanish at $(\alpha_1,\ldots,\alpha_{n-2})=(0,\ldots,0)$.

We can find now a parametrization φ of $\mathcal{Z}$ near l by associating to the parameters $\alpha_1,\ldots,\alpha_{n-2},\beta_2,\ldots,\beta_n$ the line passing through $(1:f_1:\ldots:f_n)$ and $(0:1:\beta_2:\ldots:\beta_n)$. Since we can take affine coordinates for the open set $p_{01}\neq 0$ of $G(1,n)$ such that $(a_2,\ldots,a_n,b_2,\ldots,b_n)$ represents the line generated by $(1:0:a_2:\ldots:a_n)$ and $(0:1:b_2:\ldots:b_n)$, the above parametrization φ can be written as

$$(\alpha_1,\ldots,\alpha_{n-2},\beta_2,\ldots\beta_n) \mapsto (f_2-\beta_2 f_1,\ldots,f_n-\beta_n f_1,\beta_2,\ldots,\beta_n). \tag{5}$$

Notice that in this parametrization the point l corresponds to the image of the origin. An easy verification on the Jacobian matrix shows that the corresponding branch of $\mathcal{Z}$ at l is smooth if and only if the Jacobian matrix of $(f_2,\ldots,f_n)$ with respect to $\alpha_1,\ldots,\alpha_{n-2}$ has maximal rank $n-2$, i.e. if and only if L is not contained in t_x. This proves (i).

To prove (ii) we first can assume that in our system of coordinates t_x has equations $x_1=x_2=0$. Accordingly, the parametrization (4) can be taken in such a way that

$f_3 = \alpha_1, \ldots, f_n = \alpha_{n-2}$ and f_1, f_2 has all their derivatives at $(0, \ldots, 0)$ equal to zero. Putting this condition in the parametrization (5) of $\mathcal{Z}$ we immediately see that the embedded tangent space of $\mathcal{Z}$ at l is the one defined by the Plücker coordinates $p_{01}, \ldots, p_{0n}, p_{13}, \ldots, p_{1n}$, i.e. the linear subspace in which p_{12} and any p_{ij} with $2 \leq \imath < j \leq n$ vanish.

Now the Schubert cycle of the lines passing through x is the projective space defined by the parameters $p_{01}, \ldots, p_{0n}$, i.e. the linear subspace in which all the p_{ij}'s with $1 \leq i < j \leq n$ vanish. And on the other hand, the Schubert cycle of the lines contained in T_x (which is the hyperplane $x_1 = 0$ in our coordinates) has equations $p_{12} = p_{02} = 0$ and $p_{2j} = 0$ for $j = 3, \ldots, n$ (i.e. any Plücker coordinate with 2 as a subindex is zero). The tangent space of this cycle at l (the point with all coordinates zero except p_{01}) is therefore the linear space in which p_{12}, p_{02} and all the Plücker coordinates without 0 or 1 as subindex are zero, i.e. the space defined by the Plücker coordinates $p_{01}, p_{03} \ldots, p_{0n}, p_{13}, \ldots, p_{1n}$. Hence the span of this and the Schubert cycle of lines passing through x is exactly the embedded tangent space of $\mathcal{Z}$ at l, which concludes the proof of (ii).

The proof of (iii) is an immediate consequence of the previous computations. Indeed, if we have a line $l' \in \mathbb{P}^{\frac{(n+2)(n-1)}{2}}$ (the Plücker space for $G(1,n)$) for which p_{12} and any p_{ij} with $2 \leq i < j \leq n$ vanish, we have two possibilities depending on whether p_{02} vanishes or not. If $p_{02} = 0$, then clearly l is a line contained in T_x and meeting L. If $p_{02} \neq 0$, then the Plücker equations $p_{01}p_{2j} - p_{02}p_{1j} + p_{0j}p_{12} = 0$ for $j = 3, \ldots, n$ imply that $p_{1j} = 0$; hence l represents a line passing through x, and this completes the proof. □

From this result can easily decide when an $(n-1)$-secant line represents a smooth point for the congruence X of $(n-1)$-secants to Σ and also to compute its tangent space. We will give the result in a more general context, since we will need it later on.

Lemma 5.2. . *Let X be the congruence of lines in $\mathbb{P}^n$ and let l be a line of X such that locally around l the congruence is the complete intersection of $n-1$ hypersurface branches $B_1, \ldots, B_{n-1} \subset G(1,n)$. Assume that, for $i = 1, \ldots, n-1$, the tangent space at l of the branch B_i is generated by $\Omega(x_i, \mathbb{P}^n)$ and $\Omega(L, T_i)$, for some point $x_i \in L$ and some hyperplane $T_i \supset L$. Then*

- *(i) The element l represents a smooth point in X if and only if the points $(x_1, T_1), \ldots, (x_{n-1}, T_{n-1})$ are in general position after the Segre embedding of $L \times P_l$.*
- *(ii) If $n = 4$ and the points x_1, x_2, x_3 are all different, the above condition is equivalent to the fact that T_1, T_2, T_3 are not all equal.*
- *(iii) The condition of (i) is satisfied if $T_1, \ldots, T_{n-1}$ are in general position. In this case, the tangent space to X at l is generated by the pencils $\Omega(x_1, \Pi_1), \ldots,$ $\Omega(x_{n-1}, \Pi_{n-1})$, where $\Pi_i := T_1 \cap \ldots \cap T_{i-1} \cap T_{i+1} \cap T_{n-1}$. In particular, each pair (x_i, π_i) is focal for L.*

Proof. Following the ideas of the previous proof, we choose coordinates in $\mathbb{P}^n$ such that L is the line $z_2 = \ldots = z_n = 0$. Then the span of $\Omega(x_i, \mathbb{P}^n)$ and $\Omega(L, T_i)$ will be a hyperplane of the linear space $A \subset \mathbb{P}^{\frac{(n+2)(n-1)}{2}}$ defined by the equations $p_{jk} = 0$ for $2 \leq j < k \leq n$. We can thus regard the tangent space to B_i at l as a hyperplane in the projective space A of coordinates $p_{01}, \ldots, p_{0n}, p_{12}, \ldots, p_{1n}$. More precisely, if $x_i = (a_i : b_i : 0 : \ldots : 0)$ and $H_i : c_{i2}z_2 + \ldots + c_{in}z_n = 0$, then this tangent space will have equation $b_i(c_{i2}p_{02} + \ldots + c_{in}p_{0n}) = a_i(c_{i2}p_{12} + \ldots + c_{in}p_{1n})$. The congruence will be smooth at l if and only if these $n-1$ hyperplanes are independent, i.e. if the matrix

$$\begin{pmatrix} b_1c_{12} & \dots & b_1c_{1n} & a_1c_{12} & \dots & a_1c_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_{n-1}c_{n-1,2} & \dots & b_{n-1}c_{n-1,n} & a_{n-1}c_{n-1,2} & \dots & a_{n-1}c_{n-1,n} \end{pmatrix}$$

has maximal rank, i.e. if and only if the points $(x_1, T_1), \dots, (x_{n-1}, T_{n-1})$ are in general position after the Segre embedding of $L \times P_l$. This proves (i), while (ii) and (iii) are now an immediate consequence of it. ☐

Now we know how the tangent space is made, we can say when an $(n-1)$-secant line to Σ is focal and deduce from it that there are extra components of the focal locus coming from focal lines, as it happens for $n = 3$ with the stationary bisecants to a curve (see [4]). The correct generalization to arbitrary dimension will be the following.

Definition 6. Let L be a line having exactly $n-1$ intersection points $x_1, \dots, x_{n-1}$ with the $(n-2)$-fold $\Sigma \subset \mathbb{P}^n$. We will say that L is a *stationary line* if there are tangent lines $L_1, \dots, L_{n-1}$ to Σ respectively at $x_1, \dots, x_{n-1}$ such that all of them are coplanar.

Proposition 5.1. *For any congruence X of $(n-1)$-secants of $\Sigma \subset \mathbb{P}^n$, a stationary line L is focal. Hence, if the union of the stationary lines of X has (as expected) dimension $n-1$, it produces at least one component of the focal locus F of X different from F_0 (in fact F_0 is Σ).*

Proof. Following the notations of Lemma 5.1, it is clear that L is stationary if and only if the intersection of all the T_{x_i}'s is (or contains) a plane Π. Let us consider the Schubert variety $\Omega(1, \Pi)$ of all the lines contained in Π. From Lemma 5.1, it follows that $\Omega(1, \Pi)$ is contained in the tangent of X at l. Taking any point $x \in L$, we thus see that the pencil $\Omega(x, \Pi)$ is also contained in the tangent of X at l. Now from Theorem 3.1 , we deduce that L is a focal line. For the last assertion, observe first that the family of stationary lines (if proper and not empty) has dimension $n-2$, since the dependence of the T_{x_i} imposes one condition on the lines of the congruence. ☐

5.2. The case $n = 4$. We want to explain the general situation described in the previous section in the first new case, i.e. $n = 4$. This will also give us the possibility of giving concrete examples to which apply the formulae we obtained in Section 2.

The first important remark is that, if we want to work with smooth congruences, we should expect a finite number of cases. Indeed, with the reasoning of the above section, it is easy to see that if a line L is k-secant (with $k \geq 4$) to Σ, then X has $\binom{k}{3}$ branches at l, and hence it is singular at l. Therefore, we will confine ourselves to consider the case of surfaces Σ with no quadrisecant lines and in the following we will moreover assume that Σ has at most ordinary singularities (in the classical sense, as in [12]).

We recall from [5] the list of such surfaces. For the description of these surfaces we refer also to the known classification of surfaces of small degree (e.g.[10]).

Proposition 5.2. *Let S be a surface in $\mathbb{P}^4$ with no quadrisecant lines. Then the surface S is one of the following:*

i) the projected Veronese surface S_1;

ii) *the smooth complete intersection* $S_2 = V(2,2)$ *of two quadrics;*

iii) *the Castelnuovo surface* $S_3 = Bl_{q,p_1,\ldots,p_7}(\mathbb{P}^2)$ *of degree* 5*, which is the blowing-up of* $\mathbb{P}^2$ *in eight points embedded by* $H = 4l - 2q - p_1 - \ldots - p_7$*, i.e. by plane quartics with a given double point and other seven base points;*

iv) *the smooth complete intersection* $S_4 = V(2,3)$ *of a quadric and a cubic hypersurface;*

v) *the Bordiga surface* $S_5 = Bl_{p_1,\ldots,p_{10}}(\mathbb{P}^2)$ *of degree* 6*, embedded by* $H = 4l - p_1 - \ldots - p_{10}$*, i.e. by plane quartics with ten base points;*

vi) *the inner projection* $S_6 = Bl_p(K3)$ *of the complete intersection* $V(2,2,2)$ *of three quadric hypersurfaces in* $\mathbb{P}^5$ *from a point in it;*

vii) *the smooth complete intersection* $S_7 = V(3,3)$ *of two cubic hypersurfaces;*

viii) *the elliptic quintic scroll* S_8 *;*

ix) *the rational normal scroll* S_9 *of degree* 4*.*

In the following theorem we study each of the congruences of trisecants to the above surfaces, in order to establish their smoothness and to compute their invariants.

Theorem 5.1. *The only smooth congruences* X *of trisecant lines to a surface* S *in* $\mathbb{P}^3$ *(with at most ordinary singularities) are those listed in the following, where we use the notation as in Propositions 5.2, 2.1, 2.2 and 2.3.*

i) *the congruence of trisecants to* S_1*, which is the hyperplane section of* $G(1,4)$ *and has bidegree* $(1,2)$ *and sectional genus* 1*.*

ii) *the congruence of trisecants to* S_3*, which is the dependency locus of two sections of* $\mathcal{Q}^2$ *and has bidegree* $(0,2)$ *and sectional genus* 0*.*

iii) *the congruence of trisecants to* S_4*, which is the zero locus of section of* $Sym^2\mathcal{Q}$ *and has bidegree* $(0,4)$ *and sectional genus* 1*.*

iv) *the congruence of trisecants to* S_5*, which is the dependency locus of four sections of* $\mathcal{Q}^3$ *and has bidegree* $(1,8)$ *and sectional genus* 10*.*

v) *the congruence of trisecants to* S_6*, which is the dependency locus of three sections of* $\mathcal{Q} \oplus Sym^2\mathcal{Q}$ *and has bidegree* $(2,15)$ *and sectional genus* 33*.*

vi) *the congruence of trisecants to* S_7*, which is the dependency locus of two sections of* $Sym^3\mathcal{Q}$ *and has bidegree* $(6,42)$ *and sectional genus* 181*.*

Proof. Notice that the bidegree (a,b) of the congruence X of trisecant lines to a given surface S in $\mathbb{P}^4$ is given respectively by the number a of trisecants to S passing through a general point in $\mathbb{P}^4$ and by the number b of trisecants to a hyperplane section $C = S \bigcap H$ of S intersecting a given general line of H. Hence one can compute (a,b) for each surface in Proposition 5.2 using Le Barz's formulas ([11]). We skipped the case of S_2, since a line is trisecant to a complete intersection of two quadrics if and only if it is contained in this complete intersection, hence we do not get a three-dimensional family of lines. In the cases of S_3, S_4 these surfaces are contained in one quadric, and therefore any trisecant line to any of them is contained in the quadric. We are thus in the situation of Theorem 3.3, in which all the lines are focal (or equivalently $a = 0$). Similarly the trisecants to the quintic elliptic scroll S_8 are contained in a one-dimensional family of planes, so that again $a = 0$. On the other hand, the surface S_9 has no trisecants, since it has minimal degreee.

Observe also that the geometric description of each congruence as the set of trisecants to a known surface yields the construction of each congruence as a degeneracy locus, and these constructions imply the smoothness of the congruences (and should also give the computation of some invariants). More explicitly, recalling that a section of $\mathcal{O}_{\mathbb{P}^4}(k)$ corresponds in $G(1,4)$ to a section of $Sym^k\mathcal{Q}$, the description of each congruence follows from the description of the corresponding surface in $\mathbb{P}^4$. For example the surface $S_7 = V(3,3)$ is the zero locus of a section of $\mathcal{O}_{\mathbb{P}^4}(3)^2$ and hence a line in $\mathbb{P}^4$ is a trisecant to S if and only if it is contained in one of the cubic hypersurfaces of the pencil trough S_7 i.e. if and only if it is a point in $G(1,4)$ contained in the degeneracy locus of two sections of $Sym^3\mathcal{Q}$. The other cases follow from the construction of $S_6 = Bl_p(K3)$ as degeneracy locus of three sections of $\mathcal{O}_{\mathbb{P}^4}(1) \oplus \mathcal{O}_{\mathbb{P}^4}(2)$ and of $S_5 = Bl_{p_1,\ldots,p_{10}}(\mathbb{P}^2)$ as three sections of $\mathcal{O}_{\mathbb{P}^4}(1)^4$. For instance, in the first case S_6 is defined by the three minors of a matrix with a row of linear forms and a row of quadratic forms; a line is a trisecant to S_6 if and only if (after the choice of a suitable base) there is one column of the matrix which vanishes on it. □

REMARK 5.2. In the first example of Theorem 5.1 the invariants can be easily computed; they are the following : $g' = g'' = 0, K_H^2 = 5$ and $\chi(\mathcal{O}_H)$=1. These invariants give $\deg F = 0$, $(a', b') = (0,0)$ and $\deg F_3 = 3$. It could look strange that in this example we get an apparently empty focal locus, at least since F and F_2 have their degree and bidegree zero, while F_3 has positive degree. The reason is that F is the Veronese surface S_1 itself (hence with degree zero as a hypersurface), and the focal planes are exactly the planes containing the conics of S_1 (hence again with dimension one less than expected), while F_3 is exactly the dual of S_1, hence an actual hypersurface in $\mathbb{P}^{4^*}$. This is because through a general point $x \in S_1$ there passes only one plane Π containing a conic $C \subset S_1$ such that $x \notin C$. The lines of the congruence through x are exactly those in the pencil $\Omega(x, \Pi)$. We thus get that Π is the focal plane corresponding to x. If L is a general line of the pencil $\Omega(x, \Pi)$ and x_1, x_2 are the other focal points for L (i.e. they are the intersection of L and C) denote by Π_1 and Π_2 to the respective corresponding focal planes for x_1 and x_2. Thus the span of Π_1 and Π_2 is clearly tangent to S_1 at x (because Π_1 and Π_2 are) and it is a focal hyperplane (we thus get that Theorem 4.1 is also valid in this degenerate case). Since the dual of the Veronese surface in $\mathbb{P}^4$ is a hypersurface of degree three, this explains why this last focal degree was not zero.

A similar kind of computation is meaningless in the second and third cases of Theorem 5.1 since the congruences have $a = 0$. Indeed in these cases our formula gives a negative degree for F_3.

6. Congruences of $(n-1)$-tangents to an $(n-1)$-fold in $\mathbb{P}^n$. In this section we consider the other general case in which one should expect some components of the focal locus outside F_0, i.e. the case of congruences of $(n-1)$-tangent lines to a hypersurface in $\mathbb{P}^n$. The difference now is that the hypersurface itself is a component of the focal locus of the congruence (in fact it is F_0 counted with some multiplicity). As in the previous section, we will study first the general case and will concentrate afterwards in the case $n = 4$, the first new case after [4].

6.1. General theory. Throughout this section Θ will be a smooth irreducible hypersurface of degree d in $\mathbb{P}^n$ and X will be the set of $(n-1)$-tangent lines to Θ, which we will assume to be a congruence.

In order to apply Lemma 5.2 we will need to study first the tangent space to the set of tangent lines to Θ. This is done in the following lemma.

LEMMA 6.1. *Let Θ be as above and let $\mathcal{T}$ be the variety of lines of $\mathbb{P}^n$ tangent to Θ. Let x be a point in Θ, consider a line L tangent at x and denote by T_x the tangent hyperplane of Θ at x. Then*

(i) The corresponding branch of $\mathcal{T}$ is smooth at the point represented by L if and only if the intersection multiplicity of L and Θ at x is exactly two.

(ii) *In the situation of (i), the embedded tangent space of this branch of $\mathcal{T}$ at L is generated by the Schubert cycle $\Omega(x, \mathbb{P}^n)$ and by the tangent space to the Schubert variety $\Omega(n-2, T_x)$.*

Proof. We choose homogeneous coordinates $z_0, \ldots, z_n$ in $\mathbb{P}^n$ such that x is the point $(1:0:\ldots:0)$, L is the line $z_2 = \ldots = z_n = 0$ and T_x is the hyperplane $z_n = 0$. We can also choose a local system of parameters $u_1, \ldots, u_{n-1}$ for Θ near x, such that we have a local parametrization of Θ at x given by $(u_1, \ldots, u_{n-1}) \mapsto (1 : u_1 : \ldots : u_{n-1} : f)$, where f and all its first derivatives vanish at $(u_1, \ldots, u_{n-1}) = (0, \ldots, 0)$.

We take thus a parametrization φ of $\mathcal{T}$ near l by associating to the parameters $u_1, \ldots, u_{n-1}, \lambda_2, \ldots, \lambda_{n-1}$ the line passing through $(1 : u_1 : \ldots : u_{n-1} : f)$ and $(0 : 1 : \lambda_2 : \ldots : \lambda_{n-1} : f_1 + \lambda_2 f_2 + \lambda_{n-1} f_{n-1})$ (where each f_i stands for the partial derivative of f with respect to u_i). Taking the affine coordinates $(a_2, \ldots, a_n, b_2, \ldots, b_n)$ for the open set $p_{01} \neq 0$ of $G(1, n)$ that we used in Lemma 5.1, the above parametrization φ becomes

$$\begin{cases} a_2 = & u_2 - \lambda_2 u_1 \\ \quad\vdots & \\ a_{n-1} = & u_{n-1} - \lambda_{n-1} u_1 \\ a_n = & f - u_1(f_1 + \lambda_2 f_2 + \lambda_{n-1} f_{n-1}) \\ b_2 = & \lambda_2 \\ \quad\vdots & \\ b_{n-1} = & \lambda_{n-1} \\ b_n = & f_1 + \lambda_2 f_2 + \lambda_{n-1} f_{n-1} \end{cases}.$$

Notice that in this parametrization the point l corresponds once more to the image of the origin. Computing the Jacobian matrix of the above parametrization at the origin, it follows that the corresponding branch of Θ at l is smooth if and only if the second partial derivative of f with respect to u_1 is not zero. This is equivalent to say that the intersection multiplicity at x of L and Θ is exactly two, which proves (i). Part (ii) is now identical to the one of Lemma 5.1. □

We have thus that Lemma 5.2 applies, and we have thus a precise description of the tangent space and focal points and planes at a sufficiently general $(n-1)$-tangent line. In order to find special components of F outside F_0 we can now give the following definition.

Definition 7. Let L be a line having exactly $n-1$ intersection points $x_1, \ldots, x_{n-1}$ with Θ. Then we will say that L is a stationary $(n-1)$-tangent to Θ if there is a plane Π that is tangent to Θ at $x_1, \ldots, x_{n-1}$.

Exactly in the same way as in Proposition 5.1 we can prove now the following.

Proposition 6.1. *For a given congruence X of $(n-1)$-tangent lines to a hypersurface $\Theta \subset \mathbb{P}^n$, a stationary line L is focal. Moreover the family of stationary lines of X, if filling a hypersurface of $\mathbb{P}^n$, produces at least one component of the focal locus F of X different from F_0.*

Remark 6.1. We have now an extra phenomenon (we will study it in more detail in the next subsection for $n = 4$, see Proposition 6.2): *the strict component is not reduced*. Indeed, given a point in Θ, there is a finite number N (becoming very high for $d >> 0$) of tangent lines to Θ at that point and that are tangent at other $n-2$ points of Θ. This produces that Θ counts multiplicity N as a component of the focal locus.

6.2. The case $n = 4$. As in the case of $n-1$ secants to an $(n-2)$-fold, we will find here that, imposing smoothness to the congruence, we will find finitely many cases (one in fact). The difference is that now it will be (relatively) easy to find all the numerical invariants of the congruence. The main result we will prove, following the same steps as for the case $n = 3$ in [4] is the following.

THEOREM 6.2. *The congruence X of tritangents to a general smooth hypersurface $\Theta \subset \mathbb{P}^4$ of degree d is smooth only for $d = 6$. In general, this congruence has invariants*

$$(a, b) = (\frac{1}{6}d(d-1)(d-2)(d-3)(d-4)(d-5), \frac{1}{3}d(d-3)(d-4)(d-5)(d^2+3d-2))$$

$$g' = 1 + \frac{1}{6}d(d-4)(d-5)(6d^4 - 13d^3 - 132d^2 + 365d - 330)$$

$$g'' = 1 + \frac{1}{6}d(d-4)(d-5)(3d^4 + 5d^3 - 57d^2 - 301d + 606)$$

$$\chi(\mathcal{O}_H) = \frac{1}{24}d(d-5)(31d^6 - 137d^5 - 652d^4 - 921d^3 + 32261d^2 - 74130d + 36792)$$

$$K_H^2 = \frac{1}{6}d(d-5)(60d^6 - 276d^5 - 1207d^4 - 1670d^3 + 61863d^2 - 143610d + 71928)$$

Proof. (see [4] for more details in case $n = 3$). First of all, we can assume that Θ does not contain lines, since any line contained in Θ will automatically be a singular point of the congruence (see [4] Lemma 3.4 for $n = 3$). We will use the subset of the Hilbert scheme $T \subset \mathrm{Hilb}^3\mathbb{P}^4$ parametrizing (unordered) triples of collinear points of $\mathbb{P}^4$ in order to find there the subset of those that produce a tritangent line to Θ. There is a map $q : T \to G(1, 3)$ assigning to each triple the line containing the points. The map q endows T with a projective bundle structure $T = \mathbb{P}(Sym^3Q^*)$. In this projective bundle we have the universal cubic form given by the bundle inclusion

$$\mathcal{O}_T(-1) \hookrightarrow q^*Sym^3Q \tag{6}$$

which assigns at each couple of points the cubic form (defined on the line spanned by them) vanishing on those points. We can similarly construct from this a bundle inclusion

$$\mathcal{O}_T(-2) \hookrightarrow q^*Sym^6Q$$

which corresponds for every couple to the sextic forms vanishing doubly at each of the points of the triple. The multiplication of $(d-6)$-forms by this universal form determines then another bundle inclusion i which defines the bundle R as a cokernel:

$$0 \to q^*Sym^{d-6}Q \otimes \mathcal{O}_T(-2) \xrightarrow{i} q^*Sym^dQ \to R \to 0.$$

A hypersurface $\Theta \subset \mathbb{P}^4$ of degree d corresponds to a section $\mathcal{O}_{G(1,4)} \to Sym^d Q$, and we are interested in the locus at which the pull-back of this section lies in the image of i. In other words, the zero locus of the corresponding section of R (obtained as the composition $\mathcal{O}_T \to q^* Sym^d Q \to R$) is the set $\tilde{X}$ of triples of points of Θ such that the line defined by them is tangent at those points. The congruence X is the isomorphic image by q of $\tilde{X}$. We can thus compute the invariants of X by using that $\tilde{X}$ is defined as the zero locus of the rank-six vector bundle R, of which we can compute its Chern classes from the exact sequence defining it. In particular we obtain the bidegree of the statement, and from it the double-point formula

$$a^2 + b^2 - c_3(N) = d(d-6)(5d^{10} - 72d^9 + 359d^8 - 774d^7 + 1027d^6 - 6108d^5 + 56345d^4 - 5190d^3 - 1746280d^2 + 5757600d - 5040000)$$

which implies that X is smooth only if $d = 6$ (of course the actual reason why this is so is that for $d \geq 7$ there are always singular points of the congruence corresponding to trisecant lines with multiplicity at least three at one of the tangency points; however finding this number will require a lot of computations that we preferred to skip). □

The following result requires to improve the techniques of [4].

Proposition 6.2. *The hypersurface Θ counts with multiplicity $\frac{1}{2}(d-4)(d-5)(d^2+3d+6)$ inside the focal locus of its congruence of tritangents.*

Proof. Continuing with the construction in the proof of Theorem 6.2, we want now to consider the incidence variety $I \subset \tilde{X} \times \mathbb{P}^4$ of pairs (t, x), where t is a triple of $\tilde{X}$ and x is one of the points of the triple. If we succeed in finding such a variety, we will have a map $p : I \to \mathbb{P}^4$ consisting of the second projection, whose image will be Θ, and the degree of p will be precisely the wanted multiplicity.

We first consider the projective bundle $Z := \mathbb{P}(q^*\mathcal{Q}) \xrightarrow{\phi} \tilde{X}$, which consists of pairs (t, x) such that the point x is in the line of the triple t. We have there a tautological inclusion $\mathcal{O}_Z(-1) \hookrightarrow \phi^* q^* \check{\mathcal{Q}}$ (and the universal $\mathcal{O}_Z(1)$ defines the projection p to $\mathbb{P}^4$). Dualizing its third symmetric power we get a map $\phi^* q^* Sym^3 \mathcal{Q} \to \mathcal{O}_Z(3)$, whose composition with the pull-back of the inclusion (6) is zero at the pairs (t, x) of I. Therefore our incidence variety is the zero locus in Z of a natural section of the line bundle $\phi^* \mathcal{O}_T(1) \otimes \mathcal{O}_Z(3)$. From this we get that the degree of the pull-back by p to I of a line in $\mathbb{P}^4$ is $\frac{1}{2}d(d-4)(d-5)(d^2+3d+6)$. Since Θ has degree d, we conclude the proof. □

Remark 6.3. From the invariants computed in Theorem 6.2, one can compute the (expected) invariants of the different focal loci, by using the formulae given in Propositions 2.1, 2.2 and 2.3. But observe that for instance the formula that we obtain for the degree of F_3 cannot be at all the degree of the dual of the focal hypersurface, because the strict focal hypersurface appears with a nonreduced structure. On the other hand, the degree of the total focal locus $\frac{2}{3}d(d-5)(d-4)(3d^4 - 6d^3 - 69d^2 + 188d - 168)$ contains d times the multiplicity of Θ inside the focal locus. This leads a rest of $\frac{1}{6}d(d-4)(d-5)(12d^4 - 24d^3 - 279d^2 + 743d - 690)$ for the special components of the focal locus. As it happens in the case $n = 3$ (see [4]), the singularities produce more components of the focal locus. But at least in the case $d = 6$ we can say that we have a component of degree 8184 consisting of the union of the stationary lines.

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