

WEAK WEYL'S LAW FOR CONGRUENCE SUBGROUPS*

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1. Introduction. Let G be a connected and simply connected semisimple algebraic group defined over \mathbb{Q} and let Z_G be its center. Let Γ be an arithmetic subgroup of $G(\mathbb{R})$. For simplicity we assume in the introduction that Γ is torsion free. We denote by K_∞ a maximal compact subgroup of $G(\mathbb{R})$. We endow $G(\mathbb{R})$ and K_∞ with Haar measures so that the symmetric space $X = G(\mathbb{R})/K_\infty$ has the canonical measure, i.e., the measure defined using the Killing form. Let

$$H_{\text{cusp}}^\Gamma = L^2_{\text{cusp}}(\Gamma \backslash G(\mathbb{R}))$$

denote the closure in $L^2(\Gamma \backslash G(\mathbb{R}))$ of the subspace spanned by all Γ -cuspidal automorphic forms. We denote by ρ_∞ the right regular representation of $G(\mathbb{R})$ in H_{cusp}^Γ . Let (σ, V_σ) be an irreducible unitary representation of K_∞ . Set

$$H_{\text{cusp}}^\Gamma(\sigma) = (H_{\text{cusp}}^\Gamma \otimes V_\sigma)^{K_\infty}.$$

Let Ω be the Casimir operator for $G(\mathbb{R})$. Then $\rho_\infty(\Omega) \otimes \text{Id}$ induces a selfadjoint operator in the Hilbert space $H_{\text{cusp}}^\Gamma(\sigma)$ which has pure point spectrum. We denote by $\Lambda_{\text{cusp}}(\sigma)$ the set of eigenvalues of $\rho_\infty(\Omega) \otimes \text{Id}$ in $H_{\text{cusp}}^\Gamma(\sigma)$. Let $N_{\text{cusp}}^\Gamma(T, \sigma)$ be the counting function of the cuspidal spectrum $\Lambda_{\text{cusp}}(\sigma)$, i.e., $N_{\text{cusp}}^\Gamma(T, \sigma)$ is the number of eigenvalues λ of $\rho_\infty(\Omega) \otimes \text{Id}$ in $H_{\text{cusp}}^\Gamma(\sigma)$, counted with their multiplicities, with $|\lambda| \leq T$. We shall also consider the discrete subspace

$$H_{\text{disc}}^\Gamma = L^2_{\text{disc}}(\Gamma \backslash G(\mathbb{R})),$$

and we define $H_{\text{disc}}^\Gamma(\sigma)$ and the corresponding eigenvalue counting function $N_{\text{disc}}^\Gamma(T, \sigma)$ similarly. It is an important question in the theory of automorphic forms to understand the asymptotic behavior of the counting function as $T \rightarrow \infty$. Let d be the dimension of the symmetric space $X = G(\mathbb{R})/K_\infty$ and let $\mathbf{\Gamma}$ denote the Gamma function. Let

$$c_\sigma(\Gamma) = \frac{\dim(\sigma) \text{vol}(\Gamma \backslash X)}{(4\pi)^{d/2} \mathbf{\Gamma}(d/2 + 1)}$$

be Weyl's constant. A conjecture of Sarnak [Sa] states that the counting function of the cuspidal spectrum satisfies Weyl's law, i.e.,

$$(1.1) \quad \lim_{T \rightarrow \infty} \frac{N_{\text{cusp}}^\Gamma(T, \sigma)}{T^{d/2}} = \lim_{T \rightarrow \infty} \frac{N_{\text{disc}}^\Gamma(T, \sigma)}{T^{d/2}} = c_\sigma(\Gamma).$$

This conjecture has been established in some special cases. First of all, it was Selberg [Se] who proved it for congruence subgroups of $\text{SL}(2, \mathbb{Z})$ and $\sigma = 1$. Other cases for which the conjecture has been established are Hilbert modular groups [Ef],

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congruence subgroups of $SO(n, 1)$ [Rez], $SL(3, \mathbb{Z})$ [Mil], and in particular, the conjecture was proved in [Mu2] for principal congruence subgroups of $SL(n, \mathbb{Z})$ and arbitrary σ .

A possible approach to prove (1.1) is through the Selberg trace formula applied to a function constructed from the heat kernel of the corresponding Bochner-Laplace operator. Let $E_\sigma \rightarrow X$ be the homogeneous vector bundle associated to σ and let Δ_σ be the elliptic differential operator induced by $-R(\Omega) \otimes \text{Id}$ in $C^\infty(X, E_\sigma)$, where R denotes the right regular representation of $G(\mathbb{R})$ in $C^\infty(G(\mathbb{R}))$. Using the heat kernel of Δ_σ , one produces a 1-parameter family of functions h_t on $G(\mathbb{R})$, depending on $t > 0$, such that for any irreducible unitary representation $(\pi, \mathcal{H}(\pi))$ of $G(\mathbb{R})$ one has

$$(1.2) \quad \text{Tr } \pi(h_t) = e^{t\pi(\Omega)} \dim \text{Hom}_{K_\infty}(\mathcal{H}(\pi), V_\sigma^*).$$

Given $\lambda \in \Lambda_{\text{cusp}}(\sigma)$, let $m(\lambda)$ denote its multiplicity. Then it follows from (1.2) that

$$\text{Tr}(\rho_\infty(h_t)|H_{\text{cusp}}^\Gamma) = \sum_{\lambda \in \Lambda_{\text{cusp}}(\sigma)} m(\lambda)e^{t\lambda}.$$

The idea is now to study the asymptotic behavior of the left hand side as $t \rightarrow 0$. Using the properties of the heat kernel, it follows that

$$(1.3) \quad h_t(1) \sim \frac{\dim(\sigma)}{(4\pi t)^{d/2} \text{vol}(K_\infty)}$$

as $t \rightarrow 0$. A good control of the trace formula should allow to prove that as $t \rightarrow 0$, the constant term in the asymptotic expansion of

$$t \longmapsto t^{d/2} \text{Tr}(\rho(h_t)|H_{\text{cusp}}^\Gamma)$$

is the contribution of the unit element to the trace formula:

$$t^{d/2} \text{Tr}(\rho(h_t)|H_{\text{cusp}}^\Gamma) = t^{d/2} \text{vol}(\Gamma \backslash G(\mathbb{R}))h_t(1) + o(1)$$

(we recall that we have assumed that Γ is torsion free). The desired estimate is then an immediate consequence of (1.3) combined with the Tauberian theorem. The second named author has been able to apply successfully this method for Γ a principal congruence subgroup for $G = SL(n)$. However, many technical difficulties are still to overcome in order to prove (1.1) for a general G .

The purpose of this paper is to prove a weaker result which holds for every G . Recall that an upper bound, which holds for arbitrary G and Γ , is already known thanks to Donnelly [Do]:

$$\limsup_{T \rightarrow \infty} \frac{N_{\text{cusp}}^\Gamma(T, \sigma)}{T^{d/2}} \leq c_\sigma(\Gamma).$$

By working with a simple form of the trace formula, we shall get, for a general G , a lower bound that depends on the choice of a set S of primes containing at least two finite primes. For every such set S we shall define a certain constant $c_S(\Gamma) \leq 1$, which is non zero for Γ deep enough. Let $Z_\Gamma = \Gamma \cap Z_G(\mathbb{R})$. Note that $Z_\Gamma \subset K_\infty$. Then our main result is the following theorem.

THEOREM 1.1. *Let G be an almost simple connected and simply connected semi-simple algebraic group defined over \mathbb{Q} such that $G(\mathbb{R})$ is non compact. Then for every congruence subgroup Γ and every σ such that $\sigma|_{Z_\Gamma} = \text{Id}$ we have*

$$c_\sigma(\Gamma)c_S(\Gamma) \leq \liminf_{T \rightarrow \infty} \frac{N_{\text{cusp}}^\Gamma(T, \sigma)}{T^{d/2}}.$$

This will turn out to be very easy to prove but the price to pay for simplicity is that this lower bound is far from being sharp; in particular the lower bound is non trivial, i.e. $c_S(\Gamma) > 0$, only for discrete subgroups Γ that are deep enough (see below for a more precise statement). We note that a weaker form of this result is due to Piatetski-Shapiro [PS] and for $G = \text{SL}(2, \mathbb{R})$ also to Venkov [Ve]. In [PS] it is proved that for every Γ there exists a subgroup of finite index Γ' such that the space of Maass cusp forms with respect to Γ' is infinite dimensional.

Theorem 1.1 is a consequence of Theorem 4.2. To get the lower bound we shall prove a sharp estimate for a certain strict subspace $H_{\text{cusp}}^{\Gamma, S}$ of H_{cusp}^Γ that will be defined via the adèlic picture. From now on, for simplicity of exposition, let us assume that G is absolutely almost simple and that $G(\mathbb{R})$ is non compact. We consider a discrete subgroup Γ such that

$$\Gamma = K_{\text{fin}} \cap G(\mathbb{Q}),$$

where

$$K_{\text{fin}} = \prod K_p$$

is a decomposable open compact subgroup of the group of finite adèles $G(\mathbb{A}_{\text{fin}})$. Let S be a finite set of primes of cardinality at least 2 and let

$$L_{\text{cusp}}^2(G(\mathbb{Q}) \backslash G(\mathbb{A}), S) \subset L_{\text{cusp}}^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$$

be the $G(\mathbb{A})$ -module generated by cusp forms that are orthogonal to the trivial representation of $G(\mathbb{A})$ and on which $G_S = G(\mathbb{Q}_S)$ acts by the Steinberg representation. Then we define

$$H_{\text{cusp}}^{\Gamma, S} = L_{\text{cusp}}^2(G(\mathbb{Q}) \backslash G(\mathbb{A}), S)^{K_{\text{fin}}}$$

and

$$H_{\text{cusp}}^\Gamma(\sigma, S) = (L_{\text{cusp}}^2(G(\mathbb{Q}) \backslash G(\mathbb{A}), S) \otimes V_\sigma)^K = (H_{\text{cusp}}^{\Gamma, S} \otimes V_\sigma)^{K_\infty}.$$

Let $\Lambda_{\text{cusp}}(\sigma, S)$ be the spectrum of $\rho_\infty(\Omega) \otimes \text{Id}$ in $H_{\text{cusp}}^\Gamma(\sigma, S)$. It consists of a discrete set of eigenvalues. Given $\lambda \in \Lambda_{\text{cusp}}(\sigma, S)$ denote by $m(\lambda)$ the multiplicity of the eigenvalue λ . Then it follows from (1.2) that

$$\text{Tr}(\rho_\infty(h_t)|H_{\text{cusp}}^{\Gamma, S}) = \sum_{\lambda \in \Lambda_{\text{cusp}}(\sigma, S)} m(\lambda)e^{t\lambda}.$$

Using the trace formula, we shall prove in 4.2 that

$$\text{Tr}(\rho_\infty(h_t)|H_{\text{cusp}}^{\Gamma, S}) \sim c_S(\Gamma) \frac{\dim \sigma \text{vol}(\Gamma \backslash X)}{(4\pi t)^{d/2}}$$

as $t \rightarrow 0$. For $T \geq 0$ let $\Lambda_{\text{cusp}}(\sigma, S; T)$ be the subset of all $\lambda \in \Lambda_{\text{cusp}}(\sigma, S)$ with $|\lambda| \leq T$. Set

$$N_{\text{cusp}}^\Gamma(T, \sigma, S) = \sum_{\lambda \in \Lambda_{\text{cusp}}(\sigma, S; T)} m(\lambda).$$

If $c_S(\Gamma) \neq 0$ the Tauberian theorem yields the estimate:

$$\lim_{T \rightarrow \infty} \frac{N_{\text{cusp}}^\Gamma(T, \sigma, S)}{T^{d/2}} = c_\sigma(\Gamma)c_S(\Gamma).$$

This is our Theorem 4.2. Now clearly $N_{\text{cusp}}^\Gamma(T, \sigma, S) \leq N_{\text{cusp}}^\Gamma(T, \sigma)$ and hence

$$c_\sigma(\Gamma)c_S(\Gamma) \leq \liminf_{T \rightarrow \infty} \frac{N_{\text{cusp}}^\Gamma(T, \sigma)}{T^{d/2}}.$$

We still have to observe that $c_S(\Gamma) \neq 0$ for subgroups Γ such that at primes $p \in S$ the open compact subgroup K_p is small enough, i.e., if it is a subgroup of some minimal parahoric subgroup I_p of $G_p = G(\mathbb{Q}_p)$.

2. A simple trace formula. From now on we shall work in the adèlic picture. We choose a Haar measure, for example the Tamagawa measure, on $G(\mathbb{A})$ which is given as the product of measures on each $G_p = G(\mathbb{Q}_p)$. We fix a decomposable open compact subgroup K_{fin} of $G(\mathbb{A}_{\text{fin}})$. Set

$$K = K_\infty K_{\text{fin}}.$$

Then K is a compact subgroup of $G(\mathbb{A})$. Let $f \in C_c^\infty(G(\mathbb{A}))$ be K -finite. The Arthur-Selberg trace formula in its invariant form is the equality

$$\sum_{\mathfrak{o} \in \mathcal{O}} I_{\mathfrak{o}}(f) = I(f) = \sum_{\chi \in \mathfrak{X}} I_\chi(f)$$

of a geometric and a spectral expansion for an invariant linear form $f \mapsto I(f)$ (see [Ar]).

Let Σ be some large enough set of primes, containing in particular the real place and all ramified primes. We shall consider functions f that are decomposable

$$f = \otimes f_p$$

and such that for $p \notin \Sigma$, f_p is the characteristic function of a hyperspecial maximal compact subgroup divided by its volume. We put, as usual,

$$f^\Sigma = \otimes_{p \notin \Sigma} f_p$$

and

$$f_\Sigma = \otimes_{p \in \Sigma} f_p$$

so that $f = f_\Sigma \otimes f^\Sigma$. The main observation is the following result of Arthur: when applied to a decomposable function such that f_p is cuspidal, in the sense of Arthur, at two primes $p \in S$ the trace formula is “simple”.

PROPOSITION 2.1. *If f is cuspidal at two places, the linear form $I(f)$ has a discrete spectral expansion:*

$$I(f) = \sum a(\pi) \operatorname{tr} \pi(f)$$

where the sum is over certain subrepresentations of parabolically induced representations from representations in the discrete spectrum of Levi subgroups. In particular, if π occurs in the discrete spectrum for $G(\mathbb{A})$ then $a(\pi) = m(\pi)$, where $m(\pi)$ denotes the multiplicity with which π occurs in the discrete spectrum. The geometric expansion is given by a linear combination of ordinary orbital integrals (over G_Σ)

$$I(f) = \sum_{\gamma \in \mathfrak{G}_\Sigma} a(\gamma, \Sigma) \mathcal{O}_\gamma(f_\Sigma),$$

where \mathfrak{G}_Σ is a set of representatives of conjugacy classes with elliptic semisimple part, $\mathcal{O}_\gamma(f_\Sigma)$ is the orbital integral of f_Σ and $a(\gamma, \Sigma)$ is some scalar. If γ is elliptic we have

$$a(\gamma, \Sigma) \mathcal{O}_\gamma(f_\Sigma) = a(\gamma) \mathcal{O}_\gamma(f),$$

where the scalar $a(\gamma)$ is independent of Σ . Moreover if the support of f_Σ remains in a fixed compact set then, in the sum over \mathfrak{G}_Σ , only a finite number of elliptic conjugacy classes in $G(\mathbb{Q})$ may give a non trivial contribution.

Proof. This is the content of [Ar, Corollary 7.2]. We omit the sum over $t \geq 0$ that occurs in [Ar] since the sum is now known to be absolutely convergent by [Mu1]. \square

We shall denote by $1(f)$ the operator defined by f in the trivial representation; this is a scalar operator in a one dimensional space:

$$1(f) = \int_{G(\mathbb{A})} f(x) dx.$$

Let p be a finite place. The existence of (normalized) pseudo-coefficients of the Steinberg representation is a particular case of a general result due to Kazdan, but, up to a sign, they can also be viewed as Euler-Poincaré functions at non archimedean places and an explicit construction has been given by Kottwitz [Ko]. A pseudo-coefficient of the Steinberg representation is a K_p -finite function $f_p \in C_c^\infty(G_p)$ such that

$$\operatorname{Tr} \pi_p(f_p) = 0$$

for all irreducible unitary representations π_p unless either π_p is the Steinberg representation of G_p in which case

$$\operatorname{Tr} \pi_p(f_p) = 1$$

or $\pi_p = 1_p$ the trivial representation in which case

$$1_p(f_p) = (-1)^q$$

for some integer q . Moreover, [Ko, Theorem 2] shows that orbital integrals $\mathcal{O}_\gamma(f_p)$ of non elliptic elements γ vanish.

Let S be a finite set of finite primes. Let

$$L_{\text{cusp}}^2(G(\mathbb{Q}) \backslash G(\mathbb{A}), S) \subset L_{\text{cusp}}^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$$

be the $G(\mathbb{A})$ -module generated by cusp forms that are orthogonal to the trivial representation of $G(\mathbb{A})$ and on which

$$G_S := \prod_{p \in S} G(\mathbb{Q}_p)$$

acts by the Steinberg representation. Let ρ denote the regular representation of $G(\mathbb{A})$ in $L^2_{\text{cusp}}(G(\mathbb{Q}) \backslash G(\mathbb{A}))$.

PROPOSITION 2.2. *Let G be a simply connected almost simple algebraic group defined over \mathbb{Q} . Assume that S is a set of finite primes containing at least two primes. Consider*

$$f = f_\infty \otimes f_{\text{fin},S} \otimes f_S,$$

where $f_\infty \in C_c^\infty(G_\infty)$, $f_{\text{fin},S}$ is a locally constant compactly supported function on $G(\mathbb{A}_{\text{fin}}^S)$, the group of finite adèles outside S , and f_S is a pseudo-coefficient of the Steinberg representation of G_S . Then one has

$$\sum_{\gamma \in \mathfrak{G}_e} a(\gamma) \mathcal{O}_\gamma(f) = \text{Tr}(\rho(f) | L^2_{\text{cusp}}(G(\mathbb{Q}) \backslash G(\mathbb{A}), S)) + 1(f),$$

where \mathfrak{G}_e is a set of representatives of elliptic conjugacy classes in $G(\mathbb{Q})$.

Proof. By a standard argument, automorphic representations in the discrete spectrum with a Steinberg component at some place p must be cuspidal. In fact Langlands description of the discrete spectrum implies that representations that are in the non cuspidal discrete spectrum arise from residues of Eisenstein series for non unitary parameters and hence are non tempered at all places, while the Steinberg representation is tempered [Clo, Proposition 4.10]. Moreover, automorphic representations with a Steinberg component at some place p cannot be subrepresentations of parabolically induced representations from representations in the discrete spectrum of Levi subgroups for a proper parabolic subgroup. Suppose that G_p is non compact. Since our group G is absolutely almost simple and simply connected, the strong approximation theorem holds and hence $G(\mathbb{Q})G_p$ is dense in $G(\mathbb{A})$. Therefore an automorphic representations that is trivial at p is necessarily trivial. On the other hand, if the group G_p is compact then all automorphic representations are cuspidal. Hence an automorphic representations that is trivial at one place is either trivial or cuspidal. We now use Proposition 2.1 on the simple form of the trace formula with Σ large enough, containing S . The above discussion shows that the spectral expansion of $I(f)$ is given by the sum of the trace in the cuspidal spectrum plus the trace in the trivial representation:

$$I(f) = \text{Tr}(\rho(f) | L^2_{\text{cusp}}(G(\mathbb{Q}) \backslash G(\mathbb{A}), S)) + 1(f).$$

The geometric expansion of $I(f)$ is a linear combination of orbital integrals (over G_Σ) and, as recalled above, [Ko, Theorem 2] shows that orbital integrals of non elliptic elements vanish at $p \in S$. Hence

$$I(f) = \sum_{\gamma \in \mathfrak{G}_e} a(\gamma) \mathcal{O}_\gamma(f),$$

where \mathfrak{G}_e is a set of representatives of elliptic conjugacy classes in $G(\mathbb{Q})$. \square

Let

$$H_{\text{cusp}}^{\Gamma, S} = L_{\text{cusp}}^2(G(\mathbb{Q}) \backslash G(\mathbb{A}), S)^{K_{\text{fin}}}$$

be the Hilbert subspace of $L_{\text{cusp}}^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$ generated by vectors of automorphic representations that are Steinberg at $p \in S$ and that are K_{fin} -invariant. Let e_{fin} be the characteristic function of K_{fin} , our chosen open compact subgroup at finite adèles divided by its volume:

$$e_{\text{fin}} = \frac{1}{\text{vol}(K_{\text{fin}})} \chi_{K_{\text{fin}}}.$$

Let

$$K_S = \prod_{p \in S} K_p, \quad K_{\text{fin}, S} = \prod_{p \notin S \cup \{\infty\}} K_p.$$

Let e_S (resp. $e_{\text{fin}, S}$) be the characteristic function of K_S (resp. $K_{\text{fin}, S}$) divided by its volume. Then

$$e_{\text{fin}} = e_S \otimes e_{\text{fin}, S}.$$

COROLLARY 2.3. *Assume that f is of the form*

$$f = f_{\infty} \otimes f_S \otimes e_{\text{fin}, S}$$

where f_S is a pseudo-coefficient of the Steinberg representation of G_S . Let $c(K_S)$ be the dimension of the space of K_S fixed vectors in the Steinberg representation of G_S and let

$$b(f_{\infty}, S) = \text{Tr } 1(f_{\infty} \otimes f_S),$$

where 1 is the trivial representation of $G_{\infty} \times G_S$. Then one has

$$\text{Tr}(\rho_{\infty}(f_{\infty}) | H_{\text{cusp}}^{\Gamma, S}) = c(K_S) \left(\sum_{\gamma \in \mathfrak{G}_e} a(\gamma) \mathcal{O}_{\gamma}(f) - b(f_{\infty}, S) \right).$$

Proof. We first observe that for any admissible representation τ of $G(\mathbb{A}_{\text{fin}})$, the operator $\tau(e_{\text{fin}})$ is the projection on the subspace of K_{fin} -invariant vectors. Hence,

$$\text{Tr}(\rho_{\infty}(f_{\infty}) | H_{\text{cusp}}^{\Gamma, S}) = \text{Tr}(\rho(f_{\infty} \otimes e_{\text{fin}}) | L_{\text{cusp}}^2(G(\mathbb{Q}) \backslash G(\mathbb{A}), S)).$$

However, if π_S is the Steinberg representation of G_S then

$$\text{Tr } \pi_S(e_S) = c(K_S) \text{Tr } \pi_S(f_S)$$

and hence we get

$$\text{Tr}(\rho_{\infty}(f_{\infty}) | H_{\text{cusp}}^{\Gamma, S}) = c(K_S) \text{Tr}(\rho(f) | L_{\text{cusp}}^2(G(\mathbb{Q}) \backslash G(\mathbb{A}), S)).$$

The assertion is then an immediate consequence of Proposition 2.2. \square

3. The heat kernel. In this section we establish a number of facts about the heat kernel of a Bochner-Laplace operator on the Riemannian symmetric space $X = G(\mathbb{R})/K_\infty$. Let $E_\sigma \rightarrow X$ be the homogeneous vector bundle associated to σ . Let $\Omega \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$ be the Casimir element of $G(\mathbb{R})$ and let R be the right regular representation of $G(\mathbb{R})$ on $C^\infty(G(\mathbb{R}))$. Denote by Δ_σ the second order elliptic differential operator induced by $-R(\Omega) \otimes \text{Id}$ in $C^\infty(X, E_\sigma) \cong (C^\infty(G(\mathbb{R})) \otimes V_\sigma)^{K_\infty}$. If ∇^σ denotes the canonical invariant connection on E_σ , then

$$\Delta_\sigma = (\nabla^\sigma)^* \nabla^\sigma - \lambda_\sigma \text{Id},$$

where $\lambda_\sigma = \sigma(\Omega_{K_\infty})$ is the Casimir eigenvalue of σ . Hence $\Delta_\sigma : C_c^\infty(X, E_\sigma) \rightarrow L^2(X, E_\sigma)$ is essentially selfadjoint and bounded from below. We continue to denote its unique selfadjoint extension by Δ_σ . Let $\exp(-t\Delta_\sigma)$, $t \geq 0$, be the associated heat semigroup. The heat operator is a smoothing operator on $L^2(X, E_\sigma)$ which commutes with the representation of $G(\mathbb{R})$ on $L^2(X, E_\sigma)$. Therefore, it is of the form

$$(e^{-t\Delta_\sigma} \varphi)(g) = \int_{G(\mathbb{R})} H_t(g^{-1}g_1)(\varphi(g_1)) dg_1, \quad g \in G(\mathbb{R}),$$

where $\varphi \in (L^2(G(\mathbb{R})) \otimes V_\sigma)^{K_\infty}$ and $H_t : G(\mathbb{R}) \rightarrow \text{End}(V_\sigma)$ is in $L^2 \cap C^\infty$ and satisfies the covariance property

$$H_t(g) = \sigma(k)H_t(k^{-1}gk')\sigma(k')^{-1}, \quad \text{for } g \in G(\mathbb{R}), k, k' \in K_\infty.$$

Let $\mathcal{C}^1(G(\mathbb{R}))$ be Harish-Chandra's space of integrable rapidly decreasing functions on $G(\mathbb{R})$. Then

$$(3.1) \quad H_t \in (\mathcal{C}^1(G(\mathbb{R})) \otimes \text{End}(V_\sigma))^{K_\infty \times K_\infty}$$

[BM, Proposition 2.4].

Next we need to estimate the covariant derivatives of the heat kernel. Let ∇ denote the Levi-Civita connection on $G(\mathbb{R})$ with respect to the invariant metric on $G(\mathbb{R})$. Let $d(x, y)$ be the geodesic distance of $x, y \in X$. Define the function $r(g)$ on $G(\mathbb{R})$ by

$$(3.2) \quad r(g) = d(gK_\infty, K_\infty), \quad g \in G(\mathbb{R}).$$

The following proposition is proved in [Mu2, Proposition 2.1].

PROPOSITION 3.1. *Let $a = \dim G(\mathbb{R})$, $l \in \mathbb{N}_0$ and $T > 0$. There exist $C, c > 0$ such that*

$$\|\nabla^l H_t(g)\| \leq Ct^{-(a+l)/2} \exp\left(-\frac{cr^2(g)}{t}\right)$$

for all $0 < t \leq T$ and $g \in G(\mathbb{R})$.

Define the one-parameter family of smooth functions h_t on $G(\mathbb{R})$ by

$$(3.3) \quad h_t(g) = \text{tr } H_t(g), \quad g \in G(\mathbb{R}), t \geq 0,$$

where tr denotes the trace function on $\text{End}(V_\sigma)$. Then it follows from 3.1 that h_t belongs to $\mathcal{C}^1(G(\mathbb{R}))$. In addition, h_t is both left and right K_∞ -finite.

Let π be a unitary irreducible representation of $G(\mathbb{R})$ on the Hilbert space $\mathcal{H}(\pi)$. Then $\pi(h_t)$ is trace class and it follows as in [BM] that

$$(3.4) \quad \text{Tr } \pi(h_t) = e^{t\pi(\Omega)} \dim \text{Hom}_{K_\infty}(\mathcal{H}(\pi), V_\sigma^*).$$

4. The main result. First we note that by our assumptions on G , strong approximation holds for $G(\mathbb{A})$ and hence we have

$$L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_{\text{fin}}) \cong L^2_{\text{cusp}}(\Gamma\backslash G(\mathbb{R}))$$

as $G(\mathbb{R})$ -modules. Let h_t , $t > 0$, be the one-parameter family of functions on $G(\mathbb{R})$ defined by (3.3). By [Mu1, Corollary 0.2] the restriction of $\rho_\infty(h_t)$ to $L^2_{\text{cusp}}(\Gamma\backslash G(\mathbb{R}))$ is a trace class operator. However, since the support of h_t is not compact it is not known that we may insert h_t in the trace formula for an arbitrary G (although this is likely since h_t is a very rapidly decreasing function at infinity). This difficulty is bypassed by modifying h_t in the following way. Let $\varphi \in C_c^\infty(\mathbb{R})$ be such that $\varphi(u) = 1$, if $|u| \leq 1/2$, and $\varphi(u) = 0$, if $|u| \geq 1$. Given $t > 0$, let $\varphi_t \in C_c^\infty(G(\mathbb{R}))$ be defined by

$$\varphi_t(g) = \varphi(r^2(g)/t^{1/2}),$$

where $r(g)$ is defined by (3.2). Set

$$\tilde{h}_t(g) = \varphi_t(g)h_t(g), \quad g \in G(\mathbb{R}), \quad t > 0.$$

Then $\tilde{h}_t \in C_c^\infty(G(\mathbb{R}))$. Recall that ρ_∞ denotes the right regular representation of $G(\mathbb{R})$ in H_{cusp}^Γ .

PROPOSITION 4.1. *There exist $C, c > 0$ such that*

$$|\text{Tr}(\rho_\infty(h_t)) - \text{Tr}(\rho_\infty(\tilde{h}_t))| \leq Ce^{-c/\sqrt{t}}$$

for all $0 < t \leq 1$.

Proof. For $\pi \in \Pi(G(\mathbb{R}))$ let $m_\Gamma(\pi)$ denote the multiplicity with which π occurs in the discrete subspace $L^2_{\text{cusp}}(\Gamma\backslash G(\mathbb{R}))$ and denote by λ_π the Casimir eigenvalue $\pi(\Omega)$ of π . For every $\pi \in \Pi(G(\mathbb{R}))$ with $m_\Gamma(\pi) \neq 0$ we choose an orthonormal basis $e_{\pi,i}$, $i \in I(\pi)$, for the finite-dimensional vector space $(\mathcal{H}(\pi) \otimes V_\sigma)^{K_\infty}$. Let Δ_G be the Laplace operator on $G(\mathbb{R})$ with respect to the left invariant metric. Note that

$$\Delta_G = R(-\Omega + 2\Omega_{K_\infty}),$$

where Ω_{K_∞} is the Casimir element of K_∞ . Let λ_σ be the Casimir eigenvalue of σ . We note that

$$-\lambda_\pi + 2\lambda_\sigma \geq 0$$

for all $\pi \in \Pi(G(\mathbb{R}))$ with $(\mathcal{H}(\pi) \otimes V_\sigma)^{K_\infty} \neq \{0\}$ (cf. Lemma 2.6 of [DH]). Let χ_σ be the character of σ and let $d = \dim X$. Then for every $f \in \mathcal{C}^1(G(\mathbb{R}))$ with $f = f * \bar{\chi}_\sigma$ we get

$$\begin{aligned} \text{Tr} \rho_\infty(f) &= \sum_{\pi \in \Pi(G(\mathbb{R}))} m_\Gamma(\pi) \sum_{i \in I(\pi)} (\rho_\infty(f)e_{\pi,i}, e_{\pi,i}) \\ &= \sum_{\pi \in \Pi(G(\mathbb{R}))} m_\Gamma(\pi) (1 - \lambda_\pi + 2\lambda_\sigma)^{-4d} \\ &\quad \times \sum_{i \in I(\pi)} \int_{G(\mathbb{R})} ((\text{Id} + \Delta_G)^{4d} f)(g) (\rho_\infty(g)e_{\pi,i}, e_{\pi,i}) dg. \end{aligned}$$

By [Mu1, Theorem 0.1] we have

$$\sum_{\pi \in \Pi(G(\mathbb{R}))} m_{\Gamma}(\pi) \dim(\mathcal{H}(\pi) \otimes V_{\sigma})^{K_{\infty}} (1 - \lambda_{\pi} + 2\lambda_{\sigma})^{-4d} < \infty.$$

Using this result we get

$$(4.1) \quad |\mathrm{Tr} \rho_{\infty}(f)| \leq C \|(\mathrm{Id} + \Delta_G)^{4d} f\|_{L^1(G(\mathbb{R}))}$$

for some constant $C > 0$, independent of f . Let $\psi_t = 1 - \varphi_t$, $t > 0$. Then it follows from (4.1) that

$$(4.2) \quad \begin{aligned} |\mathrm{Tr}(\rho_{\infty}(h_t)) - \mathrm{Tr}(\rho_{\infty}(\tilde{h}_t))| &= |\mathrm{Tr}(\rho_{\infty}(\psi_t h_t))| \\ &\leq C \|(\mathrm{Id} + \Delta_G)^{4d}(\psi_t h_t)\|_{L^1(G(\mathbb{R}))}. \end{aligned}$$

It remains to estimate the right hand side. Let X_1, \dots, X_a be an orthonormal basis of $\mathfrak{g}(\mathbb{R})$. Then $\Delta_G = -\sum_i X_i^2$. Denote by ∇ the canonical left invariant connection on $G(\mathbb{R})$. Then it follows that there exists $C_1 > 0$ such that for all $f \in C^{\infty}(G(\mathbb{R}))$ we have

$$(4.3) \quad |(\mathrm{Id} + \Delta_G)^{4d} f(g)| \leq C_1 \sum_{j=0}^{8d} \|\nabla^j f(g)\|, \quad g \in G(\mathbb{R}).$$

By Proposition 3.1 there exist constants $C, c > 0$ such that

$$\|\nabla^j h_t(g)\| \leq C t^{-(a+j)/2} e^{-cr^2(g)/t}, \quad g \in G(\mathbb{R}),$$

for $j \leq 8d$ and $0 < t \leq 1$. Let χ_t be the characteristic function of the complement of the interval $(-t^{1/4}/\sqrt{2}, t^{1/4}/\sqrt{2})$ in \mathbb{R} . Recall that $\psi_t(g) = (1 - \varphi)(r^2(g)/t^{1/2})$ and $(1 - \varphi)(u)$ is constant for $|u| \geq 1$. This implies that there exists a constant $C_2 > 0$ such that

$$(4.4) \quad \|\nabla^j \psi_t(g)\| \leq C_2 t^{-4d} \chi_t(r(g)), \quad g \in G(\mathbb{R}),$$

for $j \leq 8d$ and $0 < t \leq 1$. Combining (4.3) and (4.4) we obtain

$$\begin{aligned} \sum_{j=0}^{8d} \|\nabla^j(\psi_t h_t)(g)\| &\leq C_2 t^{-a/2-8d} \chi_t(r(g)) e^{-c_2 r^2(g)/t} \\ &\leq C_3 e^{-c_3/\sqrt{t}} e^{-c_3 r^2(g)} \end{aligned}$$

for all $g \in G(\mathbb{R})$ and $0 < t \leq 1$. Finally note that for every $c > 0$, $e^{-cr^2(g)}$ is an integrable function on $G(\mathbb{R})$. Together with (4.2) and (4.3) the proof follows. \square

We are now ready to apply the trace formula. Given $\pi \in \Pi(G(\mathbb{R}))$, let $m_{\Gamma}(\pi, S)$ denote the multiplicity with which π occurs in the regular representation of $G(\mathbb{R})$ on $H_{\mathrm{cusp}}^{\Gamma, S}$. Then it follows from (3.4) that

$$(4.5) \quad \mathrm{Tr}(\rho_{\infty}(h_t)|H_{\mathrm{cusp}}^{\Gamma, S}) = \sum_{\pi \in \Pi(G(\mathbb{R}))} e^{t\lambda_{\pi}} m_{\Gamma}(\pi, S) \dim \mathrm{Hom}_{K_{\infty}}(\mathcal{H}(\pi), V_{\sigma}^*).$$

We note that the λ_π 's are the eigenvalues of the operator which is induced by $\rho_\infty(\Omega) \otimes \text{Id}$ in $H_{\text{cusp}}^\Gamma(\sigma, S)$ and $m_\Gamma(\pi, S) \dim \text{Hom}_{K_\infty}(\mathcal{H}(\pi), V_\sigma^*)$ is the multiplicity of the corresponding eigenvalue. Let

$$N_{\text{cusp}}^\Gamma(T, \sigma, S) = \sum_{|\lambda_\pi| \leq T} m_\Gamma(\pi, S) \dim \text{Hom}_{K_\infty}(\mathcal{H}(\pi), V_\sigma^*).$$

We may now state and prove our main result.

THEOREM 4.2. *Let G be an almost simple, connected and simply connected algebraic group defined over \mathbb{Q} such that $G(\mathbb{R})$ is non compact. Let d_S denote the formal dimension of the Steinberg representation of G_S and let*

$$c_S(\Gamma) = c(K_S)d_S \text{vol}(K_S).$$

Then, if $Z_\Gamma = \Gamma \cap Z_G(\mathbb{R}) \subset K_\infty$ is such that $\sigma|_{Z_\Gamma} = \text{Id}$, we have

$$\lim_{T \rightarrow \infty} \frac{N_{\text{cusp}}^\Gamma(T, \sigma, S)}{T^{d/2}} = c_S(\Gamma) \frac{\text{vol}(\Gamma \backslash X)}{(4\pi)^{d/2} \mathbf{\Gamma}(d/2 + 1)}.$$

Proof. We apply the trace formula to

$$f_t = \tilde{h}_t \otimes e_{\text{fin}, S} \otimes f_S.$$

The assumptions of Corollary 2.3 are satisfied. We first observe that $b(\tilde{h}_t, S)$ remains bounded. Since $G(\mathbb{R})$ is non compact we have

$$\dim(G(\mathbb{R})/K_\infty) = d > 0$$

and hence $t^{d/2}b(\tilde{h}_t, S) \rightarrow 0$ as $t \rightarrow 0$. Then Corollary 2.3 yields

$$t^{d/2} \text{tr}(\rho_\infty(\tilde{h}_t)|H_{\text{cusp}}^{\Gamma, S}) = c(K_S)t^{d/2} \sum_{\gamma \in \mathfrak{E}_e} a(\gamma)\mathcal{O}_\gamma(f_t) + o(1)$$

as $t \rightarrow 0$. Since the support of f_S remains in a fixed compact set we already observed in Proposition 2.1 that in the sum over \mathfrak{E}_e only a finite fixed number of terms may be non zero. Hence it suffices to estimate each term. We first remark that

$$\mathcal{O}_\gamma(f_t) = \mathcal{O}_\gamma(\tilde{h}_t)\mathcal{O}_\gamma(e_{\text{fin}, S})\mathcal{O}_\gamma(f_S).$$

We shall now use results that are established for the case $G = \text{SL}(n)$ in [Mu2], but notation make sense and proofs extend readily for the general case. According to [Mu2, Proposition 7.3], we have

$$t^{d/2}J_M(\gamma, \tilde{h}_t) \rightarrow 0, \quad t \rightarrow 0,$$

for $M = G$ and $\gamma \notin Z_G(\mathbb{R})$, while we know by [Mu2, Lemma 2.3] that $t^{d/2}h_t(z)$ has a non zero positive limit for $z \in Z_G(\mathbb{R})$:

$$(4.6) \quad t^{d/2}h_t(z) \rightarrow \frac{\text{tr } \sigma(z)}{(4\pi)^{d/2} \text{vol}(K_\infty)}.$$

Since

$$J_G(\gamma, h) = \mathcal{O}_\gamma(h)$$

it follows that non central elements give a negligible contribution and, since $d_S = f_S(z)$, we get

$$\lim_{t \rightarrow 0} t^{d/2} \operatorname{Tr}(\rho_\infty(\tilde{h}_t) | H_{\text{cusp}}^{\Gamma, S}) = \frac{\operatorname{vol}(G(\mathbb{Q}) \backslash G(\mathbb{A})) c_S(\Gamma)}{\operatorname{vol}(K_{\text{fin}})} \lim_{t \rightarrow 0} t^{d/2} \sum_{z \in Z_\Gamma} h_t(z)$$

with

$$c_S(\Gamma) = c(K_S) d_S \operatorname{vol}(K_S).$$

We observe that

$$\frac{\operatorname{vol}(G(\mathbb{Q}) \backslash G(\mathbb{A}))}{\operatorname{vol}(K_\infty) \operatorname{vol}(K_{\text{fin}})} = \frac{\operatorname{vol}(\Gamma \backslash X)}{\operatorname{Card}(Z_\Gamma)}.$$

Using (4.6) and the assumption that $\sigma|_{Z_\Gamma} = \operatorname{Id}$, it follows that the above limit can be rewritten as

$$\lim_{t \rightarrow 0} t^{d/2} \operatorname{tr}(\rho_\infty(\tilde{h}_t) | H_{\text{cusp}}^{\Gamma, S}) = c_S(\Gamma) \frac{\dim \sigma \operatorname{vol}(\Gamma \backslash X)}{(4\pi)^{d/2}}.$$

Now, it follows from Proposition 4.1 that

$$\lim_{t \rightarrow 0} t^{d/2} \operatorname{Tr}(\rho_\infty(\tilde{h}_t) | H_{\text{cusp}}^{\Gamma, S}) = \lim_{t \rightarrow 0} t^{d/2} \operatorname{Tr}(\rho_\infty(h_t) | H_{\text{cusp}}^{\Gamma, S}),$$

and hence we get

$$\lim_{t \rightarrow 0} t^{d/2} \operatorname{Tr}(\rho_\infty(h_t) | H_{\text{cusp}}^{\Gamma, S}) = c_S(\Gamma) \frac{\dim \sigma \operatorname{vol}(\Gamma \backslash X)}{(4\pi)^{d/2}}.$$

We conclude using (4.5) and the Tauberian theorem. \square

As explained in the introduction, Theorem 1.1 follows immediately from Theorem 4.2.

REMARKS. 1) The constant $c_S(\Gamma)$ is non zero if and only if $c(K_S) \neq 0$ and this is true whenever K_S is small enough so that the space of the Steinberg representation of G_S contains non zero K_S -invariant vectors. This is the case if $K_p \subset I_p$ a minimal parahoric subgroup for $p \in S$.

2) If the assumption $\sigma|_{Z_\Gamma} = \operatorname{Id}$ in Theorem 4.2 is not satisfied, i.e, if there exists $z \in Z_\Gamma$ such that $\sigma(z) \neq \operatorname{Id}$, then $H_{\text{cusp}}^\Gamma(\sigma) = \{0\}$.

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