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# A ZARISKI PAIR IN AFFINE COMPLEX PLANE *

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**Abstract.** We present a Zariski pair in affine complex plane consisting of two line arrangements, each of which has six lines.

In the seminal paper [3], Zariski started the study of the fundamental groups of the complements of plane algebraic curves. Among other things, he constructed a pair of plane curves with the same degree and the same local singularities, but non-isomorphic fundamental groups, which is called in the literature a Zariski pair. The two curves in Zariski's example are sextics. Several families of Zariski pairs in projective complex plane have been found. Recently, K.-M. Fan [1] has given a Zariski pair consisting of two arrangements of lines in projective complex plane $\mathbb{C}P^2$. Each arrangement consists of seven real lines. Although there is some relationship between Zariski pairs in affine and projective planes, this relationship is not direct. Therefore, the study of Zariski pairs in affine complex plane is of independent interests. In this note, we present a Zariski pair consisting of six lines in affine complex plane $\mathbb{C}^2$. We need the following special version of Oka-Sakamoto theorem.

THEOREM 1. (Oka-Sakamoto [2]) *Let be given $m+n$ lines in $\mathbb{C}^2$: $K_1, \dots, K_m$ and $L_1, \dots, L_n$ such that $K_1 \cup \cdots \cup K_m$ intersects $L_1 \cup \cdots \cup L_n$ in $mn$ distinct points. Then*

$$\pi(\mathbb{C}^2 \setminus (K_1 \cup \cdots \cup K_m \cup L_1 \cup \cdots \cup L_n)) \cong \pi(\mathbb{C}^2 \setminus (K_1 \cup \cdots \cup K_m)) \times \pi(\mathbb{C}^2 \setminus (L_1 \cup \cdots \cup L_n)).$$

COROLLARY 2. *If $L_1, L_2, \dots, L_n$ can be divided into $m$ groups*

$$K_{11}, \dots, K_{1p_1}; \dots; K_{m1}, \dots, K_{mp_m}, \qquad (p_1 + \cdots + p_m = n),$$

*such that $(K_{i1} \cup \cdots \cup K_{ip_i}) \cap (K_{j1} \cup \cdots \cup K_{jp_j})$ consists of $p_i p_j$ distinct points, then*

$$\pi(\mathbb{C}^2 \setminus (L_1 \cup \cdots \cup L_n)) \cong \prod_{i=1}^{m} \pi(\mathbb{C}^2 \setminus (K_{i1} \cup \cdots \cup K_{ip_i})).$$

COROLLARY 3. *Let $L_1$ intersects $L_2 \cup \cdots \cup L_n$ in $n-1$ distinct points, then*

$$\pi(\mathbb{C}^2 \setminus (L_1 \cup \cdots \cup L_n)) \cong \pi(\mathbb{C}^2 \setminus (L_2 \cup \cdots \cup L_n)) \times \mathbb{Z}.$$

**A Zariski pair.** Let $\mathcal{A}$ be the arrangement in $\mathbb{C}^2$ consisting of three pairs of parallel lines with 12 normal crossings (see figure 1). Let $\mathcal{B}$ be an arrangement in $\mathbb{C}^2$ consisting of a triangle and three parallel lines which do not pass the vertices of the triangle, and are not parallel to any edge of the triangle (see figure 2). The following proposition says that these two arrangements form a Zariski pair in $\mathbb{C}^2$.

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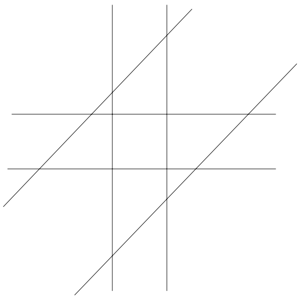

FIG. 1.

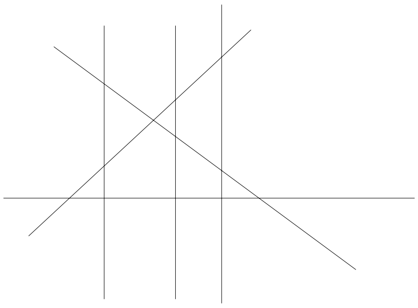

FIG. 2.

PROPOSITION. *The two arrangements $\mathcal{A}$ and $\mathcal{B}$ have the same local singularities, and the complements of them have non-isomorphic fundamental groups.*

*Proof.* Each of the two arrangements has 12 singularities of type $A_1$. Hence $\mathcal{A}$ and $\mathcal{B}$ have the same local singularities. By Corollary 2, we have

$$G_1 := \pi(M(\mathcal{A})) \cong (\pi(\mathbb{C}^2 \setminus (L_1 \cup L_2)))^{\times 3} \cong F_2 \times F_2 \times F_2,$$

where $M(\mathcal{A}) = \mathbb{C}^2 \setminus \bigcup_{L \in \mathcal{A}} L$ is the complement, $L_1, L_2$ are two parallel lines in $\mathbb{C}^2$, and $F_2$ is the free group of rank two. By Corollary 3,

$$G_2 := \pi(M(\mathcal{B})) \cong F_3 \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}.$$

Obviously $G_1$ and $G_2$ are not isomorphic. □

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