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THE FORMULA FOR THE SINGULARITY OF SZEGÖ KERNEL: II*

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Let M be a strongly pseudoconvex hypersurface in $\mathbf{C}^{n+1}$, i.e. the boundary of a domain Ω in $\mathbf{C}^{n+1}$. The Szegö kernel $K^S(x,y),\ x,y \in M$, is smooth outside of the diagonal $x = y$. The singularity at (x,x) is determined by the local datum at x of M, even though K^S itself is a global object. Our problem is to write down the singularity at (x,x) in terms of the local equation of M in $\mathbf{C}^{n+1}$. We fix a reference point, say p_*, in M and only consider the germ of M at p_*. Hence we we may shrink M near p_* without mentioning it. We use as the model structure the boundary of the Siegel upper half space.

Let a strongly pseudoconvex inside tubular neighborhood N of M in $\mathbf{C}^{n+1}$ be defined by $r > 0$. Let $r(x,x')$ be the function on $N \times N$ satisfying the conditions: (i) $r(x,x) = r(x), (ii)$ $r(x,x')$ is holomorphic in x, (iii) $r(x,x') = \overline{r(x',x)}$.

As is shown by Fefferman and Boutet de Monvel-Sjöstrand, the singularity of K^S at (p_*,p_*) is of the form:

$$K^S = Fr^{-(n+1)} + G\log r$$

where F and G are smooth functions.

In Par I [4] of this series of papers we developed a procedure to write down the above F, G near (p_*,p_*) for a specific choice of r. We used the method developed by Boutet de Monvel and Sjöstrand in [1]. Our method is based on a symplectic transformatiion χ which transforms a conic neighborhod of the characteristics of the symbols of $\overline{\partial}_\flat$ operators of the model structure to that of our structure. The symplectic transformation is defined as the solution of an ordinary differential equation of Hamilton type. However, when we try to carry out the construction explicitly in straightfoward way, the formula becomes rather cumbersome. In this paper we develop a way of calculation so that it becomes more accessible. We also construct the inverse map of χ by the similar method.

In §1 we recall the notaions and concept in Part I, which we use. Our calculation is carried out using the material develoed in §1 and we do not need Part I to follow it.

1. The notations. We consider the case M is defined by an equation: $r = 0$, where r is real valued and of the form: With $x^0 = \Re w$,

$$r = \frac{1}{i}(w - \bar{w}) - (|z|^2 + N(z,x^0)), \quad \text{where } N \equiv 0 \pmod{(z,\bar{z},x^0)^4}. \tag{1}$$

Our model case (Heisenberg structure) $\mathcal{M}$ is the case

$$N = N_0 = 0. \tag{2}$$

For $0 \le t \le 1$, denote by M_t the CR structure defined by r_t, which is the case

$$N = N_t = \frac{1}{t^2}N(tz, t^2x^0). \tag{3}$$

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N_t is a one parameter family of CR structures which is, for each $t \neq 0$, isomorphic to a neighborhood of origin in M. M_0 is the model case.

We may regard (z, x^0) as the standard chart of M as well as of M_t. They also have the real standard chart $(x^0, x^1, ..., x^{2n})$, where

$$x^{2\alpha-1} + ix^{2\alpha} = z^\alpha, \quad (\alpha = 1, ..., n). \tag{4}$$

We may thus consider M_t as CR structures defined on a same manifold $\mathbf{M}$ which is a neighborhood of the origin in the (z, x^0)-space.

Unless specified otherwise, we usually consider $M_t,\ t \neq 0$. For simplicity of notation we usually omit t. Hence M usually means M_t. Similar convention will be used for objects associated with M_t.

The $\overline{\partial}_\flat$ operators of M is generated by Q^α given, when written in the ambiant complex space $\{(z, w)\}$, by

$$Q^\alpha = \frac{\partial}{\partial \overline{z^\alpha}} - ih^\alpha \frac{\partial}{\partial \overline{w}}, \quad ih^\alpha = \frac{r_{\bar\alpha}}{r_{\bar w}}. \tag{6}$$

where $r_{\bar\alpha} = \partial r / \partial \overline{z^\alpha},\ r_{\bar w} = \partial r / \partial \overline{w}$. In the model case they are generated by

$$P^\alpha = \frac{\partial}{\partial \overline{z^\alpha}} - iz^\alpha \frac{\partial}{\partial \overline{w}}. \tag{7}$$

In the (z, x^0)-chart

$$Q^\alpha = \frac{\partial}{\partial \overline{z^\alpha}} - \frac{i}{2} h^\alpha(z, x^0) \frac{\partial}{\partial x^0}, \quad P^\alpha = \frac{\partial}{\partial \overline{z^\alpha}} - \frac{i}{2} z^\alpha \frac{\partial}{\partial x^0}. \tag{8}$$

We see easily that

$$\begin{gathered} [P^\alpha, P^\beta] = 0, \quad [P^\alpha, \overline{P^\beta}] = i\delta^\alpha_\beta \frac{\partial}{\partial x_0}, \quad [P^\alpha, \frac{\partial}{\partial x^0}] = 0, \\ [Q^\alpha, Q^\beta] = 0, \quad [Q^\alpha, \overline{Q^\beta}] = ic^{\alpha\bar\beta} \frac{\partial}{\partial x^0}, \quad [Q^\alpha, \frac{\partial}{\partial x^0}] = c^\alpha \frac{\partial}{\partial x^0} \qquad \text{where} \end{gathered} \tag{9}$$

$$c^\alpha = \frac{i}{2} \frac{\partial}{\partial x^0} h^\alpha \equiv 0 \pmod{t^5}, \tag{10}$$

$$c^{\alpha\bar\beta} = \frac{1}{2}(Q^\alpha \overline{h^\beta} + \overline{Q^\beta} h^\alpha) \equiv \delta^\alpha_\beta \pmod{t^2}. \tag{11}$$

$(c_{\alpha\bar\beta})$ is the inverse matrix of $(c^{\alpha\bar\beta})$.

Denote by $(x, \xi),\ \xi = (\xi_0, ..., \xi_{2n})$, the standard chart of $T^*\mathbf{M}$, the cotangent bundle of $(x^0, ..., x^{2n})$-space. Hence $\theta \in T^*\mathbf{M}$ has the expression:

$$\theta = \xi_j dx^j. \tag{12}$$

In terms of (z, x^0)-chart

$$\theta = \zeta_\alpha dz^\alpha + \overline{\zeta_\alpha} d\overline{z^\alpha} + \xi_0 dx^0, \quad \text{where} \tag{13}$$

$$\zeta_\alpha = \zeta_\alpha(\xi) = \frac{1}{2}(\xi_{2\alpha-1} - i\xi_{2\alpha}). \tag{14}$$

It is convenient to introduce a copy y of x-space and use y when we are working on M and use x for the model $\mathcal{M}$. The standard chart of the cotangent bundle of the y-space will be denoted by (y,η). We use $\zeta_\alpha(\eta)$ to denote the complex part of the fiber chart of the cotangent bundle of y-space. We set $z^\alpha(\xi) = \xi_{2\alpha-1} + i\xi_{2\alpha}$, $z^\alpha(\eta) = \eta_{2\alpha-1} + i\eta_{2\alpha}$.

The symbols of P^α, Q^α are

$$\begin{aligned} p^\alpha &= p^\alpha(x,\xi,t) = i\overline{\zeta_\alpha} + \frac{1}{2}z^\alpha\xi_0 = \frac{1}{2}(iz^\alpha(\xi) + z^\alpha(x)\xi_0), \\ q^\alpha &= q^\alpha(y,\eta,t) = i\overline{\zeta_\alpha}(\eta) + \frac{1}{2}h^\alpha_t(y)\eta_0 = \frac{1}{2}(iz^\alpha(\eta) + h^\alpha(y)\eta_0). \end{aligned} \tag{15}$$

In the following we work on open conic submanifolds $(T^*)'\mathcal{M}$, $(T^*)'M$ of the cotangent bundles where $\xi_0 \neq 0$, $\eta_0 \neq 0$, respectively. Hence, when we define

$$f^\alpha = \frac{1}{i}p^\alpha(x,\xi)/\xi_0, \quad g^\alpha = \frac{1}{i}q^\alpha(y,\eta)/\eta_0, \tag{16}$$

(x,ξ_0,f), (resp. (y,η_0,g)) is a chart of $(T^*)'\mathcal{M}$ (resp. $(T^*)'M$). We denote the partial derivatives with respect to the chart (y,η_0,g) by

$$\frac{\tilde\partial}{\partial y^j}, \quad \frac{\tilde\partial}{\partial \eta_0}, \quad \frac{\partial}{\partial g^\alpha}, \quad \frac{\partial}{\partial \overline{g^\alpha}} \tag{17}$$

Our reference points in the cotangent bundles are (with $e_0 = (0,...,0,1)$)

$$(x_\star,\xi_\star),\ (y_\star,\eta_\star) \text{ with } \quad x_\star = y_\star = 0,\ \xi_\star = \eta_\star = e_0. \tag{18}$$

We see easily

$$f^\alpha(x_\star,\xi_\star) = g^\alpha(y_\star,\eta_\star) = 0, \quad \dot h^\alpha(y_*) = 0 \tag{19}$$

2. Construction of the symplectic map. In [4] we constructed a generating function $S(y,\xi)$ of the symplectic map, which transforms the symbols of $\overline{\partial}_\flat$ operators of our CR-structure to the symbols of the model CR-structure. However, when we try to carry out the construction explicitly in straightfoward way, we find that the formula becomes rather cumbersome. In this section we develop a way of calculation so that it becomes more accessible. We use a number of first-order partial differential equations which are readily solvable in terms of formal power series.

A) We recall first our construction of the homogenous symplectic map χ, defined on a conic neighborhood of $(x_\star,\xi_\star)$ in $(T^*)'\mathcal{M}$ and mapped into a neighborhood of $(y_\star,\eta_\star)$ in $(T^*)'M$, with the property: For a symbol r^α_β of homogenous order 0

$$q^\alpha(\chi_t(x,\xi),t) = r^\alpha_\beta(x,\xi,t)\, p^\beta(x,\xi), \quad r^\alpha_\beta(x,\xi,0) = \delta^\alpha_\beta. \tag{1}$$

Let $(x,\xi) \to (y(x,\xi),\eta(x,\xi))$ be a symplectic map. A real valued function λ is called a potential of χ, when, for each fixed (x,ξ), $(y(x,\xi,t),\ \eta(x,\xi,t))$ is the solution of the ordinary differential equation with intial value problem:

$$\begin{aligned} &\frac{dy^j(t)}{dt} = v^j(y(t),\eta(t),t), \quad v^j(y,\eta,t) = \frac{\partial\lambda}{\partial\eta_j} \\ &\frac{d\eta_j(t)}{dt} = v_j(y(t),\eta(t),t), \quad v_j(y,\eta,t) = -\frac{\partial\lambda}{\partial y^j} \\ &(y(0),\eta(0)) = (x,\xi) \end{aligned} \tag{2}$$

It is shown in [4] that χ_t satisfies (1) for a suitable $r(x,\xi)$ when a potential of χ satisfies the equation: For a suitable s^α_β

$$\{q^\alpha, \lambda\} + \dot q^\alpha = s^\alpha_\beta q^\beta. \tag{3}$$

In fact, r and s are related by the equation:

$$\begin{aligned} \frac{dr^\alpha_\beta(x,\xi,t)}{dt} &= s^\alpha_\gamma(y_t(x,\xi),\eta_t(x,\xi),t)\ r^\gamma_\beta(x,\xi,t), \\ r^\alpha_\beta(x,\xi,0) &= \delta^\alpha_\beta, \end{aligned}, \tag{4}$$

We construct a solution λ of (3) defined on a conic neighborhood of $(y_\star . \eta_\star)$. This is of the form:

$$\lambda = \eta_0 \sum_{p\ge 1}(\lambda_{(0,p)} + \lambda_{(p,0)}), \quad \lambda_{(p,0)} = \overline{\lambda_{(0,p)}}, \quad \lambda_{(p,0)} = \lambda_{(p,0)}(y,g). \tag{5}$$

where $\lambda_{(p,q)}$ is a form of type (p,q) in $g, \bar g$ depending on y. In fact $\lambda_{(0,p)}$ are defined inductively by:

$$\lambda_{(0,1)}(y,\bar g) = \overline{g^\alpha}\lambda_{\bar\alpha}, \quad \lambda_{\bar\alpha} = \frac{1}{2}\, c_{\beta\bar\alpha}\ \dot h^\beta = -tN^{(4)}_{\bar\alpha}(z,\bar z, t^2x^0) + ... \tag{6.1}$$

where $N^{(4)}(z,\bar z,x^0)$ is the part of homogenous degree 4 in $(z,\bar z)$ of $(N)_{t=1}$, and $N_{\bar\alpha} = \partial(N)_{t=1}/\partial\overline{z^\alpha}$.

$$\begin{aligned} \lambda_{(0,p+1)}(y,\bar g) &= \frac{i}{p+1}\ \overline{g^\alpha} L^{(p-1)}_{\bar\alpha}\lambda_{(0,p)}(y,\bar g), \\ L^{(p-1)}_{\bar\alpha} &= c_{\beta\bar\alpha}\ (\tilde Q^\beta_y - (p-1)c^\beta), \end{aligned} \tag{6.p+1}$$

where $\tilde Q$ is the operator obtained by replacing $\partial/\partial y^j$ in Q^α by $\tilde\partial/\partial y^j$. It turns out

$$L^{(p-1)}_{\bar\alpha_p}...L^{(0)}_{\bar\alpha_2}\lambda_{\bar\alpha_1} \text{ is symmetric in } \bar\alpha_1,\bar\alpha_2,...,\bar\alpha_p, \quad \text{and} \tag{7}$$

$$\lambda_{(0,p)}(y,\bar g) = \overline{g^{\beta_1}}...\overline{g^{\beta_p}}\lambda_{\bar\beta_1...\bar\beta_p}(y), \tag{8}$$

$$\lambda_{\bar\beta_1...\bar\beta_p}(y) = \frac{i^{p-1}}{p!} L^{(p-2)}_{\bar\beta_p}....L^{(0)}_{\bar\beta_2}\lambda_{\bar\beta_1}. \tag{9}$$

We have

$$\lambda_{(p,0)}(y,g) = g^{\beta_1}...g^{\beta_p}\lambda_{\beta_1...\beta_p}(y), \quad \lambda_{\beta_1...\beta_p}(y) = \overline{\lambda_{\bar\beta_1...\bar\beta_p}(y)}. \tag{10}$$

We see easily

$$\lambda(y_\star,\eta_\star) = 0, \quad v^j(y_*,\eta_*) = 0, \quad v_j(y_*,\eta_*) = 0. \tag{11}$$

$$\chi(x_\star,\xi_\star) = (y_\star,\eta_\star). \tag{12}$$

s^α_β in (3) in our case is given by

$$s^\alpha_\beta = \sum_{p\geq 0} s^\alpha_{\beta(p,0)}(y,g), \quad s^\alpha_{\beta(p,0)} = (pc^\alpha - \tilde{Q}^\alpha)\lambda_{\beta\beta_1\dots\beta_p}(y)\, g^{\beta_1}\dots g^{\beta_p}. \tag{13}$$

B) χ also transforms symbols g to f. Namely, when we set

$$\hat{r}^\alpha_\beta(x,\xi) = \frac{\xi_0}{\eta_0(x,\xi)} r^\alpha_\beta(x,\xi), \tag{14}$$

$$g^\alpha \circ \chi(x,\xi) = \hat{r}^\alpha_\beta(x,\xi) f^\beta(x,\xi). \tag{15}$$

$\hat{r}^\alpha_\beta(x,\xi)$ is determined by the following equation: Set

$$\hat{s}^\alpha_\beta(y,\eta) = s^\alpha_\beta(y,\eta) - v^\flat_0(y,\eta)\delta^\alpha_\beta, \quad v^\flat_0 = \frac{v_0}{\eta_0}. \tag{16}$$

Then

$$\frac{\partial \hat{r}^\alpha_\beta(x,\xi,t)}{\partial t} = \hat{s}^\alpha_\gamma \circ \chi_t\, \hat{r}^\gamma_\beta(x,\xi,t), \quad \hat{r}^\gamma_\beta(x,\xi,0) = \delta^\alpha_\beta \tag{17}$$

Note that $g\circ\chi(x,\xi,t)$ satisfies the equation:

$$\frac{\partial g^\alpha \circ \chi(x,\xi,t)}{\partial t} = \hat{s}^\alpha_\beta \circ \chi_t(x,\xi,t)\, g^\beta \circ \chi(x,\xi,t), \quad (g^\alpha \circ \chi)_{t=0} = f^\alpha. \tag{18}$$

C) We write down $v^j(y,\eta)$ and $v_0(y,\eta)$ more explicitly. We have by definition:

$$v^0 = \sum\{(1-p)(\lambda_{(0,p)} + \lambda_{(p,0)}) + \frac{i}{2}\overline{h^\alpha}\frac{\partial\lambda_{(0.p)}}{\partial\overline{g^\alpha}} - \frac{i}{2}h^\alpha\frac{\partial\lambda_{(p,0)}}{\partial g^\alpha}\}. \tag{19}$$

$$v^j = \sum(\frac{\partial\lambda_{(0.p)}}{\partial\overline{g^\alpha}}\frac{\partial\zeta_\alpha(\eta)}{\partial\eta_j} + \frac{\partial\lambda_{(p,0)}}{\partial g^\alpha}\frac{\partial\overline{\zeta_\alpha}(\eta)}{\partial\eta_j}) \quad \text{for } j>0. \tag{20}$$

Note that

$$v^0 \equiv 0 \pmod{t^2}, \quad v^j \equiv 0 \pmod{t}. \tag{21}$$

When we set

$$v^{(\alpha)} = v^{2\alpha-1} + iv^{2\alpha} = \sum\frac{\partial\lambda_{(0.p)}}{\partial\overline{g^\alpha}}, \quad v^{(\bar\alpha)} = v^{2\alpha-1} - iv^{2\alpha} = \sum\frac{\partial\lambda_{(p,0)}}{\partial g^\alpha}, \tag{22}$$

$$\sum_{k=0}^{2n} v^k\frac{\partial}{\partial y^k} = \sum(v^{(\alpha)}\,\overline{Q}^\alpha + v^{(\bar\alpha)}\,Q^\alpha) + \sum(1-p)(\lambda_{(0,p)} + \lambda_{(p,0)})\frac{\partial}{\partial y^0}. \tag{23.1}$$

By (5) and (6) we find that

$$v^{(\alpha)} = (I + \sum_{p=1}^{\infty} \frac{i^p}{p!} g^{\bar{\alpha}_{p-1}} ... g^{\bar{\alpha}_0} L^{(p-1)}_{\bar{\alpha}_{p-1}} ... L^{(0)}_{\bar{\alpha}_0}) \lambda_{\bar{\alpha}}. \tag{23.2}$$

We have (cf. (10)§1) $v_0 = \eta_0 v_0^\flat$, where

$$v_0^\flat = \sum_p (c^\alpha \frac{\partial \lambda_{(p,0)}}{\partial g^\alpha} - \frac{\tilde{\partial} \lambda_{(p,0)}}{\partial y^0} + \overline{c^\alpha} \frac{\partial \lambda_{(0,p)}}{\partial \overline{g^\alpha}} - \frac{\tilde{\partial} \lambda_{(0,p)}}{\partial y^0}) \equiv 0 \pmod{t^3}. \tag{24}$$

$$\frac{\partial \log \eta_0(x,\xi)}{\partial t} = v_0^\flat(y(x,\xi), \eta(x,\xi)), \quad \eta(x,\xi)_{t=0} = \xi. \tag{25}$$

D) We use the first equation in (2) expressed by $(y, \eta_0, g, \bar{g})$-chart in stead of (y, η)-chart. Set

$$\begin{gathered} v^j(y,\eta) = V^j(y,g,\bar{g}) = V^j_\phi(y) + \sum (V^j_{(0,p)} + V^j_{(p,0)}), \quad V^j_{(p,0)} = \overline{V^j_{(0,p)}} \\ V^j_{(0,p)} = V^j_{(0,p)}(y,\bar{g}) = \sum V^j_{\bar{\beta}_1 ... \bar{\beta}_p}(y)\, \overline{g^{\beta_1}} ... \overline{g^{\beta_p}}. \end{gathered} \tag{26}$$

In general, for a complex vector variable g, $h(y,g,\bar{g})$ denotes a formal sum of $h_{(p,q)}(y,g,\bar{g})$ which is of type (p,q) in $g, \bar{g}$ with coeffcients in the ring of smooth functions in y. We usually write u_ϕ instead of $u_{(0,0)}$. We define $\bar{h}(y,g,\bar{g})$ by $\sum \overline{h_{(p,q)}}$. Hence $(\bar{h})_{(p,q)} = \overline{h_{(q,p)}}$. We say h is real valued when $h = \bar{h}$.

We consider an unknown vector valued function $Y(x,\gamma,\bar{\gamma},t)$ in the independent variables $(x,\gamma,\bar{\gamma},t)$. We look for a partial differential equation for $Y(x,\gamma,\bar{\gamma},t)$ so that its solutiion gives the formula:

$$y(x,\xi,t) = Y(x, g \circ \chi, \bar{g} \circ \chi). \tag{27}$$

We find by (2) and (18) that such an equation is given by

$$\begin{aligned} &\frac{\partial Y^j(x,\gamma,\bar{\gamma},t)}{\partial t} + \frac{\partial Y^j}{\partial \gamma^\alpha}(x,\gamma,\bar{\gamma},t) \hat{s}^\alpha_\beta(Y,\gamma,\bar{\gamma},t)\gamma^\beta \\ &\qquad + \frac{\partial Y^j}{\partial \overline{\gamma^\alpha}}(x,\gamma,\bar{\gamma},t) \hat{s}^{\bar{\alpha}}_{\bar{\beta}}(Y,\gamma,\bar{\gamma},t)\overline{\gamma^\beta} = V^j(Y,\gamma,\bar{\gamma},t), \quad Y(x,\gamma,\bar{\gamma},0) = x. \end{aligned} \tag{28}$$

Note that we regarded $\hat{s}^\alpha_\beta$ as a function in $(y,g,\bar{g},t)$ by (16), (13), and (19).

By the above equation $Y_\phi(x), Y_{(p,q)}(x,\gamma,\bar{\gamma})$ are determined inductively. Namely,

$$\begin{gathered} \frac{\partial Y_\phi(x)}{\partial t} = V_\phi(Y_\phi(x)), \quad (Y_\phi(x))_{t=0} = x, \\ \frac{\partial Y_{(1,0)}(x,\gamma)}{\partial t} + \frac{\partial Y_{(1,0)}}{\partial \gamma^\alpha} \hat{s}^\alpha_{\beta\phi}(Y_\phi(x))\gamma^\beta = \frac{\partial V_\phi}{\partial y^k}(Y_\phi(x)) Y^k_{(1,0)} + V_{(1,0)}(Y_\phi(x),\gamma), \end{gathered} \tag{29}$$

$(Y_{(1,0)})_{t=0} = 0$, and so on.

Since $Y(x,\gamma,\bar{\gamma})$ is determined, to determine $y(x,\xi)$ it remains to determine $\gamma = g \circ \chi$, which is a function in (x,ξ,t). However, it is more convenient to regard it as a function in $(x, f, \bar{f}, t)$. Then we see by (18) and (29) that $\gamma(x, f, \bar{f})$ is determined by the equation:

$$\frac{\partial \gamma^a(x, f, \bar{f}, t)}{\partial t} = \hat{s}^\alpha_\beta(Y(x, \gamma, \bar{\gamma}, t), \gamma, \bar{\gamma}, t)\gamma^\beta, \quad \gamma^\alpha(x, f, \bar{f}, 0) = f^\alpha. \tag{30}$$

where f is regarded as an independent complex vector variable. Therefore we reached the following conclusion:

(31) Proposition. Let $Y(x, \gamma, \bar{\gamma}, t)$ be the solution of (28). Denote by $\gamma(x, f, \bar{f}, t)$ the solution of the equation (30). Then

$$y(x, \xi) = Y(x, \gamma(x, f(x, \xi), \bar{f}(x, \xi), t), \bar{\gamma}(x, f(x, \xi), \bar{f}(x, \xi), t), t). \tag{32}$$

Note also that

$$g \circ \chi(x, \xi) = \gamma(x, f(x, \xi), \bar{f}(x, \xi), t). \tag{33}$$

3. The construction of the generating function. A) For each ξ and t let $y \to x = x(y, \xi, t)$ be the inverse map of the map :

$$x \to y = y(x, \xi, t) \quad \text{Then define} \tag{1}$$

$$S(y, \xi, t) = \xi_j x^j(y, \xi). \quad \text{We see easily} \tag{2}$$

$$S(y_*, \xi_*, t) = 0, \tag{3}$$

$$S(y, \xi, t) \equiv y \cdot \xi \pmod{t}. \tag{4}$$

For simplicity we usually omit t. We have (because χ is a symplectic map)

$$S'_\xi(y, \xi) = x(y, \xi). \tag{5}$$

$S(y, \xi)$ is the generating function of our symplectic map χ. Namely, we have the following:

(6) Proposition. (x, ξ, y, η) is in the orbit of $\chi(x, \xi)$ if and only if

$$x = S'_\xi(y, \xi), \quad \eta = S'_y(y, \xi).$$

B) To construct $x(y, \xi)$ we employ the same method we used to construct $y(x, \xi)$. Namely, we introduce complex vector valued independent variables $\omega = (..., \omega^\alpha, ...)$, $\bar{\omega}$ and consider an unknown vector valued function $x^*(y, \omega, \bar{\omega})$ in $(y, \omega, \bar{\omega})$. We then find a partial differential equation for $x^*(y, \omega, \bar{\omega}, t)$ such that its solution gives the formula:

$$x(y, \xi) = x^*(y, f(x(y, \xi), \xi), \bar{f}(x(y, \xi), \xi)). \tag{7}$$

To write down such partial differential equation, note that $x(y,\xi)$ is determined by

$$y = y(x(y,\xi),\xi), \quad \text{i.e. by(31)§2} \tag{8.1}$$

$$\begin{aligned} y = Y(x(y,\xi), \gamma(x(y,\xi), f(x(y,\xi),\xi), \bar{f}(x(y,\xi),\xi)),\\ \bar{\gamma}(x(y,\xi), f(x(y,\xi),\xi), \bar{f}(x(y,\xi),\xi))). \end{aligned} \tag{8.2}$$

In view of (7),we then see that, for our purpose, it is enough to determined $x^*(y,\omega,\bar{\omega})$ satisfying the condition:

$$y = Y(x^*(y,\omega,\bar{\omega}), \gamma(x^*(y,\omega,\bar{\omega}),\omega,\bar{\omega}), \overline{\gamma}(x^*(y,\omega,\bar{\omega}),\omega,\bar{\omega})). \tag{9}$$

We will write down the partial differential equation satisfied by the solution of the above equation.

For an unknown function $x^*(y,\omega,\bar{\omega})$ we set for simplicity

$$\gamma(x^*(y,\omega,\bar{\omega}),\omega,\bar{\omega}) = \gamma_\flat(y,\omega,\bar{\omega}). \tag{10}$$

Set

$$\begin{aligned} \tilde{Y}^i_k(y,\omega,\bar{\omega}) =& \frac{\partial Y^i}{\partial x^k}(x^*(y,\omega,\bar{\omega}), \gamma_\flat(y,\omega,\bar{\omega}), \overline{\gamma_\flat}(y,\omega,\bar{\omega}))\\ &+ \frac{\partial Y^i}{\partial \gamma^\alpha}(x^*(y,\omega,\bar{\omega}), \gamma_\flat(y,\omega,\bar{\omega}), \overline{\gamma_\flat}(y,\omega,\bar{\omega}))\frac{\partial \gamma^\alpha}{\partial x^k}(x^*(y,\omega,\bar{\omega}),\omega,\bar{\omega})\\ &+ \frac{\partial Y^i}{\partial \overline{\gamma^\alpha}}(x^*(y,\omega,\bar{\omega}), \gamma_\flat(y,\omega,\bar{\omega}), \overline{\gamma_\flat}(y,\omega,\bar{\omega}))\frac{\partial \overline{\gamma^\alpha}}{\partial x^k}(x^*(y,\omega,\bar{\omega}),\omega,\bar{\omega}). \end{aligned} \tag{11}$$

Differentiating (9) in y^j, ω^α, and in t, we find that

$$\delta^i_j = \tilde{Y}^i_k(y,\omega,\bar{\omega})\frac{\partial (x^*)^k(y,\omega,\bar{\omega})}{\partial y^j}. \tag{12}$$

$$\begin{aligned} 0 =& \tilde{Y}^i_k(y,\omega,\bar{\omega})\frac{\partial (x^*)^k(y,\omega,\bar{\omega})}{\partial \omega^\alpha}\\ &+ \frac{\partial Y^i}{\partial \gamma^\beta}(x^*(y,\omega,\bar{\omega}), \gamma_\flat(y,\omega,\bar{\omega}), \overline{\gamma_\flat}(y,\omega,\bar{\omega}))\frac{\partial \gamma^\beta}{\partial \omega^\alpha}(x^*(y,\omega,\bar{\omega}),\omega,\bar{\omega})\\ &+ \frac{\partial Y^i}{\partial \overline{\gamma^\beta}}(x^*(y,\omega,\bar{\omega}), \gamma_\flat(y,\omega,\bar{\omega}), \overline{\gamma_\flat}(y,\omega,\bar{\omega}))\frac{\partial \overline{\gamma^\beta}}{\partial \omega^\alpha}(x^*(y,\omega,\bar{\omega}),\omega,\bar{\omega}). \end{aligned} \tag{13}$$

$$\begin{aligned} 0 =& \frac{\partial Y^i}{\partial t}(x^*(y,\omega,\bar{\omega}), \gamma_\flat(y,\omega,\bar{\omega}), \overline{\gamma_\flat}(y,\omega,\bar{\omega}))\\ &+ \tilde{Y}^i_k(y,\omega,\bar{\omega})\frac{\partial (x^*)^k(y,\omega,\bar{\omega})}{\partial t} + R^i, \quad \text{where} \end{aligned} \tag{14}$$

$$\begin{aligned} R^i =& \frac{\partial Y^i}{\partial \gamma^\beta}(x^*(y,\omega), \gamma_\flat(y,\omega,\bar{\omega}), \overline{\gamma_\flat}(y,\omega,\bar{\omega}))\frac{\partial \gamma^\beta}{\partial t}(x^*(y,\omega),\omega,\bar{\omega})\\ & \frac{\partial Y^i}{\partial \overline{\gamma^\beta}}(x^*(y,\omega), \gamma_\flat(y,\omega,\bar{\omega}), \overline{\gamma_\flat}(y,\omega,\bar{\omega}))\frac{\partial \overline{\gamma^\beta}}{\partial t}(x^*(y,\omega),\omega,\bar{\omega}) \end{aligned}. \tag{15}$$

When we substitute $\frac{\partial Y}{\partial t}$ in (14) by using (28)§2, we find by (30)§2 that R^i cancels out and we obtain

$$\tilde{Y}^i_k(y,\omega,\bar{\omega})\frac{\partial (x^*)^k(y,\omega,\bar{\omega})}{\partial t}+V^i(y,\gamma_\flat(y,\omega,\bar{\omega}),\overline{\gamma_\flat}(y,\omega,\bar{\omega}))=0. \tag{16}$$

Hence, when we set

$$\mathbf{F}=\frac{\partial}{\partial t}+V^k(y,\gamma_\flat(y,\omega,\bar{\omega}),\overline{\gamma_\flat}(y,\omega,\bar{\omega}))\frac{\partial}{\partial y^k}, \tag{17}$$

we see by (13) that

$$[\mathbf{F},x^*(y,\omega,\bar{\omega})]=0,\quad x^*(y,\omega,\bar{\omega})_{t=0}=y. \tag{18}$$

In view of the formal uniqueness of Cauchy-Kowalewsky equation, this equation determines $x^*(y,\omega,\bar{\omega})$, provided we know $\gamma_\flat(y,\omega,\bar{\omega})$. We next write down the equation which characterize it. Note by (12) that

$$\frac{\partial \gamma_\flat(y,\omega,\bar{\omega})}{\partial t}=\frac{\partial \gamma}{\partial t}(x^*(y,\omega,\bar{\omega}),\omega,\bar{\omega})+\frac{\partial \gamma}{\partial x^k}(x^*(y,\omega,\bar{\omega}),\omega,\bar{\omega})\frac{\partial (x^*)^k(y,\omega,\bar{\omega})}{\partial t}. \tag{19}$$

If $\gamma_\flat(y,\omega,\bar{\omega})=(\gamma_\flat)_\phi(y)+\sum(\gamma_\flat)_{\alpha_1\ldots\alpha_p\bar{\beta}_1\ldots\bar{\beta}_q}(y)\omega^{\alpha_1}\ldots\omega^{\alpha_p}\overline{\omega^{\beta_1}}\ldots\overline{\omega^{\beta_q}}$, we set

$$\bar{\gamma}_\flat(y,\omega,\bar{\omega})=\overline{(\gamma_\flat)_\phi(y)}+\sum\overline{(\gamma_\flat)_{\alpha_1\ldots\alpha_p\bar{\beta}_1\ldots\bar{\beta}_q}(y)}\omega^{\beta_1}\ldots\omega^{\beta_q}\overline{\omega^{\alpha_1}}\ldots\overline{\omega^{\alpha_p}}. \tag{20}$$

We then see by (30)§2 and (11)

$$\begin{aligned}\frac{\partial \gamma^\alpha_\flat(y,\omega,\bar{\omega})}{\partial t}&=\hat{s}^\alpha_\beta(y,\gamma_\flat(y,\omega,\bar{\omega}),\overline{\gamma_\flat}(y,\omega,\bar{\omega}))\gamma^\beta_\flat(y,\omega,\bar{\omega})\\ &-V^k(y,\gamma_\flat(y,\omega,\bar{\omega}),\overline{\gamma_\flat}(y,\omega,\bar{\omega}))\frac{\partial \gamma_\flat(y,\omega,\bar{\omega})}{\partial y^k},\quad (\gamma_\flat(y,\omega,\bar{\omega}))_{t=0}=\omega.\end{aligned} \tag{21}$$

Therefore we have the following:

(22) Proposition. Define $\gamma_\flat(y,\omega,\bar{\omega})$ as the solution of the equation (21). Define $x^*(y,\omega,\bar{\omega})$ as the solution of the equation:

$$\begin{aligned}&\frac{\partial x^*(y,\omega,\bar{\omega})}{\partial t}+V^k(y,\gamma_\flat(y,\omega,\bar{\omega}),\overline{\gamma_\flat}(y,\omega,\bar{\omega}))\frac{\partial x^*(y,\omega,\bar{\omega})}{\partial y^k}=0,\\ &(x^*(y,\omega,\bar{\omega}))_{t=0}=y.\end{aligned} \tag{23}$$

Then $x(y,\xi)$ has the expression:

$$x(y,\xi)=x^*(y,f(x(y,\xi),\xi),\bar{f}(x(y,\xi),\xi)). \tag{24}$$

Since $x^*(y,\omega,\bar{\omega})\equiv y \pmod t$, the above formula determines $x(y,\xi)$ as a formal power series in t. Since $V^k(y,\gamma,\bar{\gamma})$ is real valued, we see easily that $x(y,\omega,\bar{\omega})$ is real valued.

We see by (21) that

$$\gamma_\flat^\alpha(y,0,\bar\omega)=0. \tag{25}$$

Hence we can write

$$\gamma_\flat^\alpha(y,\omega,\bar\omega)=\gamma_{\flat\beta}^\alpha(y,\omega,\bar\omega)\omega^\beta. \tag{26}$$

Therefore $x^*(y,\omega,0)$, $x^*(y,0,\bar\omega)$, and $\gamma_\flat(y,\omega,0)$ are determined by the equations:

$$\frac{\partial x^*(y,\omega,0)}{\partial t}+V^k(y,\gamma_\flat(y,\omega,0),0)\frac{\partial x^*(y,\omega,0)}{\partial y^k}=0, \quad (x^*(y,\omega,0)_{t=0}=y. \tag{27}$$

$$\frac{\partial x^*(y,0,\bar\omega)}{\partial t}+V^k(y,0,\overline{\gamma_\flat}(y,0,\bar\omega))\frac{\partial x^*(y,0,\bar\omega)}{\partial y^k}=0, \quad (x^*(y,0,\bar\omega))_{t=0}=y. \tag{28}$$

$$\begin{aligned}\frac{\partial \gamma_\flat^\alpha(y,\omega,0)}{\partial t}&=\hat{s}_\beta^\alpha(y,\gamma_\flat(y,\omega,0),0)\gamma_\flat^\beta(y,\omega,0)\\ &-V^k(y,\gamma_\flat(y,\omega,0),0)\frac{\partial \gamma_\flat^\alpha(y,\omega,0)}{\partial y^k}, \quad (\gamma_\flat(y,\omega,0))_{t=0}=\omega.\end{aligned} \tag{29}$$

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