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REAL AND COMPLEX HAMILTONIAN MECHANICS ON SOME SUBRIEMANNIAN MANIFOLDS *

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Dedicated to Professor Yum-Tong Siu on his sixtieth birthday

Abstract. We prove geodesic completeness and global conectivity for a step $2k+2$ subRiemannian manifold. Using complex Hamiltonian mechanics we also calculate some subRiemannian distances.

1. Introduction. SubRiemannian geometry starts with the Carathéodory's formalization of thermodynamics [6] where the quasi-static adiabatic processes are related to the integral curves of a Darboux model. The result is that any two points may be joined by a smooth integral curve. Next Chow [7] proves that any two points in a manifold with a distribution with the bracket generating property may be connected by piecewise smooth integral curves; the bracket generating property says that at every point the span of the iterated Lie brackets of vector fields tangent to the given distribution span the tangent space. Recently Gromov [9,10] proved that Chow's piecewise smooth integral curves may be replaced by smooth integral curves.

We start with a subelliptic operator Δ_X of the form

$$\Delta_X = \sum_{j=1}^{m} X_j^2$$

on a n-dimensional manifold $\mathcal{M}_n$ where the vector fields $X_1,...,X_m$ are linearly independent and bracket generating, see [7] and [13]. The subbundle X of $T\mathcal{M}_n$, X spanned by $X_1,...,X_m$ is a distribution. We introduce a metric on X by choosing $X_1,...,X_m$ for an orthonormal basis. Chow's theorem allows us to connect any two points in $\mathcal{M}_n$ with a curve tangent to X. Minimizing the lengths of such curves yields a distance function on $\mathcal{M}_n$; often referred to as the Carnot-Carathéodory distance. Thus the subelliptic operator Δ_X induces a subRiemannian geometry in an analogous to the way the elliptic Laplace-Beltrami operator induces Riemannian geometry. The step of the operator Δ_X is the minimum number of brackets necessary to obtain $T\mathcal{M}_n$ from $X_1,...,X_m$ plus one. The principal symbol of Δ_X is the Hamiltonian, and the projection of a bicharacteristic onto $\mathcal{M}_n$ is a geodesic; articles on control theory sometime refer to such geodesics as normal.

A great deal of work on these geometries is done from a control theoretic point of view. Our interests is somewhat different, we study the subRiemannian geometric concepts with a view to finding fundamental solutions, heat kernels, etc. for Δ_X in invariant terms. We mention some work relevant to this paper. Beals, Gaveau and Greiner [1] study the subRiemannian geodesics on the Heisenberg group $\mathbf{H}_n$. In particular, they show that for isotropic $\mathbf{H}_n$ the t-axis is the set of conjugate points

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for geodesics starting from the origin. Strichartz [17,18] pointed out that in a step 2 subRiemannian manifold every point has a cut point arbitrarily near it. One may say that subRiemannian geometric concepts are global. Greiner and Calin [8] discussed this more precisely on a particular step 3 example.

This paper is concerned with the geometry of the subelliptic model of [5]. In section 2 we introduce the Euler-Lagrange equations, solve them for geodesics starting from the origin in section 3 and for geodesics starting outside of the origin in section 4. In sections 5 and 6 we prove geodesic completeness and global connectivity by geodesics. Section 7 is devoted to complex Hamiltonian mechanics. In particular, we show that the critical points of the modified complex action function $f(\tau)$ yield the lengths of the geodesics starting from the origin; in the step 2 case this recovers results of [1].

2. Euler Lagrange equations. Consider the vector fields on $\mathbb{R}^3$

$$X_1 = \partial_{x_1} + 2x_2|x|^{2k}\partial_t \ , \qquad X_2 = \partial_{x_2} - 2x_1|x|^{2k}\partial_t. \tag{1}$$

The Hamiltonian H is defined as the principal symbol of the X-Laplacian

$$\Delta_X = \frac{1}{2}(X_1^2 + X_2^2) \tag{2}$$

$$H(\xi,\theta,x,t) = \frac{1}{2}\Big(\xi_1 + 2x_2|x|^{2k}\theta\Big)^2 + \frac{1}{2}\Big(\xi_2 - 2x_1|x|^{2k}\theta\Big)^2. \tag{3}$$

Definition 2.1. *A geodesic connecting the points $P(x_0,t_0)$ and $Q(x,t)$ is the projection on the (x,t)-space of the solution of the Hamilton's system*

$$\dot{x} = H_\xi, \qquad \dot{t} = H_\theta,$$

$$\dot{\xi} = -H_x, \qquad \dot{\theta} = -H_t$$

with the boundary conditions $x(0) = x_0$, $t(0) = t_0$, $x(s_f) = x$, $t(s_f) = t$.

Using the Legendre transform, one obtains the Lagrangian

$$L(x,\dot{x},\dot{t},\theta) = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \theta\dot{t} - 2\theta|x|^{2k}(x_2\dot{x}_1 - x_1\dot{x}_2), \tag{4}$$

which in polar coordinates $x_1 = r\cos\phi$, $x_2 = r\sin\phi$ takes the following form

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + \theta\dot{t} + 2\theta r^{2k+2}\dot{\phi}. \tag{5}$$

θ is constant along the solutions since $\dot{\theta} = -\partial H/\partial t = 0$. The Euler-Lagrange system of equations is

$$\begin{cases} \ddot{r} = r\dot{\phi}(\dot{\phi} + 2(2k+2)\theta r^{2k}) \\ r^2(\dot{\phi} + 2\theta r^{2k}) = C(\text{constant}) \\ \theta = \text{constant}. \end{cases} \tag{6}$$

3. Solutions which start from the origin. When $r(0) = 0$, $C = 0$ and (6) takes the form

$$\begin{cases} \ddot{r} = r\dot{\phi}(\dot{\phi} + 2(2k+2)\theta r^{2k}) \\ \dot{\phi} = -2\theta r^{2k} \\ \theta = \text{constant}. \end{cases} \tag{7}$$

Eliminating ϕ from the first two equations

$$\ddot{r} = -4(2k+1)\theta^2 r^{4k+1}. \tag{8}$$

If we set

$$V(r) = 2\theta^2 r^{2(2k+1)}, \tag{9}$$

then (8) may be written as Newton's equation

$$\ddot{r} = -V'(r), \tag{10}$$

where $V'(r) = dV(r)/dr$, $V(r)$ the potential energy. With $\dot{r} = p$ we have

$$\ddot{r} = \frac{dp}{ds} = \frac{dp}{dr}\frac{dr}{ds} = p\frac{dp}{dr} = -4(2k+1)\theta^2 r^{4k+1}.$$

Integrating,

$$\frac{1}{2}\dot{r}^2 + 2\theta^2 r^{2(2k+1)} = E \tag{11}$$

where E is a constant which denotes the total energy of the system. In fact $E = \dot{r}^2(0)/2$, and using $|\dot{x}(s)|^2 = \dot{r}^2(s) + r^2(s)\dot{\phi}(s)$, one obtains $E = \frac{1}{2}|\dot{x}(0)|^2$. Thus, we obtain the law of conservation of energy

$$\frac{1}{2}\dot{r}^2 + V(r) = E. \tag{12}$$

The radius $r(s)$ starts at 0 and increases ($\dot{r} > 0$) until $\dot{r} = 0$, when $r = r_{max}$. After that, $r(s)$ decreases ($\dot{r} < 0$) to 0. r_{max} can be found by letting $\dot{r} = 0$ in equation (12). Hence r_{max} is the positive solution of the equation

$$V(r) = E, \tag{13}$$

which is

$$r_{max} = \left(\frac{E}{2\theta^2}\right)^{\frac{1}{2(2k+1)}}, \tag{14}$$

see figure 1.

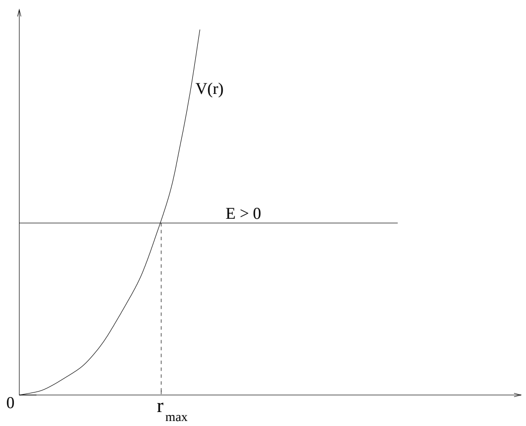

FIG. 1. r_{max} *is the solution of the equation* $V(r) = E$.

For each level of energy $E > 0$ there is a corresponding $r_{max} > 0$, which is independent of the initial direction of the solution. The solution lies in the disk $D(0, r_{max})$. The greater the energy E, the larger r_{max} is. As $E = |\dot{x}(0)|^2/2$ depends only on the magnitude of the initial speed, r_{max} will do the same. The second equation of (7) provides

$$\dot{\phi} = -2\theta r^{2k}. \tag{15}$$

As $r(0) = 0$, $\dot{\phi}(0) = 0$. For $\theta > 0$, $\dot{\phi}(s) < 0$, i.e. ϕ is decreasing. For $\theta < 0$, $\dot{\phi}(s) > 0$, i.e. ϕ is increasing. For $\theta = 0$, ϕ is constant.

Equations (11) and (15) yield

$$\frac{dr}{ds} = \sqrt{2E - 4\theta^2 r^{2(2k+1)}}, \qquad \frac{d\phi}{ds} = -2\theta r^{2k} \tag{16}$$

where $r(s)$ was supposed to be increasing from $r(0) = 0$ to r_{max}. We eliminate s by dividing the equations in (16) and obtain

$$\frac{d\phi}{dr} = \frac{-2\theta r^{2k}}{\sqrt{2E - 4\theta^2 r^{2(2k+1)}}}, \tag{17}$$

which yields

$$\phi(s) - \phi(0) = \int_0^{r(s)} \frac{-2\theta x^{2k}}{\sqrt{2E - 4\theta^2 x^{2(2k+1)}}}\, dx. \tag{18}$$

When $r = r_{max}$, the above variation of ϕ is denoted by ϕ_+, given by

$$\begin{aligned}
\phi_+ &= \int_0^{r_{max}} \frac{-2\theta x^{2k}}{\sqrt{2E - 4\theta^2 x^{2(2k+1)}}}\,dx \\
&= \frac{-2\theta}{\sqrt{2E}} \int_0^1 \Big(\frac{E}{2\theta^2}\Big)^{2k/(4k+2)} \Big(\frac{E}{2\theta^2}\Big)^{1/(4k+2)} u^{2k}\,\sqrt{1-u^{4k+2}}\,du \\
&= \frac{-\theta}{|\theta|}\,\frac{1}{2k+1}\int_0^1 \frac{v\,dv}{\sqrt{1-v^2}} \\
&= -\frac{1}{2k+1} sgn(\theta)\Big(\arcsin(1) - \arcsin(0)\Big) \\
&= -sgn(\theta)\frac{\pi}{2(2k+1)}.
\end{aligned}$$

PROPOSITION 3.1. *The angle ϕ_+ swept out by the vectorial radius between $r = 0$ and $r = r_{max}$ is*

$$\phi_+ = \begin{cases} \frac{\pi}{2(2k+1)}, & \textit{if } \theta < 0, \\ -\frac{\pi}{2(2k+1)}, & \textit{if } \theta > 0. \end{cases} \tag{19}$$

The above proposition agrees with the fact that the particle is moving clock-wise for $\theta > 0$ and counter-clock-wise for $\theta < 0$. Between $r = r_{max}$ and $r = 0$ the vectorial radius sweeps an angle ϕ_-. As r is decreasing, $\dot r < 0$, then

$$\frac{d\phi}{dr} = \frac{2\theta r^{2k}}{\sqrt{2E - 4\theta^2 r^{2(2k+1)}}}, \tag{20}$$

and

$$\begin{aligned}
\phi_- = \int_{r_{max}}^0 \frac{2\theta r^{2k}}{\sqrt{2E - 4\theta^2 r^{2(2k+1)}}}\,dr &= \int_0^{r_{max}} \frac{-2\theta r^{2k}}{\sqrt{2E - 4\theta^2 r^{2(2k+1)}}}\,dr \\
&= -sgn(\theta)\frac{\pi}{2(2k+1)} = \phi_+.
\end{aligned}$$

The solution passes through the origin again after sweeping out an angle equal to $\phi_+ + \phi_- = \frac{\pi}{2k+1}$ (clock-wise if $\theta < 0$ and counter-clock-wise if $\theta > 0$). This means, the angle between the tangents at the origin is $\frac{\pi}{2k+1}$. As this angle is commensurable with 2π, the projection of the geodesic on the x-plane will be closed and periodic.

4. Solutions which start outside the origin. The solutions with $r(0) \neq 0$ will be discussed. The system (6) is considered with the initial conditions

$$r(0) \neq 0\,, \quad \phi(0) = \phi_0. \tag{21}$$

The second equation of (6) becomes

$$\dot\phi = \frac{C}{r^2} - 2\theta r^{2k}. \tag{22}$$

Substituting in the first equation of (6), one obtains

$$\ddot r = \frac{C^2}{r^3} + 4k\theta C r^{2k-1} - 4\theta^2(2k+1) r^{4k+1}. \tag{23}$$

Let

$$V(r) = \frac{1}{2}\Big(\frac{C^2}{r^2} - 4\theta C r^{2k} + 4\theta^2 r^{4k+2}\Big) = \frac{1}{2}\left(\frac{2\theta r^{2k+2} - C}{r}\right)^2. \tag{24}$$

Then, one may rewrite equation (23) as Newton's equation

$$\ddot{r} = -V'(r). \tag{25}$$

Substituting $p = \dot{r}$, equation (23) becomes

$$p\,\frac{dp}{dr} = -\frac{dV}{dr},$$

which yields

$$\frac{p^2}{2} = -V(r) + E,$$

where E is a constant, the total energy.

Proposition 4.1. *If r is a solution of the Euler-Lagrange system with initial conditions* (21)*, then the energy is conserved:*

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(\frac{2\theta r^{2k+2} - C}{r}\right)^2 = E. \tag{26}$$

We shall give a qualitative description of the solutions of (26). There are two cases: $sgn(C) = sgn(\theta)$ and $sgn(C) \neq sgn(\theta)$. In each case we shall draw the graph of the potential energy V and the trajectory of the solution in the phase plane.

Case $\mathbf{sgn}(C) = \mathbf{sgn}(\theta)$. In this case $V(r)$ has a zero and a global minimum at

$$r_0 = \Big(\frac{C}{2\theta}\Big)^{1/2(k+1)}, \tag{27}$$

and $\lim_{x\to 0^+} V(r) = \lim_{x\to\infty} V(r) = \infty$.

Equilibrium points. r_0 corresponds to a stable equilibrium point,

$$V'(r_0) = 0 \ \text{ and } \ V''(r_0) > 0.$$

The corresponding ϕ satisfies

$$\dot{\phi} = \frac{C}{r_0^2} - 2\theta r_0^{2k} = \frac{C - 2\theta r_0^{2(k+1)}}{r_0^2} = 0$$

and hence ϕ is constant. This means that each point of the circle $\mathcal{S}(0, r_0)$ is an equilibrium point (constant solution). At each of these points the energy $E = 0$.

Tangent circles. When $E > 0$, there are exactly two positive roots r_{min} and r_{max} of the equation

$$V(r) = E, \tag{28}$$

such that

$$0 < r_{min} \leq r_0 \leq r_{max}. \tag{29}$$

The solution in the phase plane and the potential V are shown in figure 2.

PROPOSITION 4.2. *The solution of the Euler-Lagrange system (6) in the case $sgn(C) = sgn(\theta)$ lies in the circular ring $\mathcal{W}(0, r_{min}, r_{max}) = \{x \in \mathbb{R}^2; r_{min} \leq |x| \leq r_{max}\}$.*

Proof. r is equal to r_{min} or r_{max} when $\dot{r} = 0$. The equation of conservation of energy (26) shows that r_{min} and r_{max} are solutions of the equation $V(r) = E$ and the solution r has the property

$$0 < r_{min} \leq r \leq r_{max}.$$

□

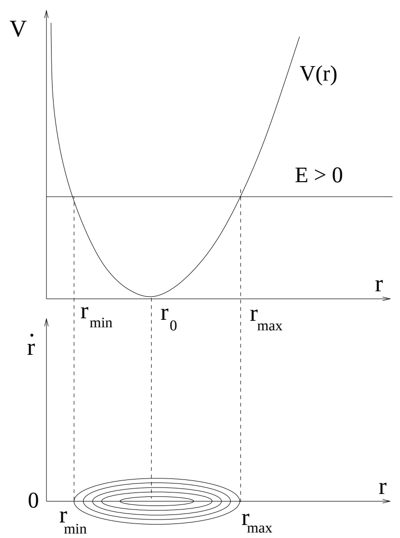

FIG. 2. *Solution in phase plane, case $sgn(C) = sgn(\theta)$.*

Existence of loops. The equation $\dot{\phi} = 0$ has the positive root

$$r = \left(\frac{C}{2\theta}\right)^{\frac{1}{2(k+1)}}. \tag{30}$$

Hence $\dot{\phi} = 0$ if and only if $r = r_0$. When the solution intersects the circle of equilibrium points $\mathcal{S}(0, r_0)$, the sign of $\dot{\phi}$ changes, i.e. the trajectory is bouncing back, making loops in the ring $\mathcal{W}(0, r_{min}, r_{max})$, see figure 3.

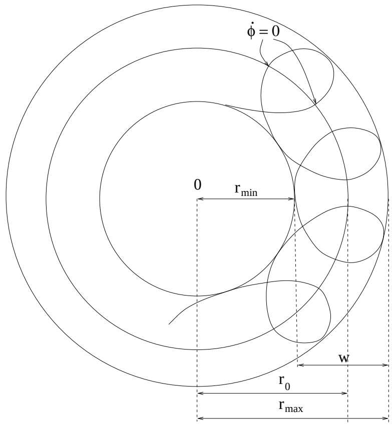

FIG. 3. *Projection of the geodesic on the x-plane, case sgn(C) = sgn(θ).*

The ring width. $w = r_{max} - r_{min}$ is a function of energy $w = w(E)$. Using properties of the function $V(r)$, we find
(i) $w(E)$ is an increasing function of energy E;
(ii) $w(0) = 0$ if and only if $r_{min} = r_0 = r_{max}$;
(iii) $\lim_{E\to\infty} r_{min} = 0$, $\lim_{E\to\infty} r_{max} = \infty$.

The t-component. From the Hamiltonian equation

$$\dot t = \frac{\partial H}{\partial \theta} = 2|x|^{2k}(x_2\dot x_1 - x_1\dot x_2) = -2r^{2(k+1)}\dot\phi,$$

$\dot\phi = 0$ on the circle $\mathcal{S}(0, r_0)$ implies that $\dot t(s)$ changes sign and reaches the local maxima and minima when the geodesic crosses the cylinder $\mathcal{S}(0, r_0) \times \mathbb{R}$.

Case sgn$(C) \neq$ **sgn**(θ), $C \neq 0$**.** Here

$$\dot\phi = \frac{C - 2\theta r^{2(k+1)}}{r^2} \neq 0,$$

which means that ϕ increases (decreases) for $\theta < 0$ $(\theta > 0)$ and hence there are no loops. In this case the potential energy is strictly positive

$$V(r) = \frac{1}{2}\Big(\frac{2\theta r^{2k+2} - C}{r}\Big)^2 > 0,$$

and has a minimum at

$$r_1 = \Big(\frac{-C}{2\theta(2k+1)}\Big)^{\frac{1}{2(k+1)}}. \tag{31}$$

Indeed, $V'(r) = 0$ if and only if

$$4\theta^2(2k+1)r^{4k+4} - 4k\theta C r^{2k+2} - C^2 = 0.$$

Substituting $u = r^{2k+2}$ we obtain a quadratic equation in u

$$4\theta^2(2k+1)u^2 - 4k\theta C u - C^2 = 0,$$

with the roots

$$u_1 = \frac{C}{2\theta} \text{ and } u_2 = -\frac{C}{2\theta(2k+1)}.$$

As r_1 is positive, we choose

$$(r_1)^{2(k+1)} = -\frac{C}{2\theta(2k+1)}.$$

The equilibrium solution. The equilibrium solution is the circle of radius r_1 centered at the origin. The corresponding ϕ can be obtained from

$$r_1^2(\dot{\phi} + 2\theta r_1^{2k}) = C$$

and it is equal to

$$\phi(s) = \Big(\frac{C}{r_1^2} - 2\theta r_1^{2k}\Big) + \phi_0.$$

Bounds for $r(s)$. There are two positive roots ρ_{min} and ρ_{max} for the equation

$$V(r) = E\ , \qquad \text{where } E \geq V(r_1)$$

and $0 < \rho_{min} \leq r_1 \leq \rho_{max}$.

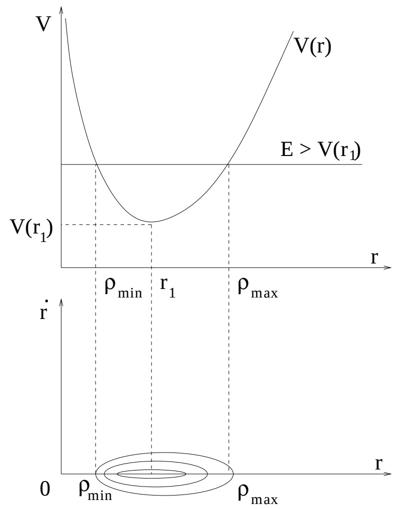

FIG. 4. *Solution in the phase plane, case $sgn(C) \neq sgn(\theta)$.*

In the phase plane the solution is rotating around the stable equilibrium point $(r_1, 0)$ such that

$$\rho_{min} \leq r(s) \leq \rho_{max}.$$

The projection of the geodesics in the x-plane is contained in the ring $\mathcal{W}(0, \rho_{min}, \rho_{max})$. The width of the ring increases as the total energy E increases, where $E = \frac{1}{2}|\dot{x}(0)|^2$. See figure 5.

The t-component. In this case $\dot{t} = -2r^{2(k+1)}\dot{\phi} \neq 0$. Hence $t(s)$ increases if $\theta > 0$, decreases if $\theta < 0$, it is constant if $\theta = 0$.

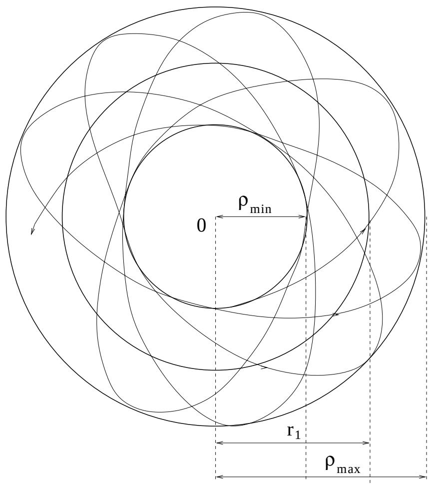

FIG. 5. *Projection of the geodesic on the x-plane, case sgn(C) ≠ sgn(θ).*

5. Geodesic completeness.

DEFINITION 5.1. *If for any point P, any geodesic $c(t)$ emanating from P is defined for all $t \in \mathbb{R}$, the geometry is called geodesically complete.*

The main result of this section is

THEOREM 5.2. *The geometry induced by the vector fields (1) is geodesically complete.*

We note that geodesic completeness does not hold in general. A counterexample in the step 2 case is provided in Calin [4]. To prove Theorem 5.2 we need a few results regarding extendability of solutions of differential equations. The following result can be found for instance in Hartman [12].

LEMMA 5.3. *Let $f(t, y)$ be a continuous function on a strip $t_0 \leq t \leq t_0 + a < \infty$, $y \in \mathbb{R}^d$ arbitrary. Let $y = y(t)$ denote a solution of*

$$y' = f(t, y), \quad y(t_0) = y_0 \tag{32}$$

on a right maximal interval J. Then either $J = [t_0, t_0 + a]$ or $J = [t_0, \delta)$, $\delta \leq t_0 + a$, and $|y(t)| \to \infty$ as $t \to \delta$.

The Hamiltonian system of equations can be written in the form (32). Consider $y \in \mathbb{R}^{2n}$, $y = (x, p)$, with $x, p \in \mathbb{R}^n$. Let $H(y)$ be the Hamiltonian function, and set

$$\nabla H(y) = \Big(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial p}\Big),$$

and

$$\mathcal{J} = \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix} \in \mathcal{M}_{2n}(\mathbb{R}).$$

Then

$$\mathcal{J}\,\nabla H(y) = \Big(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial x}\Big).$$

Hamilton' s system

$$\begin{cases} \dot{x}(s) = \partial H/\partial p \\ \dot{p}(s) = -\partial H/\partial x \end{cases} \tag{33}$$

can be written as

$$\dot{y}(s) = f(y(s)), \tag{34}$$

where $f(y) = \mathcal{J}\,\nabla H(y)$. Since f is independent of s, we may rewrite Lemma 5.3 as

Corollary 5.4. *Let $f(y)$ be an arbitrary continuous function for $y \in \mathbb{R}^{2n}$ and let $y(s)$ be a solution for*

$$\dot{y}(s) = f(y(s)), \quad y(0) = y_0$$

on a right maximal interval I. Then either $I = [0,\infty)$ or $I = [0,\delta)$ with $|y(s)| \to \infty$ as $s \to \delta$.

In other words, if the solution $y(s)$ doesn't blow up for a finite value of the parameter s, then the maximal right interval is $[0,\infty)$.

We are working in coordinates r, ϕ, t with the corresponding momenta p, η, θ. Consider a solution for the Hamilton's system

$$y(s) = \big(r(s), \phi(s), t(s); p(s), \eta(s), \theta(s)\big).$$

We shall show that $|y(s)|$ cannot blow up for finite values of s. This is equivalent to show that each component of $y(s)$ has this property. The momentum θ is bounded because it is constant. Also (5) implies that $\dot{\eta} = \partial L/\partial\phi = 0$. This implies that $\eta(s) = \eta$ is a constant and hence bounded. The momentum p is given by $p = \partial L/\partial\dot{r} = \dot{r}$. Using the conservation of energy, yields

$$|p(s)| = |\dot{r}(s)| \le \sqrt{2E}.$$

As E is constant along the solutions, $|p(s)|$ is bounded. Here we used the standard definition of the momentum

$$p = \frac{\partial L}{\partial \dot{q}},$$

and the Euler-Lagrange equation

$$\dot{p} = \frac{\partial L}{\partial q}.$$

Next, we deal with the bounds for $r(s)$. We have seen that for $E > 0$ there are exactly two positive roots r_{min}, r_{max} of the equation

$$V(r) = E,$$

and

$$0 < r_{min} \le r(s) \le r_{max},$$

so $r(s)$ is bounded. The projection of the trajectory in the x-plane is contained in the ring defined by the circles of radii r_{min} and r_{max}. See the figures 3 and 5.

In order to deal with the bounds for the ϕ component, we use

$$\dot{\phi}(s) = \begin{cases} \frac{C}{r^2(s)} - 2\theta r^{2k}(s), & \text{if } r(0) \neq 0, \\ -2\theta r^{2k}, & \text{if } r(0) = 0. \end{cases} \tag{35}$$

The inequality

$$0 \le r_{min} \le r(s) \le r_{max}$$

yields the upper bound

$$|\dot{\phi}(s)| \le C_0, \ \ \forall s,$$

with

$$C_0 = \begin{cases} |C|/r^2_{min} + 2|\theta| r^{2k}_{max}, & \text{if } r(0) \neq 0, \\ 2|\theta| r^{2k}_{max}, & \text{if } r(0) = 0. \end{cases} \tag{36}$$

Thus for all s

$$|\phi(s)| \le |\phi(s) - \phi(0)| + |\phi(0)| \le C_0|s| + |\phi(0)|,$$

so

$$\lim_{s \to \delta} |\phi(s)| \le C\delta + |\phi_0|$$

and $|\phi(s)|$ cannot blow up for finite δ.

We still need to show that $t(s)$ doesn't blow up for finite s. From Hamilton's equation

$$\dot{t}(s) = \frac{\partial H}{\partial \theta} = -2r^{2(k+1)}(s)\dot{\phi}(s).$$

Using the boundedness of $|\dot{\phi}|$,

$$\begin{aligned} |t(s)| &\le |t(s) - t(0)| + |t(0)| \le \max_{u \in [0,s]} |\dot{t}(u)||s| + |t(0)| \\ &\le 2r^{2(k+1)}_{max}|\dot{\phi}(s)| + |t(0)| \le 2C_0 r^{2(k+1)}_{max}|s| + |t(0)| \\ &= C_1|s| + |t(0)|. \end{aligned}$$

Hence $t(s)$ cannot blow up for a finite value of s. Using Corollary 5.4 the bicharacteristic solution $y(s) = (r(s), \phi(s), t(s), p(s), \eta(s), \theta(s))$ is defined on $[0, +\infty)$. Using a similar argument to the left we get $y(s)$ defined on the entire real line. In particular, the projection on the (r, ϕ)-plane has the same property.

6. Global connectivity by geodesics. Consider the subRiemannian geometry induced by the vector fields (1). In this section we shall prove the following global connectivity result:

THEOREM 6.1. *Given any two points $A, B \in \mathbb{R}^3$, there is a geodesic joining A and B.*

It is known that geodesics are locally length minimizing, see Hamenstädt [11], Lee and Markus [15], Strichartz [17,18], Belaïche [3]. However, there are examples of length minimizing curves in steps greater than 2, which are not geodesics, see Liu and Sussmann [16], Sussmann [19].

DEFINITION 6.2. *The distribution $\mathcal{H}$ generated by the vector fields X_1 and X_2 given by (1) is called the horizontal distribution. A curve c is called horizontal if $\dot{c} \in \mathcal{H}$. The Riemannian metric g defined on $\mathcal{H}$ with respect to which X_1 and X_2 are orthonormal is called the subRiemannian metric.*

By Chow's theorem [7], arbitrary points P and Q can be joined by a piece-wise horizontal curve. Using the subRiemannian metric, one may define the distance

$$d(P, Q) = \inf\{\text{length}(c); c \ \text{ horizontal, joining P and Q}\},$$

where $\text{length}(c) = \int_I \sqrt{g(\dot{c}, \dot{c})}$.

DEFINITION 6.3. *A subRiemannian manifold is called complete if it is complete as a metric space, i.e. the distance function d given by the subRiemannian metric is complete.*

The following theorem can be found in Strichartz [17,18]:

THEOREM 6.4. *Let M be a connected step 2 subRiemannian manifold.*
(a) If M is complete, then any two points can be joined by a geodesic.
(b) If there exists a point P such that every geodesic from P can be indefinitely extended, then M is complete.
(c) Every nonconstant geodesic is locally a unique length minimizing curve.
(d) Every length minimizing curve is a geodesic.

Proof of Theorem 6.1. Let $\mathcal{M} = \mathbb{R}^3 \backslash \{(0, t); t \in \mathbb{R}\}$. The subRiemannian model defined by the vector fields (1) is step 2 on $\mathcal{M}$ and step $2(k+1)$ on $\{(0, t); t \in \mathbb{R}\}$. Let A, $B \in \mathbb{R}^3$.
(i) $A, B \in \mathcal{M}$:
From Theorem 5.2, any point of $\mathcal{M}$ has the property (b) of Theorem 6.4. Therefore $\mathcal{M}$ is complete. Using Theorem 6.4 (a), any two points in M can be joined by a minimizing geodesic. By property (d) of Theorem 6.4, the minimizing geodesic is a geodesic. Hence A and B can be connected by a normal geodesic.
(ii) $A \in \{(0, t); t \in \mathbb{R}\}$ and $B \in \mathcal{M}$:
After a translation along the t-axis, Theorem 3.1 of [5] has the following

COROLLARY 6.5. *There are finitely many geodesics that join the point $(0, t_0)$ to (x, t) if and only if $x \neq 0$. These geodesics are parametrized by the solutions ψ of:*

$$\frac{|t - t_0|}{|x|^{2(k+1)}} = \mu(\psi), \tag{37}$$

There is exactly one such geodesic if and only if:

$$|t - t_0| < \mu(c_1)|x|^{2(k+1)},$$

where c_1 is the first critical point for μ and

$$\mu(x) = \frac{2}{2k+1} \frac{\int_0^{(2k+1)x} \sin^{\frac{2(k+1)}{2k+1}}(v)\,dv}{\sin^{\frac{2(k+1)}{2k+1}}(x)}. \tag{38}$$

A graph of the function μ is given in figure 6.

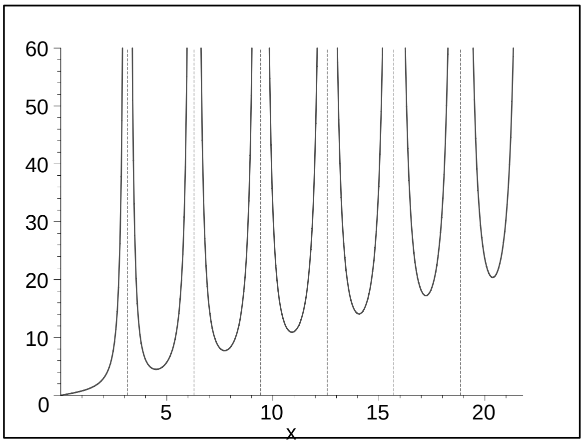

FIG. 6. *The graph of the function $\mu(x)$.*

Clearly, (ii) is an immediate consequence.
(iii) $A, B \in \{(0,t); t \in \mathbb{R}\}$:
This is the limit case, $|x| \to 0$, of Corollary 6.5. Namely, (37) becomes $\mu(\psi) = \infty$. Hence $\psi \in \frac{\pi}{2k+1}\mathbb{Z}$. There are infinitely many geodesics between A and B.

7. Complex Hamiltonian mechanics.

The complex action. The goal of this section is to describe the lengths of the geodesics starting at the origin by means of complex Hamiltonian mechanics. For Δ_X of (2), the complex action is

$$\begin{aligned} g(r,t,r_0,t_0) &= -i(t-t_0) + \frac{k}{k+1}E\tau \\ &\quad + \frac{1}{2(k+1)}\mathrm{sgn}(\tau)\Big[\sqrt{2Er^2 + W(r^2)^2} - \sqrt{2Er_0^2 + W(r_0^2)^2}\Big], \end{aligned}$$

where E is the constant of energy and W is defined below; see (4.58) of [2], where we replaced a by $1/k$ and k by $k+1$. We are interested in the action starting at the origin: $r_0 = r(0) = 0$. We set

$$\Psi(x) = \psi(x_1^2 + x_2^2) = \psi(|x|^2), \tag{39}$$

where

$$\psi(u) = \frac{u^{k+1}}{k+1}, \quad \psi'(u) = u^k. \tag{40}$$

The equations (4.11) and (4.12) of [2] yield

$$W(u) = 2\psi'(u) - \Omega = 2u^{k+1} - \Omega. \tag{41}$$

Ω, the angular momentum, is a constant of motion which satisfies the following equation

$$r^2(s)\dot{\alpha}(s) = i\Big(2r^2(s)\psi'(r^2(s)) - \Omega\Big) = iW(r^2(s)). \tag{42}$$

$r(0) = 0$ implies

$$\Omega = 0, \quad W(u) = 2u^{k+1}, \tag{43}$$

therefore

$$2Er^2 + W(r^2)^2 = 2Er^2 + \Big(2r^{2(k+1)}\Big)^2 = r^2\Big(2E + r^{2(2k+1)}\Big),$$

and

$$\sqrt{2Er^2 + W(r^2)^2} = r\sqrt{2E + r^{2(2k+1)}}. \tag{44}$$

Hence the complex action from the origin is

$$g = g(r, t, 0, t_0) = -i(t - t_0) + \frac{k}{k+1}E\tau + \frac{r}{2(k+1)}\mathrm{sgn}(\tau)\sqrt{2E + r^{2(2k+1)}}. \tag{45}$$

It satisfies the Hamilton-Jacobi equation:

$$\frac{\partial g}{\partial \tau} + \frac{1}{2}\Big[X_1(g)^2 + X_2(g)^2\Big] = 0, \tag{46}$$

or

$$\frac{\partial g}{\partial \tau} + H\left(x, \nabla_x g\right) = 0. \tag{47}$$

The classical action from the origin is given by

$$S = S(x, t, \tau, \theta) = \int_0^\tau \Big[\langle \xi, \dot{x}\rangle + \theta\dot{t} - H(s)\Big]\, ds. \tag{48}$$

S is the integral of the Lagrangian along the bicharacteristic with boundary conditions:

$$x(0) = 0, \qquad x(\tau) = x, \qquad t(0) = 0, \qquad t(\tau) = t. \tag{49}$$

Here we fix $t_0 = t(0)$, so θ cannot be chosen arbitrarily; it is a real constant along the path. In the case of the complex action, the boundary condition $t(0) = 0$ is replaced by $\theta = -i$. The complex action and the classical actions are related by the formula

$$g = S - it(0). \tag{50}$$

Using

$$\tau H_0 = \int_0^\tau H_0 = \int_0^\tau \Big[\langle \xi, \dot{x}\rangle + \theta\dot{t} - H(s)\Big]\, ds = S(x, t, \tau),$$

Equation (47) may be rewritten as

$$\frac{\partial g}{\partial \tau} + \frac{S}{\tau} = 0, \tag{51}$$

so

$$\frac{\partial g}{\partial \tau} = -\frac{g + it(0)}{\tau}. \tag{52}$$

Define the modified complex action function as

$$f(x,t,\tau) = \tau g(x,t,\tau). \tag{53}$$

We shall show that the critical points of f with respect to τ play an important role in finding the lengths of the geodesics between the origin and the point (x,t). The following two results contained in [5] describe the geodesics starting from the origin:

THEOREM 7.1. *There are finitely many geodesics that join the origin to (x,t) if and only if $x \neq 0$. These geodesics are parametrized by the solutions ψ of:*

$$\frac{|t|}{|x|^{2(k+1)}} = \mu(\psi), \tag{54}$$

see (38). There is exactly one such geodesic if and only if:

$$|t| < \mu(c_1)|x|^{2(k+1)},$$

where c_1 is the first critical point for μ. The number of geodesics increases without bound as $\frac{|t|}{|x|^{2(k+1)}} \to \infty$. If $0 \le \psi_1 < \cdots < \psi_N$ are the solutions of (54), then there are exactly N geodesics, and their lengths are given by

$$\ell_m^{2(k+1)} = \nu(\psi_m)\Big(|t| + |x|^{2(k+1)}\Big), \qquad m = 1, \dots, N \tag{55}$$

where

$$\nu(\psi) = \frac{\Big[\int_0^{(2k+1)\psi} \sin^{-\frac{2k}{2k+1}}(v)\, dv\Big]^{2(k+1)}}{(k+1)^{2(k+1)}\big(1+\mu(\psi)\big)\sin^{\frac{2(k+1)}{2k+1}}\big((2k+1)\psi\big)}. \tag{56}$$

THEOREM 7.2. *The geodesics that join the origin to a point $(0,t)$ have lengths $\ell_1, \ell_2, \dots,$ where*

$$(\ell_m)^{2(k+1)} = \Big(\frac{m}{2k+1}\Big)^{2k+1} \frac{M^{2(k+1)}}{Q}|t|, \tag{57}$$

with the constants M and Q expressed in terms of beta functions

$$M = B\Big(\frac{1}{4k+2}, \frac{1}{2}\Big),$$
$$Q = 2B\Big(\frac{4k+3}{4k+2}, \frac{1}{2}\Big).$$

For each length ℓ_m, the geodesics of that length are parametrized by the circle $\mathbb{S}^1$.

Differentiating (53), yields

$$\frac{\partial f}{\partial \tau} = \frac{\partial(\tau g)}{\partial \tau} = g + \tau \frac{\partial g}{\partial \tau} = g - (g + it(0)) = -it(0).$$

From relation (2.20) of [5],

$$t - t(0) = -\mu(\widetilde{\phi})|x|^{2(k+1)}, \qquad \widetilde{\phi} = \phi - \phi_0, \tag{58}$$

and therefore

$$\frac{\partial f}{\partial \tau} = -i(t + \mu(\widetilde{\phi})|x|^{2(k+1)}). \tag{59}$$

Since $\frac{d\widetilde{\phi}}{ds} = -2\theta r^{2k}$, see (2.14) of [5], one has

$$\widetilde{\phi} = -2\theta \int_0^{\tau} r^{2k}(s)\, ds. \tag{60}$$

Now (59) implies

PROPOSITION 7.3. *Let $x \neq 0$. τ_c is a critical point for the modified complex action $f(x,t,\tau)$ if and only if*

$$\widetilde{\phi} = 2i \int_0^{\tau_c} r^{2k}(s)\, ds$$

is a solution for the equation

$$\frac{t}{|x|^{2(k+1)}} = -\mu(\widetilde{\phi}). \tag{61}$$

Theorem 7.1 implies

PROPOSITION 7.4. *Let $\tau_1, \dots, \tau_m$ be the critical points of the modified complex action $f(x,t,\tau)$.*
The numbers

$$\zeta_j = 2i \int_0^{\tau_j} r^{2k}(s)\, ds, \quad j = 1, \dots, m,$$

satisfy the equation

$$\frac{t}{|x|^{2(k+1)}} = \mu(-\zeta_j), \tag{62}$$

and for each ζ_j we have a geodesic connecting (x,t) to the origin. The length ℓ_j of the geodesic parametrizied by ζ_j is

$$\begin{aligned} \ell_j^{2(k+1)} &= \nu(\zeta_j)\Big(|t| + |x|^{2(k+1)}\Big) \\ &= \nu\Big(2i \int_0^{\tau_j} r^{2k}(s)\, ds\Big)\Big(|t| + |x|^{2(k+1)}\Big). \end{aligned}$$

Consequently, the knowledge of the critical points τ_c of the modified complex action function gives the lengths of the geodesics. The rest of this section contains the calculation of τ_j and the geodesic lengths for the step 2 case and for the step 4 case.

Step 2 case: The Heisenberg group. Here $k = 0$. In this case $\widetilde{\phi}_j = 2i\tau_j$ and τ_j is purely imaginary. The lengths of the geodesics between the origin and (x, t), $x \neq 0$, are given by (55), that is

$$\ell_j^2 = \nu(2i\tau_j)\Big(|t| + |x|^2\Big). \tag{63}$$

Step 4 case. This is a lengthy calculation. We need to compute the integral

$$2i \int_0^\tau r^2(s)\, ds, \tag{64}$$

and we shall do that in terms of elliptic functions. The Hamiltonian function associated with the vector fields

$$X_1 = \partial_{x_1} + 2x_1|x|^2\partial_t, \quad X_2 = \partial_{x_2} - 2x_2|x|^2\partial_t \tag{65}$$

is

$$H(\xi, \theta, x, t) = \frac{1}{2}\Big(\xi_1 + 2x_2|x|^2\theta\Big)^2 + \frac{1}{2}\Big(\xi_2 - 2x_1|x|^2\theta\Big)^2. \tag{66}$$

Using Hamilton's equations

$$\dot{x}_1 = \xi_1 + 2x_2|x|^2\theta,$$

$$\dot{x}_2 = \xi_2 - 2x_1|x|^2\theta,$$

we have

$$H = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) = \frac{1}{2}, \tag{67}$$

where we have used the arc length parametrization. In polar coordinates $x_1 = r\cos\widetilde{\phi}$, $x_2 = r\sin\widetilde{\phi}$ and $\frac{d\widetilde{\phi}}{ds} = -2\theta r^{2k}$, the conservation of energy (67) takes the form

$$\dot{r}^2 + 4\theta^2 r^6 = 1. \tag{68}$$

With $\theta = -i$,

$$\dot{r}^2 - 4r^6 = 1, \tag{69}$$

and

$$\frac{dr}{ds} = \pm\sqrt{1 + 4r^6}. \tag{70}$$

Since $r(0) = 0$, $r(s)$ satisfies the integral equation

$$\int_0^r \frac{dx}{\sqrt{1 + 4x^6}} = \pm s. \tag{71}$$

The substitutions $x^2 = t$, $4t^3 = u^3$ and $u = -v$ yield

$$\begin{aligned}\int_0^r \frac{dx}{\sqrt{1+4x^6}} &= \frac{1}{2}\int_0^{r^2} \frac{dt}{\sqrt{t(1+4t^3)}}\\ &= \frac{1}{2^{4/3}}\int_0^{4^{1/3}r^2} \frac{du}{\sqrt{u(1+u^3)}}\\ &= \frac{i}{2^{4/3}}\int_0^{-4^{1/3}r^2} \frac{dv}{\sqrt{v(1-v^3)}}.\end{aligned}$$

LEMMA 7.5.

$$\int_0^a \frac{dx}{\sqrt{x(1-x^3)}} = \frac{1}{3^{1/4}} sn^{-1}\Big(\sqrt{1-\frac{h(a)^2}{b^2}}, k\Big),$$

where $b = 2-\sqrt{3}$, $k = \sqrt{b}/2$ *and*

$$h(a) = \frac{\sqrt{3}}{a+\frac{1+\sqrt{3}}{2}} - 1.$$

Proof. We need to evaluate the integral

$$\int_0^a \frac{dx}{\sqrt{f(x)}}$$

where $f(x) = x(1-x^3) = -(x^2-x)(1+x+x^2)$. We set

$$x = \frac{pt+q}{t+1}$$

with

$$p = \frac{-1-\sqrt{3}}{2}, \quad q = \frac{-1+\sqrt{3}}{2}.$$

Then

$$\begin{aligned}f &= -\frac{1}{(t+1)^4}\Big((p^2-p)t^2+q^2-q\Big)\Big((p^2+p+1)t^2+(q^2+q+1)\Big)\\ &= -\frac{1}{(t+1)^4}\Big((\frac{3}{2}+\sqrt{3})t^2+(\frac{3}{2}-\sqrt{3})\Big)\Big(\frac{3}{2}t^2+\frac{3}{2}\Big)\\ &= -\frac{1}{(t+1)^4}\,\frac{3}{2}\,(\frac{3}{2}+\sqrt{3})(t^2-b^2)(t^2+1),\end{aligned}$$

where $b^2 = 7-4\sqrt{3}$ and $b = 2-\sqrt{3}$. Thus

$$\int_0^a \frac{dx}{\sqrt{f(x)}} = \frac{2}{\sqrt{3+2\sqrt{3}}}\int_{h(a)}^b \frac{1}{\sqrt{(b^2-t^2)(t^2+1)}}\,dt.$$

By [14], p.52(3.2.4),

$$\int_x^b \frac{1}{\sqrt{(1+t^2)(b^2-t^2)}}\,dt = \frac{1}{\sqrt{1+b^2}}\,\mathrm{sn}^{-1}\Big(\frac{\sqrt{b^2-x^2}}{b}, \frac{b}{\sqrt{1+b^2}}\Big),$$

and

$$\int_0^a \frac{dx}{\sqrt{f(x)}} = 3^{-1/4}\,\mathrm{sn}^{-1}\Big(\sqrt{1-\frac{h(a)^2}{b^2}}\,,\,\frac{\sqrt{b}}{2}\Big).$$

□

Consequently,

$$\begin{aligned}\int_0^r \frac{dx}{\sqrt{1+4x^6}} &= \frac{i}{2^{4/3}}\int_0^{-4^{1/3}r^2}\frac{dv}{\sqrt{v(1-v^3)}}\\ &= \frac{i}{2^{4/3}\cdot 3^{1/4}}\mathrm{sn}^{-1}\left(\sqrt{1-\frac{h(-4^{1/3}r^2)^2}{b^2}},k\right),\end{aligned}$$

and in view of (71),

$$\mathrm{sn}^{-1}\left(\sqrt{1-\frac{h(-4^{1/3}r^2)^2}{b^2}},k\right) = \mp 2^{4/3}3^{1/4}is. \tag{72}$$

From $\mathrm{cn}^2x+\mathrm{sn}^2x=1$ (see [14], p.24(2.1.4)),

$$\mathrm{cn}^2\Big(2^{4/3}3^{1/4}is,k\Big) = \frac{h(-4^{1/3}r^2)^2}{b^2}. \tag{73}$$

Again, by [14], p.39(2.6.12):

$$\mathrm{cn}(iu,k) = \mathrm{nc}(u,k') = \frac{1}{\mathrm{cn}(u,k')},$$

where $k'=\sqrt{1-k^2}$, so (73) becomes

$$b\,\mathrm{nc}(2^{4/3}\,3^{1/4}s,k') = h(-4^{1/3}r^2), \tag{74}$$

with $k'=\sqrt{\bar{b}}/2$ and $\bar{b}=2+\sqrt{3}$. Inverting h, we obtain

$$r^2(s) = 4^{-1/3}\left(\frac{1+\sqrt{3}}{2}-\frac{\sqrt{3}}{1+b\,\mathrm{nc}(2^{4/3}\,3^{1/4}s,k')}\right). \tag{75}$$

Thus (64) can be expressed as

$$2i\int_0^\tau r^2(s)\,ds = \frac{1+\sqrt{3}}{4^{1/3}}(i\tau) - 2^{1/3}\sqrt{3}\,i\int_0^\tau\frac{ds}{1+b\ \mathrm{nc}(2^{4/3}\,3^{1/4}s,k')}. \tag{76}$$

We are left with evaluating the second term on the right hand side of (76). With $u=is$, one has

$$\int_0^\tau \frac{ds}{1+b\ \mathrm{nc}(2^{4/3}\,3^{1/4}s,k')} = \frac{1}{i}\int_0^{i\tau}\frac{du}{1+b\ \mathrm{nc}(-i\,2^{4/3}\,3^{1/4}s,k')} \tag{77}$$

$$= \frac{1}{i}\int_0^{i\tau}\frac{du}{1+b\ \mathrm{cn}(2^{4/3}\,3^{1/4}s,k)}, \tag{78}$$

since nc is even. Consequently,

$$2i \int_0^\tau r^2(s)\, ds = \mathcal{F}(i\tau), \tag{79}$$

where

$$\begin{aligned}\mathcal{F}(z) &= \frac{1+\sqrt{3}}{4^{1/3}} z - 2^{1/3}\sqrt{3}\int_0^z \frac{dv}{1+b\ \mathrm{cn}(2^{4/3}\,3^{1/4}v,k)}\\ &= \frac{1+\sqrt{3}}{4^{1/3}} z - \frac{3^{1/4}}{2}\int_0^{2^{4/3}3^{1/4}z} \frac{dv}{1+b\ \mathrm{cn}(v,k)}.\end{aligned}$$

Next we use [14], p.93(51):

$$\begin{aligned}&\int_0^u \frac{dv}{1+b\,\mathrm{cn} v}\\ &= \frac{1}{1-b^2}\Lambda\Big(u, \frac{ib}{\sqrt{1-b^2}}, k\Big) - \frac{b}{\sqrt{(1-b^2)(k^2+k'^2b^2)}}\tan^{-1}\left(\sqrt{\frac{k^2+k'^2b^2}{1-b^2}}\,\mathrm{sd}\, u\right),\\ &\quad b^2<1,\end{aligned}$$

where

$$\Lambda(u, i\beta, k) = \int_0^u \frac{dv}{1+\beta^2\,\mathrm{sn}^2 v}. \tag{80}$$

We recall that

$$b = 2-\sqrt{3},\quad k = \frac{\sqrt{b}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4},\quad k' = \frac{\sqrt{\bar{b}}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4},\quad \beta = \frac{b}{\sqrt{1-b^2}},$$

$$1-b^2 = 4\sqrt{3}-6,\quad k^2+k'^2b^2 = 1-\frac{\sqrt{3}}{2},\quad \frac{k^2+k'^2b^2}{1-b^2} = \frac{1}{4}\frac{\sqrt{3}-2}{2\sqrt{3}-3},$$

and

$$\begin{aligned}\frac{b}{\sqrt{(1-b^2)(k^2+k'^2\alpha^2)}} &= \frac{(2-\sqrt{3})\sqrt{2}}{(\sqrt{3}-1)\sqrt{2\sqrt{3}-3}}\\ &= \frac{\sqrt{2-\sqrt{3}}\cdot\sqrt{2}}{(\sqrt{3}-1)\cdot 3^{1/4}}\\ &= \frac{\sqrt{2}\,(\sqrt{3}-1)}{2}\frac{\sqrt{2}}{(\sqrt{3}-1)\,3^{1/4}}\\ &= 3^{1/4},\end{aligned}$$

$$\frac{1}{1-b^2} = \frac{1}{2}\frac{1}{2\sqrt{3}-3} = \frac{2+\sqrt{3}}{2\sqrt{3}} = \frac{1}{\sqrt{3}}+\frac{1}{2},$$

$$\sqrt{\frac{k^2+k'^2b^2}{1-b^2}} = \frac{1}{4}\frac{\sqrt{2}(\sqrt{3}-1)}{\sqrt{2\sqrt{3}-3}} = \frac{1}{4}\frac{\sqrt{2}(\sqrt{3}-1)}{3^{1/4}\sqrt{2-\sqrt{3}}} = \frac{1}{2\cdot 3^{1/4}}.$$

$$\frac{b}{\sqrt{1-b^2}} = \frac{2-\sqrt{3}}{\sqrt{2(2\sqrt{3}-3)}} = \frac{2-\sqrt{3}}{2^{1/2}3^{1/4}\sqrt{2-\sqrt{3}}} = \frac{1}{2^{1/2}3^{1/4}} \cdot \frac{\sqrt{2}(\sqrt{3}-1)}{2} = \frac{\sqrt{3}-1}{2\cdot 3^{1/4}}.$$

Consequently,

$$\int_0^u \frac{dv}{1+b\,\mathrm{cn} v} = \left(\frac{\sqrt{3}}{3}+\frac{1}{2}\right)\Lambda\Big(u, i\frac{\sqrt{3}-1}{2\cdot 3^{1/4}}, k\Big) - 3^{-1/4}\tan^{-1}\left(\frac{\mathrm{sd}\, u}{2\cdot 3^{1/4}}\right),$$

and

$$\begin{aligned} F(z) = \frac{1+\sqrt{3}}{4^{1/3}}z - \frac{2+\sqrt{3}}{4\cdot 3^{1/4}}\Lambda\Big(2^{4/3}3^{1/4}z, i\frac{\sqrt{3}-1}{2\cdot 3^{1/4}}, k\Big) \\ -\frac{1}{2}\tan^{-1}\left(\frac{\mathrm{sd}(2^{4/3}3^{1/4}z)}{2\cdot 3^{1/4}}\right). \end{aligned} \tag{81}$$

To evaluate the second term more explicitly, we use the following elliptic functions:

$$\mathrm{sn}^{-1}(x,k) = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad 0 \le x \le 1, \qquad \text{p.50(3.1.2)}$$

$$\mathrm{cn}^{-1}(x,k) = \int_x^1 \frac{dt}{\sqrt{(1-t^2)(k'^2+k^2t^2)}}, \quad 0 \le x \le 1, \qquad \text{p.52(3.2.2)}$$

$$F(\omega,k') = \mathrm{sn}^{-1}(\sin\omega, k') = \int_0^\omega (1-k'^2\sin^2\theta)^{-1/2}\,d\theta, \qquad \text{p.51(3.1.8)}$$

$$\mathcal{E}(u,k) = \int_0^u \mathrm{dn}^2(v,k)\,dv, \qquad \text{p.62(3.4.25)}$$

$$D(\omega,k') = \int_0^\omega \sqrt{1-k'^2\sin^2\theta}\,d\theta = \int_0^\omega \sqrt{\cos^2\theta - k^2\sin^2\theta}\,d\theta, \qquad \text{p.63(3.4.26)}$$

$$\mathcal{E} = \mathcal{E}(k) = \mathcal{E}(K,k) = \int_0^{\pi/2}(1-k^2\sin^2\theta)^{1/2}\,d\theta = \int_0^K \mathrm{dn}^2 v\,dv, \qquad \text{p.63(3.5.4)}$$

$$\begin{aligned} K = K(k) &= \int_0^{\pi/2}(1-k^2\sin^2\theta)^{-1/2}\,d\theta \\ &= \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \qquad \text{p.73(3.8.1)} \end{aligned}$$

where numbers refer to [14]. In particular,

$$\begin{aligned} \Lambda(u,i\beta,k) &= \frac{k^2}{\beta^2+k^2}u + \frac{\beta}{\sqrt{(\beta^2+k^2)(\beta^2+1)}}\Big[u\big\{D(\omega,k') - F(\omega,k') \\ &\quad + \frac{\mathcal{E}}{K}F(\omega,k')\big\} - \tan^{-1}(Y/X)\Big] \\ &= \frac{k^2}{\beta^2+k^2}u + \frac{\beta}{\sqrt{(\beta^2+k^2)(\beta^2+1)}}\Big[u\big\{D(\omega,k') - F(\omega,k') \\ &\quad + \frac{\mathcal{E}}{K}F(\omega,k')\big\} + \frac{i}{2}\log\frac{\theta_4(x-iy)}{\theta_4(x+iy)}\Big], \end{aligned}$$

where

$$X = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} \cos 2nx \, \cosh 2ny,$$

$$Y = 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} \sin 2nx \, \sinh 2ny,$$

with $q = \exp(-\pi K'/K)$, $x = \pi u/2K$ and $y = \pi F(\omega, k')/2K$ and $\sin\omega = \beta/\sqrt{\beta^2 + k^2}$, see [14], p.72(3.7.47). θ_4 stands for Jacobi's zeta function. In our case

$$\beta = \frac{\sqrt{3}-1}{2\cdot 3^{1/4}}, \quad k = \frac{\sqrt{6}-\sqrt{2}}{4}, \quad k' = \frac{\sqrt{6}+\sqrt{2}}{4}, \quad b = 2-\sqrt{3}.$$

An algebraic computation yields

$$\frac{k^2}{\beta^2+k^2} = b\sqrt{3}, \quad \frac{\beta}{\sqrt{(\beta^2+k^2)(\beta^2+1)}} = 2\cdot 3^{1/4}\, b, \quad \frac{\beta}{\sqrt{\beta^2+k^2}} = \sqrt{3}-1.$$

We shall show the function Λ depends only on the Jacobi's epsilon function $\mathcal{E}(u,k)$. We have

$$F(\omega, k') = \mathrm{sn}^{-1}(\sin\omega, k') = \mathrm{sn}^{-1}(\sqrt{3}-1, k').$$

From [14], p.41(2.7.14) and p.63(3.4.27), $\mathrm{am}\, v = \int_0^v \mathrm{dn}\,\gamma\, d\gamma$, and $E(v,k') = D(\mathrm{am}\, v, k')$, with $\mathrm{sn}(v) = \sin(\mathrm{am}\, v)$. With $\omega = \mathrm{am}\, v$, yields $v = \mathrm{sn}^{-1}(\sqrt{3}-1, k')$, and hence

$$D(\omega, k') = \mathcal{E}(\mathrm{sn}^{-1}(\sqrt{3}-1, k'), k').$$

If $k = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$, then $K'/K = \sqrt{3}$ see [14], p.86(10). Hence

$$q = e^{-\pi\sqrt{3}}.$$

Also

$$K = K(k) = \frac{3^{1/4}}{4\sqrt{\pi}}\frac{1}{\sqrt{3}}\,\Gamma(1/6)\Gamma(1/3),$$

and

$$K[2\sqrt{3}E - (\sqrt{3}+1)K] = \frac{\pi}{2}.$$

Dividing by $K^2 = \frac{1}{16\pi\sqrt{3}}\Gamma^2(1/6)\Gamma^2(1/3)$ we get

$$\frac{\mathcal{E}}{K} = \frac{\pi}{4\sqrt{3}K^2} + \frac{\sqrt{3}+1}{2\sqrt{3}} = \left(\frac{2\pi}{\Gamma(1/6)\Gamma(1/3)}\right)^2 + \frac{3+\sqrt{3}}{6}.$$

Hence,

$$\Lambda(u, i\beta, k) = b\sqrt{3}u + 2\cdot 3^{1/4}\, b\Bigg[u\Bigg\{E(am^{-1}\omega, k') - am^{-1}\omega$$
$$+\left[\left(\frac{2\pi}{\Gamma(1/6)\Gamma(1/3)}\right)^2 + \frac{3+\sqrt{3}}{6}\right]am^{-1}\omega\Bigg\} + \frac{i}{2}\log\frac{\theta_4(x-iy)}{\theta_4(x+iy)}\Bigg],$$

with $am^{-1}\omega = \mathrm{sn}^{-1}(\sqrt{3}-1, k')$. Hence, the formula (81) becomes

$$\begin{aligned} \digamma(z) &= \frac{1+\sqrt{3}}{4^{1/3}} z - 3^{1/2}\, 2^{-2/3} z \\ &+ \frac{1}{2}\left[u \left\{ E(am^{-1}\omega, k') + \left[\left(\frac{2\pi}{\Gamma(1/6)\Gamma(1/3)} \right)^2 - \frac{3-\sqrt{3}}{6} \right] am^{-1}\omega \right\} \right. \\ &\left. + \frac{i}{2} \log \frac{\theta_4(x-iy)}{\theta_4(x+iy)} \right] - \frac{1}{2} \tan^{-1} \left(\frac{\mathrm{sd}(2^{4/3}\, 3^{1/4} z)}{2 \cdot 3^{1/4}} \right), \end{aligned}$$

with $u = 2^{4/3}\, 3^{1/4}\, z$ and $am^{-1}\omega = \mathrm{sn}^{-1}(\sqrt{3}-1, k')$. Finally, we restate the last part of Proposition 7.4 for the step 4 case, $k = 1$.

COROLLARY 7.6. *Let τ_j be the critical points of the modified complex action $f(\tau) = \tau g(\tau)$. Setting $\zeta_j = \digamma(i\tau_j)$, the lengths of the geodesics between the origin and the point (x, t), $|x| \neq 0$ are given by*

$$\ell_j^4 = \nu(\zeta_j)\big(|t| + |x|^4\big).$$

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