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P-TH MEAN PSEUDO ALMOST AUTOMORPHIC MILD SOLUTIONS TO SOME NONAUTONOMOUS STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract

In this paper we first introduce and study the concepts of p-th mean pseudo almost automorphy and that of p-th mean pseudo almost periodicity for $p \geq 2$. Next, we make extensive use of the well-known Schauder fixed point principle to obtain the existence of p-th mean pseudo almost automorphic mild solutions to some nonautonomous stochastic differential equations.

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1 Introduction

Let $\mathbb{H}$ be a Hilbert space. In a recent paper by Fu and Liu [37], the concept of square-mean almost automorphy was introduced. Such a notion generalizes in a natural fashion the notion of square-mean almost periodicity, which has been studied in various situations by Bezandry and Diagana [9, 10, 11, 12, 13]. In [37], the authors made use of the Banach

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fixed principle to obtain the existence of a square-mean almost automorphic solution to the autonomous stochastic differential equation

$$dX(t) = AX(t)dt + f(t)dt + g(t)dW(t), \ t \in \mathbb{R}$$

where $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is a linear operator which generates an exponentially stable C_0-semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ and $f, g : \mathbb{R} \to L^2(\Omega, \mathbb{H})$ are square-mean almost automorphic stochastic processes, and $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, \mathcal{F}_t)$ where

$$\mathcal{F}_t = \sigma\Big\{W(u) - W(v) : u, v \leq t\Big\}.$$

Recently, Liang, Xiao, and Zhang [42, 61, 62] introduced the concept of pseudo almost automorphy, which is a powerful generalization of both the notion of almost automorphy due to Bochner [15] and that of pseudo almost periodicity due to Zhang (see [27]). Such a concept has recently generated several developments, see, e.g., [17], [30], [35], [36], and [43].

Motivated by the above mentioned papers, the present article is aimed at introducing some new classes of stochastic processes called respectively p-th mean pseudo almost automorphic stochastic processes and p-th mean pseudo almost periodic stochastic processes for $p \geq 2$. It should be mentioned that the notion of p-th mean pseudo almost automorphy generalizes in a natural fashion both the notion of square-mean almost periodicity and that of square-mean almost automorphy.

Since the concept of p-th mean pseudo almost automorphy is also a generalization of the p-th mean pseudo almost periodicity, our main focus throughout this paper will be on the p-th mean pseudo almost automorphy rather than on latter. In particular, properties of p-th mean pseudo almost automorphic stochastic processes will be discussed in the second section.

Applications include use of Schauder fixed point theorem to study the existence of square-mean pseudo almost automorphic solutions to the nonautonomous stochastic differential equations

$$dX(t) = A(t)X(t)\,dt + F_1(t, X(t))\,dt + F_2(t, X(t))\,d\mathbb{W}(t), \ \ t \in \mathbb{R}, \tag{1.1}$$

where $(A(t))_{t\in\mathbb{R}}$ is a family of densely defined closed linear operators satisfying Acquistapace and Terreni conditions, the functions $F_i(i = 1, 2) : \mathbb{R} \times L^p(\Omega, \mathbb{H}) \to L^p(\Omega, \mathbb{H})$ are jointly continuous satisfying some additional conditions, and $\mathbb{W}$ is a one-dimensional Wiener process.

2 Preliminaries

Most of the material of this Section, except those on the concepts of p-th mean pseudo almost automorphy and that of p-th mean pseudo almost periodicity and their properties, is taken from Bezandry and Diagana [14].

Let $(\mathbb{B}, \|\cdot\|)$ be a Banach space. If L is a linear operator on the Banach space $\mathbb{B}$, then $D(L)$, $\rho(L)$, $\sigma(L)$, $N(L)$, and $R(L)$ stand respectively for the domain, resolvent, spectrum, null space, and the range of L. Also, we set $R(\lambda, L) := (\lambda I - L)^{-1}$ for all $\lambda \in \rho(L)$. If P

is a projection, we then set $Q = I - P$, where I is the identity operator of $\mathbb{B}$; . If $\mathbb{B}_1$, $\mathbb{B}_2$ are Banach spaces, then the space $B(\mathbb{B}_1, \mathbb{B}_2)$ denotes the collection of all bounded linear operators from $\mathbb{B}_1$ into $\mathbb{B}_2$ equipped with its natural topology. This is simply denoted by $B(\mathbb{B}_1)$ when $\mathbb{B}_1 = \mathbb{B}_2$.

2.1 Evolution Families

Let $\mathbb{B}$ be a Banach space equipped with the norm $\|\cdot\|$.

The family of closed linear operators $A(t)$ for $t \in \mathbb{R}$ on $\mathbb{B}$ with domain $D(A(t))$ (possibly not densely defined) is said to satisfy Acquistapace-Terreni conditions if: there exist constants $\omega \geq 0$, $\theta \in \left(\frac{\pi}{2}, \pi\right)$, $K, L \geq 0$ and $\mu, \nu \in (0,1]$ with $\mu + \nu > 1$ such that

$$S_\theta \cup \left\{0\right\} \subset \rho\Big(A(t) - \omega\Big) \ni \lambda, \qquad \|R(\lambda, A(t) - \omega)\| \leq \frac{K}{1 + |\lambda|} \tag{2.1}$$

and

$$\|(A(t) - \omega) R(\lambda, A(t) - \omega)\ [R(\omega, A(t)) - R(\omega, A(s))]\| \leq L\, |t - s|^\mu\, |\lambda|^{-\nu} \tag{2.2}$$

for $t, s \in \mathbb{R}$, $\lambda \in S_\theta := \left\{\lambda \in \mathbb{C} \setminus \{0\} : \left|\arg \lambda\right| \leq \theta\right\}$.

It should be mentioned that the conditions (2.1) and (2.2) were introduced in the literature by Acquistapace and Terreni in [2, 3] for $\omega = 0$. Among other things, it ensures that there exists a unique evolution family $\mathcal{U} = U(t,s)$ on $\mathbb{B}$ associated with $A(t)$ satisfying

(a) $U(t,s)U(s,r) = U(t,r)$ for all $t, s, r \in \mathbb{R}$ with $t \geq s \geq r$;

(b) $U(t,t) = I$ for $t \in \mathbb{R}$ where I;

(c) $(t,s) \mapsto U(t,s) \in B(\mathbb{B})$ is continuous for $t > s$;

(d) $U(\cdot, s) \in C^1((s,\infty), B(\mathbb{B}))$, $\dfrac{\partial U}{\partial t}(t,s) = A(t)U(t,s)$ and

$$\left\|A(t)^k U(t,s)\right\| \leq K\,(t-s)^{-k}$$

for $0 < t - s \leq 1$, $k = 0, 1$; and

(e) $\partial_s^+ U(t,s)x = -U(t,s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in \overline{D(A(s))}$.

Definition 2.1. One says that an evolution family $\mathcal{U}$ has an *exponential dichotomy* (or is *hyperbolic*) if there are projections $P(t)$ $(t \in \mathbb{R})$ that are uniformly bounded and strongly continuous in t and constants $\delta > 0$ and $N \geq 1$ such that

(f) $U(t,s)P(s) = P(t)U(t,s)$;

(g) the restriction $U_Q(t,s) : Q(s)\mathbb{B} \to Q(t)\mathbb{B}$ of $U(t,s)$ is invertible (we then set $\widetilde{U}_Q(s,t) := U_Q(t,s)^{-1}$); and

(h) $\|U(t,s)P(s)\| \leq Ne^{-\delta(t-s)}$ and $\left\|\widetilde{U}_Q(s,t)Q(t)\right\| \leq Ne^{-\delta(t-s)}$ for $t \geq s$ and $t, s \in \mathbb{R}$.

As in [14], this setting requires some estimates related to $U(t,s)$. For that, we introduce the interpolation spaces for $A(t)$. We refer the reader to the following excellent books [34], and [48] for proofs and further information on theses interpolation spaces.

Let A be a sectorial operator on $\mathbb{B}$ (for that, in (2.1)-(2.2), replace $A(t)$ with A) and let $\alpha \in (0,1)$. Define the real interpolation space

$$\mathbb{B}_\alpha^A := \left\{x \in \mathbb{B} : \|x\|_\alpha^A := \sup_{r>0} \left\| r^\alpha (A-\omega) R(r, A-\omega) x \right\| < \infty \right\},$$

which, by the way, is a Banach space when endowed with the norm $\|\cdot\|_\alpha^A$. For convenience we further write

$$\mathbb{B}_0^A := \mathbb{B},\ \|x\|_0^A := \|x\|,\ \mathbb{B}_1^A := D(A)$$

and

$$\|x\|_1^A := \|(\omega - A)x\|.$$

Moreover, let $\widehat{\mathbb{B}}^A := \overline{D(A)}$ of $\mathbb{B}$. In particular, we have the following continuous embedding

$$D(A) \hookrightarrow \mathbb{B}_\beta^A \hookrightarrow D((\omega - A)^\alpha) \hookrightarrow \mathbb{B}_\alpha^A \hookrightarrow \widehat{\mathbb{B}}^A \hookrightarrow \mathbb{B}, \tag{2.3}$$

for all $0 < \alpha < \beta < 1$, where the fractional powers are defined in the usual way.

In general, $D(A)$ is not dense in the spaces $\mathbb{B}_\alpha^A$ and $\mathbb{B}$. However, we have the following continuous injection

$$\mathbb{B}_\beta^A \hookrightarrow \overline{D(A)}^{\|\cdot\|_\alpha^A} \tag{2.4}$$

for $0 < \alpha < \beta < 1$.

Given the family of linear operators $A(t)$ for $t \in \mathbb{R}$, satisfying (2.1)-(2.2), we set

$$\mathbb{B}_\alpha^t := \mathbb{B}_\alpha^{A(t)}, \quad \widehat{\mathbb{B}}^t := \widehat{\mathbb{B}}^{A(t)}$$

for $0 \leq \alpha \leq 1$ and $t \in \mathbb{R}$, with the corresponding norms. Then the embedding in Eq. (2.3) holds with constants independent of $t \in \mathbb{R}$. These interpolation spaces are of class $\mathcal{J}_\alpha$ ([48, Definition 1.1.1]) and hence there is a constant $c(\alpha)$ such that

$$\|y\|_\alpha^t \leq c(\alpha) \|y\|^{1-\alpha} \|A(t)y\|^\alpha, \quad y \in D(A(t)). \tag{2.5}$$

We have the following fundamental estimates for the evolution family $U(t,s)$.

Proposition 2.2. *[7] Suppose the evolution family $U = U(t,s)$ has exponential dichotomy. For $x \in \mathbb{B}$, $0 \leq \alpha \leq 1$ and $t > s$, the following hold:*

(i) There is a constant $c(\alpha)$, such that

$$\left\| U(t,s)P(s)x \right\|_\alpha^t \leq c(\alpha) e^{-\frac{\delta}{2}(t-s)} (t-s)^{-\alpha} \left\| x \right\|. \tag{2.6}$$

(ii) There is a constant $m(\alpha)$, such that

$$\left\| \widetilde{U}_Q(s,t) Q(t) x \right\|_\alpha^s \leq m(\alpha) e^{-\delta(t-s)} \|x\|. \tag{2.7}$$

Throughout the paper, we adopt the following assumption.

(H.1) The family of operators $A(t)$ satisfies Acquistpace-Terreni conditions and the evolution family $\mathcal{U} = \{U(t,s),\ t \geq s\}$ associated with $A(t)$ is exponentially stable, that is, there exist constant $M, \delta > 0$ such that

$$\left\|U(t,s)\right\| \leq M\, e^{-\delta(t-s)}$$

for all $t \geq s$.

We need the following technical lemma:

Lemma 2.3. *[28, Diagana] For each $x \in \mathbb{B}$, suppose that the family of operators $A(t)$ $(t \in \mathbb{R})$ satisfy Acquistapce-Terreni conditions, assumption* (H.1) *holds. Let μ, α, β be real numbers such that* $0 \leq \mu < \alpha < \beta < 1$ *with* $2\alpha > \mu + 1$. *Then there is a constant* $r(\mu, \alpha) > 0$ *such that*

$$\|A(t)U(t,s)x\|_\alpha \leq r(\mu,\alpha)e^{-\frac{\delta}{4}(t-s)}(t-s)^{-\alpha}\,\|x\|. \tag{2.8}$$

for all $t > s$.

Proof. Let $x \in \mathbb{B}$. First of all, note that $\|A(t)U(t,s)\|_{B(\mathbb{B},\mathbb{B}_\alpha)} \leq K(t-s)^{-(1-\alpha)}$ for all t,s such that $0 < t-s \leq 1$ and $\alpha \in [0,1]$.

Letting $t - s \geq 1$ and using (H.1) and the above-mentioned estimate, we obtain

$$\begin{aligned}
\|A(t)U(t,s)x\|_\alpha &= \|A(t)U(t,t-1)U(t-1,s)x\|_\alpha \\
&\leq \|A(t)U(t,t-1)\|_{B(\mathbb{B},\mathbb{B}_\alpha)}\,\|U(t-1,s)x\| \\
&\leq MKe^{\delta}e^{-\delta(t-s)}\,\|x\| \\
&= K_1 e^{-\delta(t-s)}\,\|x\| \\
&= K_1 e^{-\frac{3\delta}{4}(t-s)}(t-s)^{\alpha}(t-s)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\,\|x\|.
\end{aligned}$$

Now since $e^{-\frac{3\delta}{4}(t-s)}(t-s)^{\alpha} \to 0$ as $t \to \infty$ it follows that there exists $c_4(\alpha) > 0$ such that

$$\|A(t)U(t,s)x\|_\alpha \leq c_4(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\,\|x\|.$$

Now, let $0 < t-s \leq 1$. Using Eq. (2.6) and the fact $2\alpha > \mu + 1$, we obtain

$$\begin{aligned}
\|A(t)U(t,s)x\|_\alpha &= \left\|A(t)U(t,\frac{t+s}{2})U(\frac{t+s}{2},s)x\right\|_\alpha \\
&\leq \left\|A(t)U(t,\frac{t+s}{2})\right\|_{B(\mathbb{B},\mathbb{B}_\alpha)}\left\|U(\frac{t+s}{2},s)x\right\| \\
&\leq k_1\left\|A(t)U(t,\frac{t+s}{2})\right\|_{B(\mathbb{B},\mathbb{B}_\alpha)}\left\|U(\frac{t+s}{2},s)x\right\|_\mu \\
&\leq k_1 K\Big(\frac{t-s}{2}\Big)^{\alpha-1} c(\mu)\Big(\frac{t-s}{2}\Big)^{-\mu} e^{-\frac{\delta}{4}(t-s)}\,\|x\| \\
&\leq c_5(\alpha,\mu)(t-s)^{\alpha-1-\mu}e^{-\frac{\delta}{4}(t-s)}\,\|x\| \\
&\leq c_5(\alpha,\mu)(t-s)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\,\|x\|.
\end{aligned}$$

Therefore there exists $r(\alpha,\mu) > 0$ such that

$$\|A(t)U(t,s)x\|_\alpha \le r(\alpha,\mu)(t-s)^{-\alpha} e^{-\frac{\delta}{4}(t-s)} \|x\|$$

for all $t,s \in \mathbb{R}$ with $t \ge s$.

□

It should be mentioned that if $U(t,s)$ is exponentially stable, then $P(t) = I$ and $Q(t) = I - P(t) = 0$ for all $t \in \mathbb{R}$. In that case, Eq. (2.6) still holds and can be rewritten as follows: for all $x \in \mathbb{B}$,

$$\|U(t,s)x\|_\alpha^t \le c(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha} \|x\|. \tag{2.9}$$

2.2 P-th Mean Pseudo Almost Automorphic and Pseudo Almost Periodic Stochastic Processes

Throughout this paper, $\mathbb{H}$ will denote a real separable Hilbert space with norms $\|\cdot\|$ and $(\Omega,\mathcal{F},\mathbf{P})$ a complete probability space.

Let $p \ge 1$. The collection of all strongly measurable, p^{th} or p-th integrable $\mathbb{H}$-valued random variables, denoted by $L^p(\Omega,\mathbb{H})$, is a Banach space equipped with norm

$$\|X\|_{L^p(\Omega,\mathbb{H})} = (\mathbf{E}\|X\|^p)^{1/p},$$

where the expectation $\mathbf{E}$ is defined by

$$\mathbf{E}[g] = \int_\Omega g(\omega)d\mathbf{P}(\omega).$$

Definition 2.4. A stochastic process $X : \mathbb{R} \to L^p(\Omega;\mathbb{B})$ is said to be continuous whenever

$$\lim_{t\to s} \mathbf{E}\Big\|X(t) - X(s)\Big\|^p = 0.$$

Definition 2.5. A stochastic process $X : \mathbb{R} \to L^p(\Omega;\mathbb{B})$ is said to be stochastically bounded whenever

$$\lim_{N\to\infty} \sup_{t\in\mathbb{R}} \mathbf{P}\Big\{\Big\|X(t)\Big\| > N\Big\} = 0.$$

2.2.1 P-th Mean Pseudo Almost Periodic Stochastic Processes

In this subsection and through the paper unless otherwise, $p \ge 2$, is a real number.

Definition 2.6. A continuous stochastic process $X : \mathbb{R} \to L^p(\Omega;\mathbb{B})$ (for $p \ge 1$) is said to be a p-th mean almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least a number τ for which

$$\sup_{t\in\mathbb{R}} \mathbf{E}\|X(t+\tau) - X(t)\|^p < \varepsilon.$$

The collection of all stochastic processes $X : \mathbb{R} \to L^p(\Omega;\mathbb{B})$ which are p-th mean almost periodic is then denoted by $AP(\mathbb{R};L^p(\Omega;\mathbb{B}))$.

The next lemma provides with some properties of the p-th mean almost periodic processes.

Lemma 2.7. *If X belongs to $AP(\mathbb{R};L^p(\Omega;\mathbb{B}))$, then*

(i) the mapping $t \to \mathbf{E}\|X(t)\|^p$ is uniformly continuous;

(ii) there exists a constant $M > 0$ such that $\mathbf{E}\|X(t)\|^p \leq M$, for all $t \in \mathbb{R}$.

Let $BC(\mathbb{R}; L^p(\Omega;\mathbb{B}))$ denote the collection of all stochastic processes $X : \mathbb{R} \mapsto L^p(\Omega;\mathbb{B})$, which are bounded and continuous. Similarly, Let $\mathbf{CUB}(\mathbb{R}; L^p(\Omega;\mathbb{B}))$ denote the collection of all stochastic processes $X : \mathbb{R} \mapsto L^p(\Omega;\mathbb{B})$, which are continuous and uniformly bounded. It is then easy to check that $\mathbf{CUB}(\mathbb{R}; L^p(\Omega;\mathbb{B})) \subset BC(\mathbb{R}; L^p(\Omega;\mathbb{B}))$ is a Banach space when it is equipped with the sup norm:

$$\|X\|_\infty = \sup_{t\in\mathbb{R}} \left(\mathbf{E}\|X(t)\|^p\right)^{\frac{1}{p}}.$$

Lemma 2.8. $AP(\mathbb{R};L^p(\Omega;\mathbb{B})) \subset \mathbf{CUB}(\mathbb{R};L^p(\Omega;\mathbb{B}))$ *is a closed subspace.*

In view of the above, the space $AP(\mathbb{R};L^p(\Omega;\mathbb{B}))$ of p-th mean almost periodic processes equipped with the norm $\|\cdot\|_\infty$ is a Banach space.

Let $(\mathbb{B}_1, \|\cdot\|_1)$ and $(\mathbb{B}_2, \|\cdot\|_2)$ be Banach spaces and let $L^p(\Omega;\mathbb{B}_1)$ and $L^p(\Omega;\mathbb{B}_2)$ be their corresponding L^p-spaces, respectively.

Definition 2.9. A function $F : \mathbb{R}\times L^p(\Omega;\mathbb{B}_1) \to L^p(\Omega;\mathbb{B}_2))$, $(t,Y) \mapsto F(t,Y)$, which is jointly continuous, is said to be p-th mean almost periodic in $t \in \mathbb{R}$ uniformly in $Y \in \mathbb{K}$ where $\mathbb{K} \subset L^p(\Omega;\mathbb{B}_1)$ is a compact if for any $\varepsilon > 0$, there exists $l(\varepsilon,\mathbb{K}) > 0$ such that any interval of length $l(\varepsilon,\mathbb{K})$ contains at least a number τ for which

$$\sup_{t\in\mathbf{R}} \mathbf{E}\|F(t+\tau,Y) - F(t,Y)\|_2^p < \varepsilon$$

for each stochastic process $Y : \mathbb{R} \to \mathbb{K}$.

The proof of the next composition is a straightforward consequence of the classical composition of almost periodic functions involving Lipschitz condition.

Theorem 2.10. *Let $F : \mathbb{R}\times L^p(\Omega;\mathbb{B}_1) \to L^p(\Omega;\mathbb{B}_2)$, $(t,Y) \mapsto F(t,Y)$ be a p-th mean almost periodic process in $t \in \mathbb{R}$ uniformly in $Y \in \mathbb{K}$, where $\mathbb{K} \subset L^p(\Omega;\mathbb{B}_1)$ is compact. Suppose that F is Lipschitz in the following sense:*

$$\mathbf{E}\|F(t,Y) - F(t,Z)\|_2^p \leq M\, \mathbf{E}\|Y - Z\|_1^p$$

for all $Y,Z \in L^p(\Omega;\mathbb{B}_1)$ and for each $t \in \mathbb{R}$, where $M > 0$. Then for any p-th mean almost periodic process $\Phi : \mathbb{R} \to L^p(\Omega;\mathbb{B}_1)$, the stochastic process $t \mapsto F(t,\Phi(t))$ is p-th mean almost periodic.

Define $PAP_0(\mathbb{R}; L^p(\Omega;\mathbb{B}))$ to be the collection of all $X \in BC(\mathbb{R}, L^p(\Omega;\mathbb{B}))$ such that

$$\lim_{T\to\infty}\left[\frac{1}{2T}\int_{-T}^{T}\mathbf{E}\,\|X(s)\|^p\,ds\right]^{1/p} = 0.$$

Equivalently, $PAP_0(\mathbb{R}; L^p(\Omega;\mathbb{B}))$ is the collection of all $X \in BC(\mathbb{R}, L^p(\Omega;\mathbb{B}))$ such that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathbf{E}\,\|X(s)\|^p\,ds = 0.$$

Similarly, we define $PAP_0(\mathbb{R}\times L^p(\Omega;\mathbb{B}_1); L^p(\Omega;\mathbb{B}_2))$ to be the collection of all bounded jointly continuous stochastic processes $F : \mathbb{R}\times L^p(\Omega;\mathbb{B}_1) \to L^p(\Omega;\mathbb{B}_2)$ such that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathbf{E}\,\|F(s,X)\|_2^p\,ds = 0$$

uniformly in $X \in K$, where $K \subset L^p(\Omega;\mathbb{B}_1)$ is any bounded subset.

Definition 2.11. A stochastic process $X \in BC(\mathbb{R}; L^p(\Omega;\mathbb{B}))$ is called p-th pseudo almost periodic if it can be expressed as $X = Y + \Phi$, where $Y \in AP(\mathbb{R}; L^p(\Omega;\mathbb{B}))$ and $\Phi \in PAP_0(\mathbb{R}; L^p(\Omega;\mathbb{B}))$. The collection of such functions will be denoted by $PAP(\mathbb{R}; L^p(\Omega;\mathbb{B}))$.

Definition 2.12. A bounded continuous stochastic process $F : \mathbb{R}\times L^p(\Omega;\mathbb{B}_1) \to L^p(\Omega;\mathbb{B}_2)$ is called p-th mean pseudo almost periodic whenever it can be expressed as $F = G + \Phi$, where $G \in AP(\mathbb{R}\times L^p(\Omega;\mathbb{B}_1); L^p(\Omega;\mathbb{B}_2))$ and $\Phi \in PAP_0(\mathbb{R}\times L^p(\Omega;\mathbb{B}_1); L^p(\Omega;\mathbb{B}_2))$. The collection of such processes will be denoted by $PAP_0(\mathbb{R}\times L^p(\Omega;\mathbb{B}_1); L^p(\Omega;\mathbb{B}_2))$.

The decomposition of p-th pseudo almost periodic stochastic processes given in Definition 2.11 and Definition 2.12 is unique.

The next composition result is a consequence of a composition result from [61].

Theorem 2.13. *Let $F : \mathbb{R}\times L^p(\Omega;\mathbb{B}_1) \to L^p(\Omega;\mathbb{B}_2)$, $(t,Y) \mapsto F(t,Y)$ be a p-th mean pseudo almost periodic process in $t \in \mathbb{R}$ uniformly in $Y \in K$, where $K \subset L^p(\Omega;\mathbb{B}_1)$ is any compact subset. Suppose that $F(t,\cdot)$ is uniformly continuous on bounded subsets $K' \subset L^p(\Omega;\mathbb{B}_1)$ in the following sense: for all $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that $X, Y \in K'$ and $\mathbf{E}\,\|X - Y\|_1^p < \delta_\varepsilon$, then*

$$\mathbf{E}\,\|F(t,Y) - F(t,Z)\|_2^p < \varepsilon, \quad \forall t \in \mathbb{R}.$$

Then for any p-th mean pseudo almost periodic process $\Phi : \mathbb{R} \to L^p(\Omega;\mathbb{B}_1)$, the stochastic process $t \mapsto F(t,\Phi(t))$ is p-th mean pseudo almost periodic.

Using the composition of classical pseudo almost periodic functions [27] we deduce the following composition result.

Theorem 2.14. *Let $F : \mathbb{R}\times L^p(\Omega;\mathbb{B}_1) \to L^p(\Omega;\mathbb{B}_2)$, $(t,Y) \mapsto F(t,Y)$ be a p-th mean pseudo almost periodic process in $t \in \mathbb{R}$ uniformly in $Y \in \mathbb{K}$, where $\mathbb{K} \subset L^p(\Omega;\mathbb{B}_1)$ is compact. Suppose that F is Lipschitz in the following sense:*

$$\mathbf{E}\,\|F(t,Y) - F(t,Z)\|_2^p \le M\,\mathbf{E}\|Y - Z\|_1^p$$

for all $Y, Z \in L^p(\Omega;\mathbb{B}_1)$ and for each $t \in \mathbb{R}$, where $M > 0$. Then for any p-th mean pseudo almost periodic process $\Phi : \mathbb{R} \to L^p(\Omega;\mathbb{B}_1)$, the stochastic process $t \mapsto F(t,\Phi(t))$ is p-th mean pseudo almost periodic.

2.2.2 P-th Mean Pseudo Almost Automorphic Stochastic Processes

Definition 2.15. A continuous stochastic process $X : \mathbb{R} \to L^p(\Omega;\mathbb{B})$ is said to be p-th mean almost automorphic if if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exist a subsequence $(s_n)_{n\in\mathbb{N}}$ and a stochastic process $\tilde{X}$ such that

$$\lim_{n\to\infty} \mathbf{E}\left\|\tilde{X}(t) - X(t+s_n)\right\|^p = 0$$

for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} \mathbf{E}\left\|\tilde{X}(t-s_n) - X(t)\right\|^p = 0$$

for each $t \in \mathbb{R}$.

Clearly, our definition (Definition 2.15) is more general than that given in [37, Definition 2.5]. Note that a continuous stochastic process X, which is $2-$nd mean almost automorphic will be called *square-mean almost automorphic*.

The collection of all p-th mean almost automorphic stochastic processes will be denoted by $AA(\mathbb{R};L^p(\Omega;\mathbb{B}))$ and is a Banach space when equipped with the supnorm.

The proof of the next theorem follows along the same as that of the classical case and hence omitted.

Theorem 2.16. *If $f, f_1, f_2 \in AA(\mathbb{R};L^p(\Omega;\mathbb{B}))$, then*

(i) $f_1 + f_2 \in AA(\mathbb{R};L^p(\Omega;\mathbb{B}))$,

(ii) $\lambda f \in AA(\mathbb{R};L^p(\Omega;\mathbb{B}))$ *for any scalar* λ,

(iii) $X_\alpha \in AA(\mathbb{R};L^p(\Omega;\mathbb{B}))$ *where* $X_\alpha : \mathbb{R} \to \mathbb{B}$ *is defined by* $X_\alpha(\cdot) = X(\cdot + \alpha)$,

(iv) the range $\mathcal{R}_X := \{X(t) : t \in \mathbb{R}\}$ *is relatively compact in* $L^p(\Omega;\mathbb{B})$*, thus* X *is bounded in norm,*

(v) if $X_n \to X$ *uniformly on* $\mathbb{R}$ *where each* $X_n \in AA(\mathbb{R};L^p(\Omega;\mathbb{B}))$*, then* $X \in AA(\mathbb{R};L^p(\Omega;\mathbb{B}))$ *too.*

Definition 2.17. A stochastic process $F : \mathbb{R}\times L^p(\Omega;\mathbb{B}_1) \to L^p(\Omega;\mathbb{B}_2))$, $(t,Y) \mapsto F(t,Y)$, which is jointly continuous, is said to be p-th mean almost automorphic in $t \in \mathbb{R}$ uniformly in $Y \in K$ where $K \subset L^p(\Omega;\mathbb{B}_1)$ is a compact if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exist a subsequence $(s_n)_{n\in\mathbb{N}}$ and a stochastic process $\tilde{F} : \mathbb{R}\times L^p(\Omega;\mathbb{B}_1) \to L^p(\Omega;\mathbb{B}_2))$ such that

$$\lim_{n\to\infty} \mathbf{E}\left\|\tilde{F}(t,X) - F(t+s_n,X)\right\|^p = 0$$

is well defined in $t \in \mathbb{R}$ and for each $X \in K$, and

$$\lim_{n\to\infty} \mathbf{E}\left\|\tilde{F}(t-s_n,X) - F(t,x)\right\|^p = 0$$

for all $t \in \mathbb{R}$ and $X \in K$.

The collection of those stochastic processes is denoted $AA(\mathbb{R}\times L^p(\Omega;\mathbb{B}_1);L^p(\Omega;\mathbb{B}_2))$.

The next composition result is a consequence of a composition result from [61].

Theorem 2.18. *Let $F : \mathbb{R}\times L^p(\Omega;\mathbb{B}_1) \to L^p(\Omega;\mathbb{B}_2)$, $(t,Y)\mapsto F(t,Y)$ be a p-th mean almost automorphic process in $t\in\mathbb{R}$ uniformly in $Y\in K$, where $K\subset L^p(\Omega;\mathbb{B}_1)$ is any compact subset. Suppose that $F(t,\cdot)$ is uniformly continuous on bounded subsets $K'\subset L^p(\Omega;\mathbb{B}_1)$ in the following sense: for all $\varepsilon>0$ there exists $\delta_\varepsilon>0$ such that $X,Y\in K'$ and $\mathbf{E}\|X-Y\|_1^p<\delta_\varepsilon$, then*

$$\mathbf{E}\|F(t,Y)-F(t,Z)\|_2^p<\varepsilon,\ \ \forall t\in\mathbb{R}.$$

Then for any p-th mean almost automorphic process $\Phi:\mathbb{R}\to L^p(\Omega;\mathbb{B}_1)$, the stochastic process $t\mapsto F(t,\Phi(t))$ is p-th mean almost automorphic.

Definition 2.19. A stochastic process $X\in BC(\mathbb{R};L^p(\Omega;\mathbb{B}))$ is called p-th pseudo almost automorphic if it can be expressed as $X=Y+\Phi$, where $Y\in AA(\mathbb{R};L^p(\Omega;\mathbb{B}))$ and $\Phi\in PAP_0(\mathbb{R};L^p(\Omega;\mathbb{B}))$. The collection of such functions will be denoted by $PAA(\mathbb{R};L^p(\Omega;\mathbb{B}))$.

Definition 2.20. A bounded continuous stochastic process $F:\mathbb{R}\times L^p(\Omega;\mathbb{B}_1)\to L^p(\Omega;\mathbb{B}_2)$ is called p-th mean pseudo almost automorphic if it can be expressed as $F=G+\Phi$, where $G\in AA(\mathbb{R}\times L^p(\Omega;\mathbb{B}_1);L^p(\Omega;\mathbb{B}_2))$ and $\Phi\in PAP_0(\mathbb{R}\times L^p(\Omega;\mathbb{B}_1);L^p(\Omega;\mathbb{B}_2))$. The collection of such processes will be denoted by $PAP_0(\mathbb{R}\times L^p(\Omega;\mathbb{B}_1);L^p(\Omega;\mathbb{B}_2))$.

The next theorem, which is a straightforward consequence of a result due to Liang et al. [61].

Theorem 2.21. *The space $PAA(\mathbb{R};L^p(\Omega;\mathbb{B}))$ equipped with the sup norm $\|\cdot\|_\infty$ is a Banach space.*

The next composition result is a consequence of [43, Theorem 2.4].

Theorem 2.22. *Suppose $F:\mathbb{R}\times L^p(\Omega;\mathbb{B}_1)\to L^p(\Omega;\mathbb{B}_2)$ belongs to $PAA(\mathbb{R}\times L^p(\Omega;\mathbb{B}_1),L^p(\Omega;\mathbb{B}_2))$; $F=G+H$, with $X\mapsto G(t,X)$ being uniformly continuous on any bounded subset K of $L^p(\Omega;\mathbb{B}_1)$ uniformly in $t\in\mathbb{R}$. Furthermore, we suppose that there exists $L>0$ such that*

$$\mathbf{E}\|F(t,x)-F(t,y)\|_2^p\le L\mathbf{E}\|x-y\|_1^p$$

for all $X,Y\in L^p(\Omega;\mathbb{B}_1)$ and $t\in\mathbb{R}$.

Then the function defined by $H(t)=F(t,\Phi(t))$ belongs to $PAA(\mathbb{R};L^p(\Omega;\mathbb{B}_2))$ provided $\Phi\in PAA(\mathbb{R};L^p(\Omega;\mathbb{B}_1))$.

The next composition result is a consequence of a composition result from [61].

Theorem 2.23. *If $F\in PAA(\mathbb{R}\times L^p(\Omega;\mathbb{B}_1),L^p(\Omega;\mathbb{B}_2))$ and if $X\mapsto F(t,X)$ is uniformly continuous on any bounded subset K of $L^p(\Omega;\mathbb{B}_1)$ for each $t\in\mathbb{R}$, then the stochastic process defined by $H(t)=F(t,\Phi(t))$ belongs to $PAA(\mathbb{R};L^p(\Omega;\mathbb{B}_2))$ provided $\Phi\in PAA(\mathbb{R};L^p(\Omega;\mathbb{B}_1))$.*

3 Main Results

In this section, we study the existence of p-th mean pseudo almost automorphic solutions to the class of nonautonomous stochastic differential equations of type (1.1) where $(A(t))_{t\in\mathbb{R}}$ is a family of closed linear operators on $L^p(\Omega;\mathbb{H})$ satisfying (2.1)-(2.2), and the stochastic processes $F_i(i=1,2):\mathbb{R}\times L^p(\Omega,\mathbb{H})\to L^p(\Omega,\mathbb{H})$ are p-th mean pseudo almost automorphic in $t \in \mathbb{R}$ uniformly in the second variable, and $\mathbb{W}$ is one-dimensional Wiener with the real number line as time parameter.

In order to deal with the existence and uniqueness of a p-th mean pseudo almost automorphic solution to (1.1), we make extensive use of ideas and techniques utilized in [38], [28], [14], and the Schauder fixed-point theorem.

Our setting requires the following assumptions:

(H.2) The injection $\mathbb{H}_\alpha \hookrightarrow \mathbb{H}$ is compact.

(H.3) Fix μ, α, β be real numbers such that $0 \le \mu < \alpha < \beta < 1$ with $2\alpha > \mu + 1$. Moreover, the following holds:

$$\mathbb{H}_\alpha^t = \mathbb{H}_\alpha \text{ and } \mathbb{H}_\beta^t = \mathbb{H}_\beta$$

for all $t \in \mathbb{R}$, with uniform equivalent norms.

(H.4) $R(\zeta, A(\cdot)) \in AA(\mathbb{R}; B(L^p(\Omega;\mathbb{H})))$.

(H.5) The function $F_i(i = 1,2) : \mathbb{R} \times L^p(\Omega;\mathbb{H}) \to L^p(\Omega,\mathbb{H})$ is p-th mean pseudo almost automorphic in the first variable uniformly in the second variable. Moreover, $X \to F_i(t, X)$ is uniformly continuous on any bounded subset O_i of $L^p(\Omega;\mathbb{H})$ for each $t \in \mathbb{R}$. Finally,

$$\sup_{t\in\mathbb{R}} \mathbf{E}\,\|F_i(t,X)\|^p \le \mathcal{M}_1\left(\|X\|_\infty\right)$$

where $\mathcal{M}_i : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying

$$\lim_{r\to\infty} \frac{\mathcal{M}_i(r)}{r} = 0\,.$$

In this section, Γ_1 and Γ_2 stand respectively for the nonlinear integral operators defined by

$$(\Gamma_1 X)(t) := \int_{-\infty}^t U(t,s)F_1(s,X(s))\,ds \;\text{ and }\; (\Gamma_2 X)(t) := \int_{-\infty}^t U(t,s)F_2(s,X(s))\,d\mathbb{W}(s)\,.$$

In addition to the above-mentioned assumptions, we assume that $\alpha \in (0, \frac{1}{2} - \frac{1}{p})$ if $p > 2$ and $\alpha \in (0, \frac{1}{2})$ if $p = 2$.

Lemma 3.1. *[14] Under assumptions* (H.1)-(H.3)-(H.4)-(H.5)*, the mappings* $\Gamma_i(i = 1,2) : BC(\mathbb{R}, L^p(\Omega,\mathbb{H})) \to BC(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha))$ *are well defined and continuous.*

Lemma 3.2. *Under assumptions* (H.1)-(H.3)-(H.4)-(H.5)*, the integral operator* $\Gamma_i(i = 1,2)$ *maps* $PAA(\mathbb{R}, L^p(\Omega,\mathbb{H}))$ *into itself.*

Proof. Let $X \in PAA(\mathbb{R}, L^p(\Omega,\mathbb{H}))$. Using the composition result Theorem 2.23 it follows that $f_1(t) := F_1(t, X(t)) \in PAA(\mathbb{R}, L^p(\Omega,\mathbb{H}))$. Then write $f_1(t) = f(t) + g(t)$ where $f \in AA(\mathbb{R}, L^p(\Omega,\mathbb{H}))$ and $g \in PAP_0(\mathbb{R}, L^p(\Omega,\mathbb{H}))$.

Now set

$$(M_1)(t) := \int_{-\infty}^t U(t,s)f(s)ds$$

and

$$(N_1)(t) := \int_{-\infty}^{t} U(t,s)g(s)ds$$

To complete the proof we have to show that $M_1 \in AA(\mathbb{R}, L^p(\Omega,\mathbb{H}))$ and $N_1 \in PAP_0(\mathbb{R}, L^p(\Omega,\mathbb{H}))$. Now since $f \in AA(\mathbb{R}, L^p(\Omega,\mathbb{H}))$, for every sequence of real numbers $(\tau'_n)_{n\in\mathbb{N}}$ there exist a stochastic process $\tilde{f} : \mathbb{R} \mapsto L^p(\Omega,\mathbb{H})$ and a subsequence $(\tau_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \mathbf{E}\left\|\tilde{f}(t) - f(t+\tau_n)\right\|^p = 0$$

for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} \mathbf{E}\left\|\tilde{f}(t-\tau_n) - f(t)\right\|^p = 0$$

for each $t \in \mathbb{R}$.

Set $\tilde{M}_1(t) = \int_{-\infty}^{t} U(t,s)\tilde{f}(s)ds$ for all $t \in \mathbb{R}$.

Now

$$\begin{aligned}
M_1(t+\tau_n) - \tilde{M}_1(t) &= \int_{-\infty}^{t+\tau_n} U(t+\tau_n,s)f(s)ds - \int_{-\infty}^{t} U(t,s)\tilde{f}(s)ds \\
&= \int_{-\infty}^{t} U(t+\tau, s+\tau_n)f(s+\tau_n)ds \\
&- \int_{-\infty}^{t} U(t,s)\tilde{f}(s)ds \\
&= \int_{-\infty}^{t} U(t+\tau_n, s+\tau_n)\Big(f(s+\tau_n) - \tilde{f}(s)\Big)ds \\
&+ \int_{-\infty}^{t} \Big(U(t+\tau_n, s+\tau_n) - U(t,s)\Big)\tilde{f}(s)ds.
\end{aligned}$$

Using the exponential stability of $U(t,s)$ and the Lebesgue Dominated Convergence Theorem, one can easily see that

$$\mathbf{E}\left\|\int_{-\infty}^{t} U(t+\tau_n, s+\tau_n)\Big(f(s+\tau_n) - \tilde{f}(s)\Big)ds\right\|^p \to 0 \ \text{ as } \ n\to\infty, \ t\in\mathbb{R}.$$

Similarly, from [8] it follows that

$$\mathbf{E}\left\|\int_{-\infty}^{t} \left(U(t+\tau_n, s+\tau_n) - U(t,s)\right)\tilde{f}(s)ds\right\|^p \to 0 \ \text{ as } \ n\to\infty, \ t\in\mathbb{R}.$$

Therefore,

$$\lim_{n\to\infty} \mathbf{E}\left\|\tilde{M}_1(t) - M_1(t+\tau_n)\right\|^p = 0, \ \ t\in\mathbb{R}.$$

Using similar ideas as the previous ones, one can easily see that

$$\lim_{n\to\infty} \mathbf{E}\left\|M_1(t) - \tilde{M}_1(t-\tau_n)\right\|^p = 0, \ \ t\in\mathbb{R}.$$

Let $T > 0$. Again using the fact $U(t,s)$ is exponentially stable, we have

$$
\begin{aligned}
\frac{1}{2T}\int_{-T}^{T}\mathbf{E}\|N_1(t)\|^p\,dt &\le \frac{1}{2T}\int_{-T}^{T}\int_0^{+\infty}\mathbf{E}\|U(t,s)g(t-s)\|^p\,dsdt\\
&\le \frac{M}{2T}\int_{-T}^{T}\int_0^{+\infty}e^{-\delta s}\mathbf{E}\|g(t-s)\|^p\,dsdt\\
&\le M\int_0^{+\infty}e^{-\delta s}\left(\frac{1}{2T}\int_{-T}^{T}\mathbf{E}\|g(t-s)\|^p\,dt\right)ds.
\end{aligned}
$$

Now using the fact $PAP_0(\mathbb{R}, L^p(\Omega,\mathbb{H}))$ is translation-invariant it follows that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathbf{E}\|g(t-s)\|^p\,dt = 0,$$

as $t \mapsto g(t-s) \in PAP_0(\mathbb{R}, L^p(\Omega,\mathbb{H}))$ for every $s \in \mathbb{R}$.

Using Lebesgue Dominated Convergence Theorem it follows that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathbf{E}\|N_1(t)\|^p\,dt = 0$$

and hence $N_1 \in PAP_0(\mathbb{R}, L^p(\Omega,\mathbb{H}))$.

As to Γ_2, we use again the composition result Theorem 2.23. It follows that $f_2(t) := F_2(t, X(t)) \in PAA(\mathbb{R}, L^p(\Omega,\mathbb{H}))$. Now, write $f_2(t) = h(t) + l(t)$ where $h \in AA(\mathbb{R}, L^p(\Omega,\mathbb{H}))$ and $l \in PAP_0(\mathbb{R}, L^p(\Omega,\mathbb{H}))$.

Then, set

$$(S_1)(t) := \int_{-\infty}^{t} U(t,s)h(s)d\mathbb{W}(s)$$

and

$$(T_1)(t) := \int_{-\infty}^{t} U(t,s)l(s)d\mathbb{W}(s)\,.$$

To complete the proof we need to show that $S_1 \in AA(\mathbb{R}, L^p(\Omega,\mathbb{H}))$ and $T_1 \in PAP_0(\mathbb{R}, L^p(\Omega,\mathbb{H}))$. Now since $h \in AA(\mathbb{R}, L^p(\Omega,\mathbb{H}))$, for every sequence of real numbers $(\tau'_n)_{n\in\mathbb{N}}$ there exist a stochastic process $\tilde{h} : \mathbb{R} \mapsto L^p(\Omega,\mathbb{H})$ and a subsequence $(\tau_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty}\mathbf{E}\left\|\tilde{h}(t) - h(t+\tau_n)\right\|^p = 0$$

for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty}\mathbf{E}\left\|\tilde{h}(t-\tau_n) - h(t)\right\|^p = 0$$

for each $t \in \mathbb{R}$.

Set $\tilde{S}(t) = \displaystyle\int_{-\infty}^{t} U(t,s)\tilde{h}(s)d\mathbb{W}(s)$ for all $t \in \mathbb{R}$.

Now

$$\begin{aligned}
S_1(t+\tau_n)-\tilde{S}(t) &= \int_{-\infty}^{t+\tau_n} U(t+\tau_n,s)h(s)d\mathbb{W}(s)-\int_{-\infty}^{t} U(t,s)\tilde{h}(s)d\mathbb{W}(s)\\
&= \int_{-\infty}^{t} U(t+\tau,s+\tau_n)h(s+\tau_n)d\mathbb{W}(s)\\
&- \int_{-\infty}^{t} U(t,s)\tilde{h}(s)d\mathbb{W}(s)\\
&= \int_{-\infty}^{t} U(t+\tau_n,s+\tau_n)\Big(h(s+\tau_n)-\tilde{h}(s)\Big)d\mathbb{W}(s)\\
&+ \int_{-\infty}^{t} (U(t+\tau_n,s+\tau_n)-U(t,s))\,\tilde{h}(s)d\mathbb{W}(s).
\end{aligned}$$

Using the exponential stability of $U(t,s)$ and the Lebesgue Dominated Convergence Theorem, one can see that

$$\begin{aligned}
&\mathbf{E}\left\|\int_{-\infty}^{t} U(t+\tau_n,s+\tau_n)\Big(h(s+\tau_n)-\tilde{h}(s)\Big)d\mathbb{W}(s)\right\|^p\\
&\le C_p\mathbf{E}\left[\int_{-\infty}^{t}\left\|U(t+\tau_n,s+\tau_n)\Big(h(s+\tau_n)-\tilde{h}(s)\Big)\right\|^2 ds\right]^{p/2}\to 0 \text{ as } n\to\infty,\ t\in\mathbb{R}.
\end{aligned}$$

Similarly, from [8] it follows that

$$\begin{aligned}
&\mathbf{E}\left\|\int_{-\infty}^{t} (U(t+\tau_n,s+\tau_n)-U(t,s))\,\tilde{h}(s)d\mathbb{W}(s)\right\|^p\\
&\le C_p\mathbf{E}\left[\int_{-\infty}^{t}\left\|(U(t+\tau_n,s+\tau_n)-U(t,s))\,\tilde{h}(s)\right\|^2 ds\right]^{p/2}\to 0 \text{ as } n\to\infty,\ t\in\mathbb{R}.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\mathbf{E}\left\|\tilde{S}(t)-S_1(t+\tau_n)\right\|^p=0,\ \ t\in\mathbb{R}.$$

Using similar ideas as the previous ones, one can easily see that

$$\lim_{n\to\infty}\mathbf{E}\left\|S_1(t)-\tilde{S}(t-\tau_n)\right\|^p=0,\ \ t\in\mathbb{R}.$$

Let $T>0$. Again using the fact $U(t,s)$ is exponentially stable, we have for $p>2$,

$$\begin{aligned}
\frac{1}{2T}\int_{-T}^{T}\mathbf{E}\,\|T_1(t)\|^p\,dt &\le C_p\cdot M^p\,\frac{1}{2T}\int_{-T}^{T}\mathbf{E}\left[\int_0^\infty e^{-2\delta s}\,\|l(t-s)\|^2\,ds\right]^{p/2}dt\\
&\le C_p\cdot M^p\Big(\int_0^\infty e^{-2\delta s}\,ds\Big)^{\frac{p-2}{2}}\times\\
&\quad\times\int_0^\infty e^{-2\delta s}\left(\frac{1}{2T}\int_{-T}^{T}\mathbf{E}\,\|l(t-s)\|^p\,dt\right)ds\,.
\end{aligned}$$

For $p=2$, we have

$$\frac{1}{2T}\int_{-T}^{T}\mathbf{E}\,\|T_1(t)\|^2\,dt \le \int_0^\infty e^{-2\delta s}\,\frac{1}{2T}\int_{-T}^{T}\mathbf{E}\,\|l(t-s)\|^2\,dt\,ds\,.$$

Now using the fact $PAP_0(\mathbb{R}, L^p(\Omega, \mathbb{H}))$ is translation-invariant it follows that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathbf{E}\,\|l(t-s)\|^p\, dt = 0,$$

as $t \mapsto l(t-s) \in PAP_0(\mathbb{R}, L^p(\Omega, \mathbb{H}))$ for every $s \in \mathbb{R}$.

Using Lebesgue Dominated Convergence Theorem it follows that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathbf{E}\,\|T_1(t)\|^p\, dt = 0$$

and hence $T_1 \in PAP_0(\mathbb{R}, L^p(\Omega, \mathbb{H}))$.

□

Let $\gamma \in (0,1]$ and let

$$BC^\gamma(\mathbb{R}, L^p(\Omega, \mathbb{H}_\alpha)) = \Big\{X \in BC(\mathbb{R}, L^p(\Omega, \mathbb{H}_\alpha)) : \big\|X\big\|_{\alpha,\gamma} < \infty\Big\},$$

where

$$\big\|X\big\|_{\alpha,\gamma} = \sup_{t\in\mathbb{R}} \Big[\mathbf{E}\big\|X(t)\big\|_\alpha^p\Big]^{\frac{1}{p}} + \gamma \sup_{t,\, s\in\mathbb{R},\, s\neq t} \frac{\Big[\mathbf{E}\big\|X(t)-X(s)\big\|_\alpha^p\Big]^{\frac{1}{p}}}{\big|t-s\big|^\gamma}.$$

Clearly, the space $BC^\gamma(\mathbb{R}, L^p(\Omega, \mathbb{H}_\alpha))$ equipped with the norm $\|\cdot\|_{\alpha,\gamma}$ is a Banach space, which is in fact the Banach space of all bounded continuous Holder functions from $\mathbb{R}$ to $L^p(\Omega, \mathbb{H}_\alpha)$ whose Holder exponent is γ.

Lemma 3.3. *[14] Under assumptions* (H.1)-(H.5)*, the mapping* Γ_1 *defined previously maps bounded sets of* $BC(\mathbb{R}, L^p(\Omega, \mathbb{H}))$ *into bounded sets of* $BC^\gamma(\mathbb{R}, L^p(\Omega, \mathbb{H}_\alpha))$ *for some* $0 < \gamma < 1$.

Lemma 3.4. *[14] Let* $\alpha, \beta \in (0, \frac{1}{2})$ *with* $\alpha < \beta$. *Under assumptions* (H.1)-(H.6)*, the mapping* Γ_2 *defined previously maps bounded sets of* $BC(\mathbb{R}, L^p(\Omega, \mathbb{H}))$ *into bounded sets of* $BC^\gamma(\mathbb{R}, L^p(\Omega, \mathbb{H}_\alpha))$ *for some* $0 < \gamma < 1$.

Lemma 3.5. *The nonlinear integral operators* $\Gamma_i (i = 1, 2)$ *map bounded sets of* $PAA(\Omega, L^p(\Omega, \mathbb{H}))$ *into bounded sets of* $BC^\gamma(\mathbb{R}, L^p(\Omega, \mathbb{H}_\alpha)) \cap PAA(\mathbb{R}, L^p(\Omega, \mathbb{H}))$ *for* $0 < \gamma < \alpha$.

Proof. The proof follows along the same lines as that of Lemma 3.3 and hence omitted. □

Similarly, the next lemma is a consequence of [38, Proposition 3.3]. Note in this context that $\mathbb{X} = L^p(\Omega, \mathbb{H})$ and $\mathbb{Y} = L^p(\Omega, \mathbb{H}_\alpha)$.

Lemma 3.6. *For* $0 < \gamma < \alpha$, $BC^\gamma(\mathbb{R}, L^p(\Omega, \mathbb{H}_\alpha))$ *is compactly contained in* $BC(\mathbb{R}, L^p(\Omega, \mathbb{H}))$*, that is, the canonical injection*

$$id : BC^\gamma(\mathbb{R}, L^p(\Omega, \mathbb{H}_\alpha)) \hookrightarrow BC(\mathbb{R}, L^p(\Omega, \mathbb{H}))$$

is compact, which yields

$$id : BC^\gamma(\mathbb{R}, L^p(\Omega, \mathbb{H}_\alpha)) \cap PAA(\mathbb{R}, L^p(\Omega, \mathbb{H})) \to PAA(\mathbb{R}, L^p(\Omega, \mathbb{H}))$$

is compact, too.

Theorem 3.7. *Suppose assumptions* (H.1)-(H.5) *hold, then the nonautonomous differential equation Eq.* (1.1) *has at least one p-th mean pseudo almost automorphic solution.*

Proof. Let us recall that in view of Lemmas 3.6 and 3.2, we have

$$\left\|(\Gamma_1+\Gamma_2)X\right\|_{\alpha,\infty} \le d(\beta,\delta)\Big(\mathcal{M}_1(\left\|X\right\|_\infty)+\mathcal{M}_2(\left\|X\right\|_\infty)\Big)$$

and

$$\mathbf{E}\left\|(\Gamma_1+\Gamma_2)X(t_2)-(\Gamma_1+\Gamma_2)X(t_1)\right\|_\alpha^p \le s(\alpha,\beta,\delta)\Big[\mathcal{M}_1(\left\|X\right\|_\infty)+\mathcal{M}_2(\left\|X\right\|_\infty)\Big]\left|t_2-t_1\right|^\gamma$$

for all $X \in BC(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha))$, t_1, $t_2 \in \mathbb{R}$ with $t_1 \neq t_2$, where $d(\beta,\delta)$ and $s(\alpha,\beta,\delta)$ are positive constants. Consequently, $X \in BC(\mathbb{R}, L^p(\Omega,\mathbb{H}))$ and $\left\|X\right\|_\infty < R$ yield $(\Gamma_1+\Gamma_2)X \in BC^\gamma(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha))$ and $\left\|(\Gamma_1+\Gamma_2)X\right\|_{\alpha,\infty}^p < R_1$ where $R_1 = c(\alpha,\beta,\delta)\Big(\mathcal{M}_1(R)+\mathcal{M}_2(R)\Big)$. since $\mathcal{M}_i(R)/R \to 0$ as $R \to \infty$, and since $\mathbf{E}\left\|X\right\|^p \le c\mathbf{E}\left\|X\right\|_\alpha^p$ for all $X \in L^p(\Omega,\mathbb{H}_\alpha)$, it follows that exists an $r > 0$ such that for all $R \ge r$, the following hold

$$(\Gamma_1+\Gamma_2)\Big(B_{PAA(\mathbb{R},L^p(\Omega,\mathbb{H}))}(0,R)\Big) \subset B_{BC^\gamma(\mathbb{R},L^p(\Omega,\mathbb{H}_\alpha))} \cap B_{PAA(\mathbb{R},L^p(\Omega,\mathbb{H}))}(0,R)\,.$$

In view of the above, it follows that $(\Gamma_1+\Gamma_2) : D \to D$ is continuous and compact, where D is the ball in $PAA(\mathbb{R}, L^p(\Omega,\mathbb{H}))$ of radius R with $R \ge r$. Using the Schauder fixed point it follows that $(\Gamma_1+\Gamma_2)$ has a fixed point, which is obviously a p-th mean pseudo almost automorphic mild solution to Eq. (1.1).

□

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