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Flow-Box Theorem and Beyond

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Abstract

For a given vector field $\nu(x)$ around a nonsingular point x_0, we provide explicit coordinates $z = \varphi(x)$ in which the vector field is straightened out, i. e., $\varphi_*(\nu)(z) = \frac{\partial}{\partial z_1}$. The procedure is generalized to Frobënius Theorem, namely, for an involutive distribution $\Delta = \text{span}\,\{\nu_1, \ldots, \nu_m\}$ around a nonsingular point x_0, we give explicit coordinates $z = \varphi(x)$ in which

$$\varphi_*\Delta = \text{span}\left\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_m}\right\}.$$

The method is illustrated by several examples and is applied to the linearization of control systems.

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1 Introduction

The theory of differential equations is one of the most productive and useful contributions of our modern times. Its applications are widespread in all branches of natural sciences, particulary, in Physics, Biology, Chemistry, Engineering, Ecology, and in Weather Predictions, just to name few. It plays the role of a connector between abstract mathematical theories and applications in real world problems. Paraphrasing Newton quoted as saying that ”it is useful to solve differential equations”, a lot has been deserved in solving differential equations with various methods and techniques provided in the literature. Existence and uniqueness of solutions have been addressed in many scientific papers and textbooks. Consider the simplest expression of a linear partial differential equation

$$\nu_1(x)\frac{\partial u}{\partial x_1} + \cdots + \nu_n(x)\frac{\partial u}{\partial x_n} = b(x)$$

where $\nu_1(x), \cdots, \nu_n(x)$ and $b(x)$ are smooth or analytic functions in the variable x. This partial differential equation is referred to as a homogeneous (resp. nonhomogeneous) linear first order partial differential equation when $b = 0$ (resp. $b \neq 0$). The vector field $\nu(x)$ whose

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components are $\nu_1(x), \cdots, \nu_n(x)$ is called the characteristic vector field of the homogeneous equation and the corresponding dynamical system $\dot{x} = \nu(x)$, its characteristic equation. The solutions of the system are the integral curves of the characteristic equation and are often obtained by solving the so-called Lagrange subsidiary equation (also called characteristic equation)

$$\frac{dx_1}{\nu_1(x)} = \cdots = \frac{dx_n}{\nu_n(x)} = \frac{du}{b(x)}.$$

Several methods have been devoted to the solving of such system among them Euler's method and Natani's method [3], [5]. Most of the work on ordinary differential equations have been done around equilibrium points (non regular or singular point), that is, a point x_0 where $\nu(x_0) = 0$. The reason being that regular points, that is, where $\nu(x_0) \neq 0$ are not topologically reach because in their neighborhoods all trajectories are straight parallel lines (straightening theorem). Though this fact remains true and hence often neglected, the straightening theorem has many important applications. Indeed, a solution of the nonhomogeneous partial differential equation above can be easily found around a regular point x_0 of ν by simple quadrature in new coordinates: If $z = \varphi(x)$ is a change of coordinates around x_0 that rectifies the vector field ν, i.e., such that $\varphi_* \nu = \frac{\partial}{\partial z_1}$, then the nonhomogeneous equation simplifies as $\frac{\partial \tilde{u}}{\partial z_1} = \tilde{b}(z)$, where $u(x) = \tilde{u}(\varphi(x))$ and $b(x) = \tilde{b}(\varphi(x))$. A solution $\tilde{u}$ (yielding $u = \tilde{u} \circ \varphi$) is given by $\tilde{u}(z) = a(z_2, \ldots, z_n) + \int_0^{z_1} \tilde{b}(\varepsilon, z_2 \ldots, z_n)\, d\varepsilon$. The only difficulty in applying the straightening theorem is in finding the straightening diffeomorphism. The main focus of this paper is to provide an explicit algorithm allowing to compute the straightening diffeomorphism.

2 Definitions and Notations

This section deals with basic notations and definitions. We will first start by recalling the notion of vector fields and flow; then we will give a version of the flow box theorem. In Section 3, we will give our main results, that is, explicit formulas for computing both the rectifying change of coordinates and its inverse. We generalize those results to the Frobënius Theorem and we discuss the convergence of the power series. We illustrate by taking several examples in Section 5. The last Section 6 gives applications to the linearization of control systems.

2.1 Vector Fields and Flows.

An analytic vector field, ν, is an analytic mapping from a manifold $\mathcal{M}$ to its tangent fiber-space $T\mathcal{M}$ that associates to each point $x \in \mathcal{M}$, a tangent vector $\nu|_x \in T_x\mathcal{M}$. In local coordinates $x = (x_1, \ldots, x_n)$, the vector field ν is simply written

$$\nu(x) \triangleq \nu|_x = \nu_1(x) \frac{\partial}{\partial x_1}\Big|_x + \cdots + \nu_n(x) \frac{\partial}{\partial x_n}\Big|_x,$$

where $\left\{ \frac{\partial}{\partial x_1}\Big|_x, \ldots, \frac{\partial}{\partial x_n}\Big|_x \right\}$ form a basis of $T_x\mathcal{M}$. In what follows we will omit the subscript $|_x$ and use the more compact notation $\partial_{x_i} \triangleq \frac{\partial}{\partial x_i}\Big|_x$ and we will denote by $\mathcal{V}(\mathcal{M})$ the set of all vector fields on $\mathcal{M}$.

A curve $\gamma : I \subseteq R \longmapsto \mathcal{M}$ is an *integral curve* for the vector field $\nu \in \mathcal{V}(\mathcal{M})$ if for any $t \in I$ we have $\gamma'(t) = \nu(\gamma(t))$. The *flow* of ν is an analytic map $\phi : I \times \mathcal{M} \longmapsto \mathcal{M}$ such that for any $x \in \mathcal{M}$ the curve $\phi_x : t \longmapsto \phi(t,x)$ is an integral curve of ν. In other words, the flow is a solution of the differential equation

$$\frac{d\phi(t,x)}{dt} = \nu(\phi(t,x)),\ \phi(0,x) = x.$$

For further details about the existence conditions and uniqueness of solutions, we refer the readers to the existing literature (e.g. [1], [15], [16] and the references therein). Above and throughout the paper, all objects considered are analytic except otherwise stated. As all results are intended locally, we will set $\mathcal{M} = R^n$ without loss of generality. For a vector field $\nu(x) = \nu_1(x)\partial_{x_1} + \cdots + \nu_n(x)\partial_{x_n}$ and a function h in x-coordinates, we denote by

$$L_\nu(h)(x) = \frac{\partial h}{\partial x}\nu(x) = \frac{\partial h}{\partial x_1}\nu_1(x) + \cdots + \frac{\partial h}{\partial x_n}\nu_n(x)$$

the Lie-derivative of h along the vector field ν. Recursive Lie-derivatives are defined by setting

$$L_\nu^0(h)(x) = h(x),\ \ L_\nu^{j+1}(h)(x) = L_\nu(L_\nu^j(h))(x),\ \ j = 0,1,\ldots,\infty.$$

Given another vector field $\mu(x) = \mu_1(x)\partial_{x_1} + \cdots + \mu_n(x)\partial_{x_n}$, the Lie-bracket between μ and ν defines a new vector field

$$[\mu,\nu] = (L_\mu(\nu_1) - L_\nu(\mu_1))\partial_{x_1} + \cdots + (L_\mu(\nu_n) - L_\nu(\mu_n))\partial_{x_n}.$$

Let $\varphi : R^n \longmapsto R^n$ be a (local) diffeomorphism, $\varphi(0) = 0$, giving rise to new coordinates system $z = \varphi(x)$. The vector field ν is transformed via φ into a new vector field given by

$$\varphi_*\nu(z) = L_\nu(\varphi_1)(z)\partial_{z_1} + \cdots + L_\nu(\varphi_n)(z)\partial_{z_n},$$

where, by abuse of notation, we put

$$L_\nu(\varphi_j)(z) = \frac{\partial \varphi_j}{\partial x}(\varphi^{-1}(z))\nu(\varphi^{-1}(z)),\ \text{for all } 1 \le j \le n.$$

In the next subsection we will recall a version of the flow box theorem before the main results. First we introduce some notation that will be useful in the sequel. For any $x \in R^n$ we put $x = (x_1,\ldots,x_n)$. For the set of n-tuples of integers, i.e., the subset $N^n \subset R^n$, we will use a bolded variable to denote its elements. Given two n-tuples $\boldsymbol{m} = (m_1,\ldots,m_n)$ and $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n)$ we say that $\boldsymbol{m} \succeq \boldsymbol{\alpha}$ if and only if $m_i \ge \alpha_i$ for all $1 \le i \le n$ and we denote by $\boldsymbol{m}! = m_1!\cdots m_n!$ and $\boldsymbol{m}^{\boldsymbol{\alpha}} = m_1^{\alpha_1}\cdots m_n^{\alpha_n}$. By extension, for $x = (x_1,\ldots,x_n) \in R^n$ we put $x^{\boldsymbol{m}} = x_1^{m_1}\cdots x_n^{m_n}$ and set $|\boldsymbol{m}| = m_1 + \cdots + m_n$. Let f be an analytic function with $f(x) = \sum f_{\boldsymbol{m}} \times x^{\boldsymbol{m}}$ its Taylor series expansion, where $f_{\boldsymbol{m}} = \frac{1}{\boldsymbol{m}!}\frac{\partial^{\boldsymbol{m}} f}{\partial x^{\boldsymbol{m}}}(0)$ are constant coefficients, and let $\nu = \nu_1\partial_{x_1} + \cdots + \nu_n\partial_{x_n}$ be an analytic vector field. For any $\rho > 0$, we define the norm $||\cdot||_\rho$ by $||f||_\rho = \sum |f_{\boldsymbol{m}}|\cdot\rho^{|\boldsymbol{m}|}$ and extend the norm to vector fields by $||\nu||_\rho = \max\left\{||\nu_1||_\rho,\ldots,||\nu_n||_\rho\right\}.$

2.2 Flow Box Theorem.

The flow-box theorem is a very well-known result in differential geometry and dynamical systems. A simple version of that theorem is stated as follows.

Theorem 2.1. (Flow-Box Theorem) *Let* ν *be a nonsingular vector field at* $x_0 \in R^n$*, i.e.,* $\nu(x_0) \neq 0$*. There exist a local change of coordinates* $z = \varphi(x)$ *in a neighborhood* $U \ni x_0$ *such that* $\varphi_*(\nu)(z) = \partial_{z_1}$ *for all* $z \in \varphi(U)$*.*

In very simple terms, the flow-box theorem states that new coordinates $z = \mathbf{col}\,(z_1, \ldots, z_n)$ can be found in which the integral curves of a nonsingular vector field are (locally) parallel straight lines $\phi(t,z) = (z_1 + t, z_2, \ldots, z_n)$. The existence and proof of this theorem, as well as its general form, can be found in the literature (e. g. [1], [15]).

There are few methods (method of characteristics, integration of differential one-forms) dealing with solving partial differential equations but, to the author's knowledge, none gives explicit formulas for rectifying a general vector field. We propose here a systematic way of finding the rectifying change of coordinates as well as its inverse by giving explicit formulas for both the diffeomorphism and its inverse. We will illustrate the results with several examples. We will apply the results in finding feedback linearizing coordinates for control systems that can be linearized. Notice that finding the rectifying coordinates for a given vector field ν is equivalent of solving the system of partial differential equations

$$\begin{cases} \nu_1 \dfrac{\partial \varphi_1}{\partial x_1} + \cdots + \nu_n \dfrac{\partial \varphi_1}{\partial x_n} & = & 0 \\ & \cdots & \\ \nu_1 \dfrac{\partial \varphi_{n-1}}{\partial x_1} + \cdots + \nu_n \dfrac{\partial \varphi_{n-1}}{\partial x_n} & = & 0 \\ \nu_1 \dfrac{\partial \varphi_n}{\partial x_1} + \cdots + \nu_n \dfrac{\partial \varphi_n}{\partial x_n} & = & 1. \end{cases}$$

The first $n-1$ equations are equivalent to the fact that $\nu^\perp = \text{span}\,\{\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_{n-1}\}$, that is, the co-distribution associated with ν is generated by the *exact* differential 1-forms $\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_{n-1}$. The characteristic equation (or subsidiary equation) associated to that co-distribution is defined by

$$\frac{dx_1}{\nu_1(x)} = \cdots = \frac{dx_n}{\nu_n(x)}.$$

Each solution is called a first integral of the characteristic equation. The importance of rectifying the vector field can be in solving the Cauchy problem; in finding closed form solutions of ordinary differential equations, and finding the symmetries of vector fields.

3 Main Result

In this section we give the main results of this paper which are explicit coordinates change (and their inverse) normalizing any non vanishing (nonsingular) vector field.

Theorem 3.1. *Let* ν *be a nonsingular vector field on* R^n*,* $1 \leq k \leq n$ *any integer such that* $\nu_k(0) \neq 0$*. Assume* $\nu(0) = \partial_{z_k}$ *and let* $\sigma_k(x) = 1/\nu_k(x)$*.*

(i) *The diffeomorphism* $z = \varphi(x) = \mathbf{col}\,(\varphi_1(x), \ldots, \varphi_n(x))$ *defined by*

$$\begin{aligned} \varphi_j(x) &= x_j + \sum_{s=1}^{\infty} \frac{(-1)^s x_k^s}{s!} L_{\sigma_k \nu}^{s-1}(\sigma_k \nu_j)(x) \quad j \neq k \\ \varphi_k(x) &= \sum_{s=1}^{\infty} \frac{(-1)^{s+1} x_k^s}{s!} L_{\sigma_k \nu}^{s-1}(\sigma_k)(x) \end{aligned} \tag{3.1}$$

satisfies $\varphi_*(\nu)(z) = \partial_{z_k}$ *in a neighborhood of the origin* $U \ni 0 \subseteq R^n$.

(ii) *The diffeomorphism* $x = \psi(z) = \mathbf{col}\,(\psi_1(z), \ldots, \psi_n(z))$ *given by*

$$\begin{aligned} \psi_j(z) &= z_j + \sum_{s=1}^{\infty} \frac{z_k^s}{s!} \left(\sum_{i=0}^{s-1} (-1)^i C_s^i \partial_{z_k}^i \cdot L_{\nu}^{s-i-1}(\nu_j)(z) \right) \quad j \neq k \\ \psi_k(z) &= \sum_{s=1}^{\infty} \frac{z_k^s}{s!} \left(\sum_{i=0}^{s-1} (-1)^i C_s^i \partial_{z_k}^i \cdot L_{\nu}^{s-i-1}(\nu_k)(z) \right) \end{aligned} \tag{3.2}$$

is an inverse of $z = \varphi(x)$, *that is, it satisfies* $\dfrac{\partial \psi(z)}{\partial z_k} = \nu(\psi(z))$.

Before we prove this result, let us first make the following remarks:

R1. The expressions above are not Taylor series around the origin or in the variable x_k as the coefficients $L_{\sigma_k \nu}^s(\sigma_k \nu_j)(x)$ are evaluated at $x = (x_1, \ldots, x_n)$ and might well depend on x_k (see later for justification).

R2. If the vector field ν is independent of some variable x_j $(j \neq k)$, the diffeomorphism $\varphi(x)$ is also independent of the variable x_j (except a linear dependence for the component $\varphi_j(x)$).

R3. If any of the components of $\nu(x)$ is zero, say $\nu_j(x) = 0$, then $\varphi_j(x) = x_j$.

To summarize the remarks above, let us mention that the method used here is different from the classical series expansion for solving ordinary differential equations for which both $\nu(x) = \sum \nu_{m_1 \cdots m_n} x_1^{m_1} \cdots x_n^{m_n}$ and $\varphi_j(x) = \sum \varphi^j_{q_1 \cdots q_n} x_1^{q_1} \cdots x_n^{q_n}$ are expanded in power series and a recursive relationship between the constant coefficients $\nu_{m_1 \cdots m_n}$ and $\varphi^j_{q_1 \cdots q_n}$ is thought. Here the expressions of φ and its inverse depend on the entire vector field ν (not on the coefficient of its series) and the coefficients of the series are functions rather than constants. The problem of convergence is of paramount importance for the validation of the results and is addressed below

Theorem 3.2. *Let* $\rho > 0$ *be a positive number such that* $||\nu||_\rho = \kappa(\rho) < \infty$. *For any* $0 < \hat{\rho} < \rho e^{-1-\kappa(\rho)}$, *the series (3.1) and (3.2) converge inside the ball of radius* $\hat{\rho}$. *Moreover, we have*

(i) $||\varphi_j(x)||_{\hat{\rho}} \leq \hat{\rho} \left[1 + \dfrac{\kappa(\rho)}{1 - \kappa(\rho)/\ln(\rho/\hat{\rho})} \right]$

(ii) $||\psi_j(z)||_{\hat{\rho}} \leq \hat{\rho} \left[1 + \dfrac{\kappa(\rho)\,(\ln(\rho/\hat{\rho}))^2}{(\ln(\rho/\hat{\rho}) - 1)\,(\ln(\rho/\hat{\rho}) - 1 - \kappa(\rho))} \right]$

An extension of Theorem 3.1 is obtained for the Frobënius theorem and is given below. The version of the Frobënius theorem used here can be found in many textbooks [1, 14, 15, 16].

Theorem 3.3. (Frobënius) (i) *Let* $\nu^1(x),\dots,\nu^m(x)$ *be a set of analytic vector fields on* R^n *such that the distribution* $\mathcal{D}(x) = \text{span}\left\{\nu^1(x),\dots,\nu^m(x)\right\}$ *is involutive and of maximal rank* $m \le n$ *in a neighborhood* $U \subset R^n$ *of the origin. There exist an open neighborhood* $0 \in \Omega \subset U$ *and a change of coordinates* $z = \varphi(x)$ *such that* $\varphi_* \mathcal{D}(z)$=span $\{\partial_{z_1},\dots,\partial_{z_m}\}$ $\forall\, z \in \varphi(\Omega)$.
(ii) *There exists a sequence of* explicit *coordinates changes* $\varphi^k(x) = x + \phi^k(x_k,\dots,x_n)$, *for* $k = 1,\dots,n$ *whose composition* $\varphi(x) = \varphi^m \circ \cdots \circ \varphi^1(x)$ *rectifies the distribution* $\mathcal{D}$.

4 Proofs

In this section we give proofs of our results, namely, Theorems 3.1, 3.2, and 3.3. We start with Theorem 3.1 and give a constructive proof.
Proof-Sketch of Theorem 3.1 (i). Without loss of generality we assume $k = n$. The general case will follow. Notice that for any diffeomorphism $z = \varphi(x)$ the two following conditions are equivalent.
(a) $\varphi_*(\nu)(z) = \partial_{z_n}$.
(b) $L_\nu(\varphi_j)(x) = 0$ and $L_\nu(\varphi_n)(x) = 1$ for $1 \le j \le n-1$.

For that reason we will show that condition (b) holds. To start let us take $1 \le j \le n-1$. It follows directly

$$\begin{aligned}
L_\nu(\varphi_j)(x) &= L_\nu(x_j) + \sum_{s=1}^{\infty} L_\nu\left(\frac{(-1)^s x_n^s}{s!} L_{\sigma_n \nu}^{s-1}(\sigma_n \nu_j)\right) \\
&= \nu_j(x) + \sum_{s=1}^{\infty} \frac{(-1)^s x_n^s}{s!} L_\nu L_{\sigma_n \nu}^{s-1}(\sigma_n \nu_j) + \sum_{s=1}^{\infty} \frac{(-1)^s x_n^{s-1}}{(s-1)!} \nu_n(x) L_{\sigma_n \nu}^{s-1}(\sigma_n \nu_j) \\
&= \nu_j(x) + \sum_{s=1}^{\infty} \frac{(-1)^s x_n^s}{s!} \nu_n(x) L_{\sigma_n \nu}^{s}(\sigma_n \nu_j) - \nu_j(x) - \sum_{s=1}^{\infty} \frac{(-1)^s x_n^s}{s!} \nu_n(x) L_{\sigma_n \nu}^{s}(\sigma_n \nu_j) = 0.
\end{aligned}$$

A direct computation shows that

$$\begin{aligned}
L_\nu \varphi_n(x) &= \sum_{s=1}^{\infty} L_\nu\left(\frac{(-1)^{s-1} x_n^s}{s!} L_{\sigma_n \nu}^{s-1}(\sigma_n)\right) \\
&= \sum_{s=1}^{\infty} \frac{(-1)^{s-1} x_n^s}{s!} L_\nu L_{\sigma_n \nu}^{s-1}(\sigma_n) + \sum_{s=1}^{\infty} \frac{(-1)^{s-1} x_n^{s-1}}{(s-1)!} \nu_n(x) L_{\sigma_n \nu}^{s-1}(\sigma_n) \\
&= \sum_{s=1}^{\infty} \frac{(-1)^{s-1} x_n^s}{s!} \nu_n(x) L_{\sigma_n \nu}^{s}(\sigma_n) + \nu_n(x)\sigma_n(x) + \sum_{s=1}^{\infty} \frac{(-1)^s x_n^s}{s!} \nu_n(x) L_{\sigma_n \nu}^{s}(\sigma_n) = 1.
\end{aligned}$$

This ends the sketch of proof of Theorem 3.1 (i). The assumptions $\nu(0) = \partial_{x_k}$ is not restrictive. Indeed, if $\nu(0) = c_1 \partial_{x_1} + \cdots + c_n \partial_{x_n}$, then by a linear change of coordinates we can always get $\nu(0) = \partial_{x_k}$. The proof of the general case follows by first applying the linear change of coordinates (permutation)

$$\tau(x) \triangleq \begin{cases} \tilde{x}_j &= \tau_j(x) = x_j \qquad j \ne k, j \ne n \\ \tilde{x}_k &= \tau_k(x) = x_n \\ \tilde{x}_n &= \tau_n(x) = x_k. \end{cases}$$

Under these new coordinates the transformed vector field $\tilde{\nu} = \tau_*(\nu)$ is obtained from ν by permuting the components ν_k and ν_n and the variables x_k and x_n. Thus we have $\tilde{\nu}_n(0) = \partial_{\tilde{x}_n}$ and the proof above yields

$$\begin{aligned}\tilde{\varphi}_j(\tilde{x}) &= \tilde{x}_j + \sum_{s=1}^{\infty} \frac{(-1)^s \tilde{x}_n^s}{s!} L_{\tilde{\sigma}_n \tilde{\nu}}^{s-1}(\tilde{\sigma}_n \tilde{\nu}_j)(\tilde{x}) \;\; j \neq n \\ \tilde{\varphi}_n(\tilde{x}) &= \sum_{s=1}^{\infty} \frac{(-1)^{s-1} \tilde{x}_n^s}{s!} L_{\tilde{\sigma}_n \tilde{\nu}}^{s-1}(\tilde{\sigma}_n)(\tilde{x}),\end{aligned} \tag{4.1}$$

such that $\tilde{\varphi}_*(\tilde{\nu}) = \partial_{\tilde{z}_n}$. Because $\tau \circ \tau = \mathrm{Id}$ and τ is linear (hence $\frac{\partial \tau}{\partial x} \equiv \tau$), it thus follows that $\tilde{\nu} = \tau \circ \nu \circ \tau$ which is equivalent to $\tau \circ \tilde{\nu} = \nu \circ \tau$. It is enough to show that the expressions of φ and $\tilde{\varphi}$ given respectively by (3.1) and (4.1) are related by $\tau \circ \tilde{\varphi} = \varphi \circ \tau$. Indeed, for $j \neq k$ and $j \neq n$, we get

$$\varphi_j(\tau(\tilde{x})) = \tau_j(\tilde{x}) + \sum_{s=1}^{\infty} \frac{(-1)^s (\tau_k(\tilde{x}))^s}{s!} L_{\sigma_k \nu}^{s-1}(\sigma_k \nu_j)(\tau(\tilde{x})) = \tilde{x}_j + \sum_{s=1}^{\infty} \frac{(-1)^s \tilde{x}_n^s}{s!} L_{\tilde{\sigma}_n \tilde{\nu}}^{s-1}(\tilde{\sigma}_n \tilde{\nu}_j)(\tilde{x}) = \tilde{\varphi}_j(\tilde{x})$$

using the fact that $L_{\tilde{\sigma}_n \tilde{\nu}}^{s-1}(\tilde{\sigma}_n \tilde{\nu}_j)(\tilde{x}) = L_{\sigma_k \nu}^{s-1}(\sigma_k \nu_j)(\tau(\tilde{x}))$ for any $s \geq 1$. On the other hand, for $j = n$, we get

$$\varphi_n(\tau(\tilde{x})) = \tau_n(\tilde{x}) + \sum_{s=1}^{\infty} \frac{(-1)^s (\tau_k(\tilde{x}))^s}{s!} L_{\sigma_k \nu}^{s-1}(\sigma_k \nu_j)(\tau(\tilde{x})) = \tilde{x}_k + \sum_{s=1}^{\infty} \frac{(-1)^s \tilde{x}_n^s}{s!} L_{\tilde{\sigma}_n \tilde{\nu}}^{s-1}(\tilde{\sigma}_n \tilde{\nu}_k)(\tilde{x}) = \tilde{\varphi}_k(\tilde{x})$$

and, for $j = k$, we have

$$\varphi_k(\tau(\tilde{x})) = \sum_{s=1}^{\infty} \frac{(-1)^s (\tau_k(\tilde{x}))^s}{s!} L_{\sigma_k \nu}^{s-1}(\sigma_k)(\tau(\tilde{x})) = \sum_{s=1}^{\infty} \frac{(-1)^s \tilde{x}_n^s}{s!} L_{\tilde{\sigma}_n \tilde{\nu}}^{s-1}(\tilde{\sigma}_n)(\tilde{x}) = \tilde{\varphi}_n(\tilde{x}).$$

This is the same as having $\tau \circ \tilde{\varphi}(\tilde{x}) = \varphi \circ \tau(\tilde{x})$. (ii) The proof of the inverse is constructive. Because of the linear transformation mentioned above, it is enough to show the proof in the case $k = n$, that is, we suppose $\nu(0) = \partial_{z_n}$. We look for a change of coordinates $x = \psi(z)$ that satisfies $\frac{\partial \psi(z)}{\partial z_n} = \nu(\psi(z))$. First, we extend ν in R^{n+1} as

$$\hat{\nu}(x,y) = \hat{\nu}_1(x,y)\partial_{x_1} + \cdots + \hat{\nu}_n(x,y)\partial_{x_n} + \hat{\nu}_{n+1}(x,y)\partial_y,$$

where $\hat{\nu}_j = \nu_j(x)$ for $1 \leq j \leq n$, and $\hat{\nu}_{n+1} = \nu_n(x)$. We want emphasize here the fact that the components $\hat{\nu}_n(x,y)$ and $\hat{\nu}_{n+1}(x,y)$ are both equal to $\nu_n(x)$. Because $\hat{\nu}(0) \neq 0$ there exist a change of coordinates $(z,w) = \hat{\varphi}(x,y)$ such that $\hat{\varphi}_* \hat{\nu} = \partial_{z_n} + \partial_w$. An inverse $(x,y) = \hat{\psi}(z,w)$ should thus satisfy

$$\frac{\partial \hat{\psi}}{\partial z_n} + \frac{\partial \hat{\psi}}{\partial w} = \hat{\nu}(\hat{\psi}(z,w)). \tag{4.2}$$

Define the operator $\nabla \triangleq \partial_{z_n} + \partial_w$ and rewrite (4.2) as $\nabla \cdot \hat{\psi} = \hat{\nu}(\hat{\psi}(z,w))$. Apply the operator ∇ again on both side and get (we put $\nabla^2 \triangleq \nabla \circ \nabla$)

$$\nabla^2 \cdot \hat{\psi}(z,w) = \nabla \cdot \hat{\nu}(\hat{\psi}(z,w)) = \frac{\partial \hat{\nu}}{\partial(x,y)}(\hat{\psi}(z,w)) \nabla \cdot \hat{\psi}(z,w) = \frac{\partial \hat{\nu}}{\partial(x,y)}(\hat{\psi}(z,w)) \hat{\nu}(\hat{\psi}(z,w)) = L_{\hat{\nu}}(\hat{\nu})(\hat{\psi}(z,w)).$$

A simple recurrence argument yields

$$\nabla^s \cdot \hat{\psi}(z,w) = L_{\hat{\nu}}^{s-1}(\hat{\nu})(\hat{\psi}(z,w)), \quad \text{for all } s \geq 1.$$

Define $\partial_w^s \triangleq \frac{\partial^s}{\partial w^s}$, and $\partial_{z_n}^s \triangleq \frac{\partial^s}{\partial z_n^s}$ for all $s \geq 1$. Since on the one hand side, $\partial_w = -\partial_{z_n} + \nabla$ and on the other hand side $\nabla \circ \partial_{z_n} = \partial_{z_n} \circ \nabla$, it follows that

$$\partial_w^s = \sum_{i=0}^{s} (-1)^i C_s^i \partial_{z_n}^i \circ \nabla^{s-i},$$

where $\partial_{z_n}^s \circ \nabla^0 = \partial_{z_n}^s$ and $\partial_{z_n}^0 \circ \nabla^s = \nabla^s$. We then deduce that

$$\partial_w^s \hat{\psi} \triangleq \frac{\partial^s \hat{\psi}}{\partial w^s} = \sum_{i=0}^{s} (-1)^i C_s^i \partial_{z_n}^i \cdot \left(\nabla^{s-i} \hat{\psi}(z,w)\right) = (-1)^s \partial_{z_n}^s \cdot \hat{\psi}(z,w) + \sum_{i=0}^{s-1} (-1)^i C_s^i \partial_{z_n}^i \cdot L_{\hat{\nu}}^{s-i-1}(\hat{\nu})(\hat{\psi}(z,w)).$$

Taking $\hat{\psi}(z,0) = (z,0)$, we get

$$\frac{\partial^s \hat{\psi}}{\partial w^s}\Big|_{w=0} = \sum_{i=0}^{s-1} (-1)^i C_s^i \partial_{z_n}^i \cdot L_{\hat{\nu}}^{s-i-1}(\hat{\nu})(z,0).$$

A Taylor series expansion of $\hat{\psi}(z,w)$ with respect to w at $w=0$ is

$$\hat{\psi}(z,w) = \begin{pmatrix} z \\ 0 \end{pmatrix} + \sum_{s=1}^{\infty} \frac{w^s}{s!} \left(\sum_{i=0}^{s-1} (-1)^i C_s^i \partial_{z_n}^i \cdot L_{\hat{\nu}}^{s-i-1}(\hat{\nu})(z,0) \right)$$

Let us define $\psi(z)$ by its components in the following way: for any $1 \leq j \leq n$ we set $\psi_j(z) = \hat{\psi}_j(z,w)|_{w=z_n}$. Since $\hat{\nu}_j(x,y) = \nu_j(x)$ is independent of the variable y, it follows that $L_{\hat{\nu}}^s \hat{\nu}_j = L_{\nu}^s \nu_j$ for all $s \geq 0$. We then deduce that

$$\psi_j(z) = z_j + \sum_{s=1}^{\infty} \frac{z_n^s}{s!} \left(\sum_{i=0}^{s-1} (-1)^i C_s^i \partial_{z_n}^i \cdot L_{\nu}^{s-i-1}(\nu_j)(z) \right).$$

To complete the proof we will show that $\dfrac{\partial \psi_j(z)}{\partial z_n} = \nu_j(\psi(z))$ for all $1 \leq j \leq n$; which indeed follows from the fact that

$$\frac{\partial \psi_j(z)}{\partial z_n} = \frac{\partial}{\partial z_n} \hat{\psi}_j(z,z_n) = \frac{\partial \hat{\psi}_j}{\partial z_n}(z,z_n) + \frac{\partial \hat{\psi}_j}{\partial w}(z,z_n) = \hat{\nu}_j(\hat{\psi}(z,z_n)) = \nu_j(\psi(z)).$$

This ends the proof-sketch of Theorem 3.1. □

Proof of Theorem 3.2. In the following we prove Theorem 3.2, that is, the convergence of the series (3.1)-(3.2). Before we proceed we need to introduce more notation. Recall the notation introduced previously in subsection 2.1 and denote by $\partial_i : \mathcal{C}^\omega(R^n) \longrightarrow \mathcal{C}^\omega(R^n)$ the derivation operator defined by $\partial_i(f) = \frac{\partial f}{\partial x_i}$. For a n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in N^n$ we get

$$\partial^\alpha(f) = \partial_1^{\alpha_1} \circ \cdots \circ \partial_n^{\alpha_n}(f) = \frac{\partial^\alpha f}{\partial x^\alpha} = \frac{\partial^{\alpha_1 + \cdots + \alpha_n} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

For the vector field ν we have $\partial^\alpha(\nu) = \partial^\alpha(\nu_1)\partial_{x_1} + \cdots + \partial^\alpha(\nu_n)\partial_{x_n}$. It is easy to see that

$$L_\nu(f) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\nu_j = \sum_{j_1=1}^{n} \partial^{\alpha_0}(f) \times \partial^{\alpha_1}(\nu_{j_1})$$

where $|\alpha_0| = 1$ and $|\alpha_1| = 0$ with α_0 an n-tuple whose components are zero except the $(j_1)^{th}$ component. By an inductive argument we check that for any $s \geq 1$ the successive Lie derivatives yield

$$L_\nu^s(f) = \sum_{j_1,\ldots,j_s=1}^{n} \sum \partial^{\alpha_0}(f) \times \partial^{\alpha_1}(\nu_{j_1}) \times \cdots \times \partial^{\alpha_{s-1}}(\nu_{j_{s-1}}) \times \partial^{\alpha_s}(\nu_{j_s}), \tag{4.3}$$

where the second summation is taken over some n-tuples $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{in})$, $i = 0, 1, \ldots, s$ with $\alpha_s = 0$, $|\alpha_0| \geq 1$ and $|\alpha_0| + |\alpha_1| + \cdots + |\alpha_s| = s$. The proof uses the following lemma.

Lemma 4.1. *Let f (resp. ν) be an analytic function (resp. vector field). Let $s \geq 1$ and $t \geq 0$ be given integers and $0 < \hat{\rho} < \rho$ two positive real numbers. Define*

$$M = \sup_{m_i \succeq \alpha_i} \left\{ \sum_{|\alpha|=s} (m_0!/\alpha_0!) \cdots (m_s!/\alpha_s!)(\hat{\rho}/\rho)^{|\mathbf{m}|} \right\}.$$

Then we have the following inequalities

(i) $M \leq s!\Big(\ln(\rho/\hat{\rho})\Big)^{-s}$

(ii) $||L_\nu^s(f)||_{\hat{\rho}} \leq s!\Big(\hat{\rho}\ln(\rho/\hat{\rho})\Big)^{-s}||f||_\rho||\nu||_\rho^s$

(iii) $||\partial_{z_n}^i L_\nu^{t-i}(f)||_{\hat{\rho}} \leq t!(\hat{\rho}\ln(\rho/\hat{\rho}))^{-t}||f||_\rho||\nu||_\rho^{t-i}$

Proof of Theorem 3.2 (i). Replacing s by $s-1$, the vector field ν by $\sigma_k\nu$, and the function f by $\sigma_k\nu_j$ in Lemma 4.1 (ii) and taking into account the fact that $||f||_\rho \leq \kappa(\rho)$ we obtain

$$||L_{\sigma_k\nu}^{s-1}(f)||_{\hat{\rho}} \leq (s-1)!\hat{\rho}^{-s+1} \times ||f||_\rho \times \left(\frac{\kappa(\rho)}{\ln(\rho/\hat{\rho})}\right)^{s-1} \leq (s-1)!(\hat{\rho}\ln(\rho/\hat{\rho}))^{-s+1} \times (\kappa(\rho))^s.$$

Thus an approximation of the series $\varphi_j(x) = x_j + \sum_{s=1}^{\infty} \frac{(-1)^s x_k^s}{s!} L_{\sigma_k\nu}^{s-1}(\sigma_k\nu_j)(x)$ is given by

$$\begin{aligned}
||\varphi_j(x)||_{\hat{\rho}} &\leq \hat{\rho} + \sum_{s=1}^{\infty} \tfrac{\hat{\rho}^s}{s!}\Big|\Big|L_{\sigma_k\nu}^{s-1}(\sigma_k\nu_j)(x)\Big|\Big|_{\hat{\rho}} \leq \hat{\rho} + \hat{\rho}\kappa(\rho) \times \sum_{s=1}^{\infty} \tfrac{1}{s}\Big(\kappa(\rho)/\ln(\rho/\hat{\rho})\Big)^{s-1} \\
&\leq \hat{\rho} + \hat{\rho}\kappa(\rho) \times \sum_{s=1}^{\infty} \Big(\kappa(\rho)/\ln(\rho/\hat{\rho})\Big)^{s-1}
\end{aligned}$$

The series converges and is bounded by $\hat{\rho} + \frac{\hat{\rho}\kappa(\rho)}{1-\kappa(\rho)/\ln(\rho/\hat{\rho})}$ if $\kappa(\rho)/\ln(\rho/\hat{\rho}) < 1$, i.e., if $\hat{\rho} < \rho e^{-\kappa(\rho)}$.

Proof of Theorem 3.2 (ii). To prove the convergence of the series

$$\psi_j(z) = z_j + \sum_{s=1}^{\infty} \frac{z_k^s}{s!}\left(\sum_{i=0}^{s-1}(-1)^i C_s^i \partial_{z_k}^i L_{\sigma_k\nu}^{s-i-1}(\sigma_k\nu_j)(z)\right)$$

we replace f by $\sigma_k\nu_j$ and ν by $\sigma_k\nu$ in Lemma 4.1 (iii). We can thus estimate the component ψ_j as

$$\begin{aligned}
||\psi_j(z)||_{\hat\rho} &\le \hat\rho + \sum_{s=1}^{\infty}\tfrac{\hat\rho^s}{s!}\left(\sum_{i=0}^{s-1} C_s^i ||\partial_{z_k}^i L_{\sigma_k\nu}^{s-i-1}(\sigma_k\nu_j)(z)||_{\hat\rho}\right)\\
&\le \hat\rho + \sum_{s=1}^{\infty}\tfrac{\hat\rho^s}{s!}\left(\sum_{i=0}^{s-1} C_s^i (s-1)!(\hat\rho\ln(\rho/\hat\rho))^{-s+1}||\sigma_k\nu_j||_\rho ||\sigma_k\nu||_\rho^{s-i-1}\right)\\
&\le \hat\rho + \hat\rho||\sigma_k\nu_j||_\rho \sum_{s=1}^{\infty}\frac{\ln(\rho/\hat\rho)^{-s+1}}{s}\left(\sum_{i=0}^{s-1} C_s^i \kappa(\rho)^{s-i-1}\right)\\
&\le \hat\rho + \hat\rho||\sigma_k\nu_j||_\rho \sum_{s=1}^{\infty}\frac{\ln(\rho/\hat\rho)^{-s+1}}{s}\frac{(1+\kappa(\rho))^s-1}{\kappa(\rho)}\\
&\le \hat\rho + \hat\rho\ln(\rho/\hat\rho)\sum_{s=1}^{\infty}\ln(\rho/\hat\rho)^{-s}\left[(1+\kappa(\rho))^s-1\right].
\end{aligned}$$

The series is convergent if $\frac{1+\kappa(\rho)}{\ln(\rho/\hat\rho)} < 1$, i.e., if $\hat\rho < \rho e^{-1-\kappa(\rho)}$ and is bounded by

$$\hat\rho\left[1 + \frac{\kappa(\rho)\,(\ln(\rho/\hat\rho))^2}{(\ln(\rho/\hat\rho)-1)\,(\ln(\rho/\hat\rho)-1-\kappa(\rho))}\right].$$

Proof of Lemma 4.1 (i) Because $m^i!/\alpha^i! \le (m^i)^{\alpha^i}$ for all $0 \le i \le s$ we deduce that

$$M \le \sup_{m_i \succeq \alpha_i}\left\{\sum_{|\alpha_0|+\cdots+|\alpha_s|=s}(m_0)^{\alpha_0}(m_1)^{\alpha_1}\cdots(m_s)^{\alpha_s}\times(\hat\rho/\rho)^{|m_0|+|m_1|+\cdots+|m_s|}\right\}.$$

On the other side, $\sum\limits_{|\alpha_0|+\cdots+|\alpha_s|=s}(m_0)^{\alpha_0}(m_1)^{\alpha_1}\cdots(m_s)^{\alpha_s} \le \Big(|m_0|+|m_1|+\cdots+|m_s|\Big)^s$, which implies that

$$\begin{aligned}
M &\le \sup_{m_i\succeq\alpha_i}\left\{\Big(|m_0|+|m_1|+\cdots+|m_s|\Big)^s\times(\hat\rho/\rho)^{|m_0|+|m_1|+\cdots+|m_s|}\right\}\\
&\le \sup_{m_i\succeq\mathbf{0}}\left\{\Big(|m_0|+|m_1|+\cdots+|m_s|\Big)^s\times(\hat\rho/\rho)^{|m_0|+|m_1|+\cdots+|m_s|}\right\}.
\end{aligned}$$

The inequality follows from Stirling formula $s! = \sqrt{2\pi s}(s/e)^s e^{\lambda_s}$ where $\lambda_s > 0$, and the fact that the maximum of $x^s(\hat\rho/\rho)^x$ is $\left(\frac{s}{\ln(\rho/\hat\rho)}\right)^s e^{-s}$.

(ii) Let the Taylor expansions of the analytic functions $f, \nu_{j_1}, \ldots, \nu_{j_s}$ be represented by

$$f(x) = \sum_{m_0\succeq 0} f_{m_0}\times x^{m_0} \quad \text{and} \quad \nu_{j_i}(x) = \sum_{m_i\succeq 0}(\nu_{j_i})_{m_i}\times x^{m_i}, \ \text{ for all } 1 \le i \le s.$$

It follows easily that

$$\partial^{\alpha_0}(f) = \sum_{m_0 \succeq \alpha_0} (m_0!/\alpha_0!) f_{m_0} \times x^{m_0-\alpha_0}$$

and for any $1 \leq i \leq s$

$$\partial^{\alpha_i}(\nu_{j_i}) = \sum_{m_i \succeq \alpha_i} (m_i!/\alpha_i!)(\nu_{j_i})_{m_i} \times x^{m_i-\alpha_i}.$$

Consequently

$$L_\nu^s(f) = \sum_{j_1,\ldots,j_s=1}^{n} \sum_{|\alpha|=s} \sum_{m_i \succeq \alpha_i} f_{m_0} \times (\nu_{j_1})_{m_1} \times \cdots \times (\nu_{j_s})_{m_s} \times (m_0!/\alpha_0!) \cdots (m_s!/\alpha_s!) \times x^{\mathbf{m}-\alpha}$$

where, for convenience, we put $\mathbf{m} = m_0 + m_1 + \cdots + m_s$ and $\alpha = \alpha_0 + \alpha_1 + \cdots + \alpha_s$. For any $0 < \hat{\rho} < \rho$ we have the following estimates (set $\mathcal{J} = (j_1, \ldots, j_s)$)

$$||L_\nu^s(f)||_{\hat{\rho}} = \sum_{\mathcal{J}=1}^{n} \sum_{|\alpha|=s} \sum_{m_i \geq \alpha_i} |f_{m_0}| \times |(\nu_{j_1})_{m_1}| \times \cdots \times |(\nu_{j_s})_{m_s}| (m_0!/\alpha_0!) \cdots (m_s!/\alpha_s!) \times \hat{\rho}^{|\mathbf{m}-\alpha|}$$

$$= \hat{\rho}^{-|\alpha|} \sum_{\mathcal{J}=1}^{n} \sum_{|\alpha|=s} \sum_{m_i \geq \alpha_i} |f_{m_0}| \times |(\nu_{j_1})_{m_1}| \times \cdots \times |(\nu_{j_s})_{m_s}| (m_0!/\alpha_0!) \cdots (m_s!/\alpha_s!) \times \rho^{|\mathbf{m}|} (\hat{\rho}/\rho)^{|\mathbf{m}|}$$

$$= \hat{\rho}^{-s} \sum_{\mathcal{J}=1}^{n} \sum_{|\alpha|=s} \sum_{m_i \geq \alpha_i} |f_{m_0}| \rho^{|m_0|} \cdot |(\nu_{j_1})_{m_1}| \rho^{|m_1|} \cdots |(\nu_{j_s})_{m_s}| \rho^{|m_s|} (m_0!/\alpha_0!) \cdots (m_s!/\alpha_s!) \cdot (\hat{\rho}/\rho)^{|\mathbf{m}|}$$

$$\leq \hat{\rho}^{-s} \sum_{\mathcal{J}=1}^{n} \sum_{|\alpha|=s} \sum_{m_i \geq \alpha_i} |f_{m_0}| \rho^{|m_0|} \cdot |(\nu_{j_1})_{m_1}| \rho^{|m_1|} \cdots |(\nu_{j_s})_{m_s}| \rho^{|m_s|} (m_0!/\alpha_0!)| \cdots (m_s!/\alpha_s!) \cdot (\hat{\rho}/\rho)^{|\mathbf{m}|}$$

$$\leq \hat{\rho}^{-s} ||f||_\rho \times ||\nu||_\rho^s \times \sup_{m_i \succeq \alpha_i} \left\{ \sum_{|\alpha_0|+\cdots+|\alpha_s|=s} (m_0!/\alpha_0!)(m_1!/\alpha_1!) \cdots (m_s!/\alpha_s!) \times (\hat{\rho}/\rho)^{|\mathbf{m}|} \right\}.$$

Using Lemma 4.1 (i), the item (ii) follows directly.

(iii) Consider (4.3) where s is replaced by $t-i$, that is,

$$L_\nu^{t-i}(f) = \sum_{\mathcal{J}} \sum \partial^{\alpha_0}(f) \times \partial^{\alpha_1}(\nu_{j_1}) \times \cdots \times \partial^{\alpha_{t-i}}(\nu_{j_{t-i}})$$

with $\alpha_{t-i} = 0$, $|\alpha_0| \geq 1$ and $|\alpha_0| + |\alpha_1| + \cdots + |\alpha_{t-i}| = t - i$. Differentiating i times with respect to x_n we get

$$\partial_{x_n}^i L_\nu^{t-i}(f) = \sum_{\mathcal{J}} \sum \partial^{\hat{\alpha}_0}(f) \times \partial^{\hat{\alpha}_1}(\nu_{j_1}) \times \cdots \times \partial^{\hat{\alpha}_{t-i}}(\nu_{j_{t-i}}) \tag{4.4}$$

with $|\hat{\alpha}_0| \geq 1$ and $|\hat{\alpha}_0| + |\hat{\alpha}_1| + \cdots + |\hat{\alpha}_{t-i}| = t$. Following the same steps in Lemma 4.1 (ii) we get $||\partial_{x_n}^i L_\nu^{t-i}(f)||_{\hat{\rho}} \leq t!(\hat{\rho}\ln(\rho/\hat{\rho}))^{-t} ||f||_\rho ||\nu||_\rho^{t-i}$. Notice that the power $t-i$ on the last term is due to the fact there are $t-i$ factors only that involve the components of the vector field ν. □

Proof of Theorem 3.3. A formal proof of this theorem can be found in the literature. We are here interested on a constructive proof, that is, item (ii). We start with the simplest case where the vector fields commute, that is, their pairwise Lie-brackets are zero.

(a) *Commutative Case*. Take $\nu^1(x),\ldots,\nu^m(x)$ such that

$$\begin{cases} \dim\operatorname{span}\left\{\nu^1(0),\ldots,\nu^m(0)\right\} = m, \\ \left[\nu^k,\nu^l\right] = 0, \qquad 1 \le k,l \le m. \end{cases}$$

Without loss of generality we can assume that $\nu^k(0) = \partial_{x_k}$ for $1 \le k \le m$. We apply Theorem 3.1 to the vector field ν^1 and define a change of coordinates φ^1 such that $\varphi^1_*\nu^1 = \partial_{z_1}$. Under this change of coordinates the distribution $\mathcal{D}(x) = \operatorname{span}\left\{\nu^1(x),\ldots,\nu^m(x)\right\}$ is transformed as $\mathcal{D}^1 = \varphi^1_*\mathcal{D} = \operatorname{span}\left\{\partial_{z_1},\varphi^1_*\nu^2,\ldots,\varphi^1_*\nu^m\right\}$. Since $\varphi^1_*[\nu^1,\nu^l] = [\varphi^1_*\nu^1,\varphi^1_*\nu^l] = 0$, then the vector fields $\varphi^1_*\nu^2,\ldots,\varphi^1_*\nu^m$ are independent of the variable z_1. Thus we can apply Theorem 3.1 again to $\varphi^1_*\nu^2$ to define φ^2 such that $\varphi^2_*(\varphi^1_*\nu^2) = \partial_{z_2}$. Moreover, $\varphi^2_*(\varphi^1_*\nu^1) = \partial_{z_1}$ because $\varphi^2(z) = z + \phi^2(z_2,\ldots,z_n)$. We denote the new distribution as

$$\mathcal{D}^2 = \varphi^2_*\mathcal{D}^1 = \operatorname{span}\left\{\partial_{z_1},\partial_{z_2},\varphi^2_*(\varphi^1_*\nu^3),\ldots,\varphi^2_*(\varphi^1_*\nu^m)\right\}.$$

Let us assume that the original distribution has been transformed, via changes of coordinates $\varphi^1,\ldots,\varphi^{k-1}$, into

$$\mathcal{D}^{k-1} = (\varphi^{k-1}\circ\cdots\circ\varphi^1)_*\mathcal{D} = \operatorname{span}\left\{\partial_{z_1},\ldots,\partial_{z_{k-1}},\tilde{\nu}^k,\ldots,\tilde{\nu}^m\right\},$$

where $\tilde{\nu}^l = (\varphi^{k-1}\circ\cdots\circ\varphi^1)_*\nu^l$ for $k \le l \le m$. Because of the commutativity it follows that the vector fields $\tilde{\nu}^k,\ldots,\tilde{\nu}^m$ are independent of the variables $z_1,\ldots,z_{k-1}$. Thus we can apply Theorem 3.1 to the vector field $\tilde{\nu}^k$ and find a change of coordinates φ^k such that $\varphi^k_*\tilde{\nu}^k = \partial_{z_k}$. Moreover, such change of coordinates can be chosen in such a way that $\varphi^k_*\partial_{z_i} = \partial_{z_i}$ for $1 \le i \le k-1$. It thus takes the distribution $\mathcal{D}^{k-1}$ into

$$\mathcal{D}^k = (\varphi^k\circ\cdots\circ\varphi^1)_*\mathcal{D} = \operatorname{span}\left\{\partial_{z_1},\ldots,\partial_{z_k},\tilde{\nu}^{k+1},\ldots,\tilde{\nu}^m\right\},$$

where $\tilde{\nu}^l = (\varphi^k\circ\cdots\circ\varphi^1)_*\nu^l$ for $k+1 \le l \le m$.

(b) *Involutive Case.* In this case we consider, for simplicity, two vector fields ν^1 and ν^2 and we suppose that

$$\begin{cases} \dim\operatorname{span}\left\{\nu^1(0),\nu^2(0)\right\} = 2, \\ \left[\nu^1,\nu^2\right](x) = \gamma_1(x)\nu^1(x) + \gamma_2(x)\nu^2(x). \end{cases}$$

Without loss of generality we assume that $\nu^1(0) = \partial_{x_1}$ and $\nu^2(0) = \partial_{x_2}$. We apply Theorem 3.1 to the vector field ν^1 and define a change of coordinates φ^1 such that $\varphi^1_*\nu^1 = \partial_{z_1}$. Because the involutitivity is invariant by change of coordinates, then the transformed vector fields $\tilde{\nu}^1 = \varphi^1_*\nu^1 = \partial_{z_1}$ and $\tilde{\nu}^2 = \varphi^1_*\nu^2$ satisfy

$$\left[\partial_{z_1},\tilde{\nu}^2\right](z) = \tilde{\gamma}_1(z)\partial_{z_1} + \tilde{\gamma}_2(z)\tilde{\nu}^2(z), \tag{4.5}$$

where $\tilde{\gamma}_1(z) = \varphi^1_*\gamma_1(z)$ and $\tilde{\gamma}_2(z) = \varphi^1_*\gamma_2(z)$. The condition (4.5) implies, in particular, that

$$\frac{\partial\tilde{\nu}^2_j}{\partial z_1} = \tilde{\gamma}_2(z)\tilde{\nu}^2_j(z), \;\; 2 \le j \le n.$$

The value of $\tilde{\gamma}_2$ is thus uniquely determined as $\frac{\partial \tilde{v}_2^2}{\partial z_1} \Big/ \tilde{v}_2^2(z)$, and the condition above can be integrated to yield

$$\tilde{v}_j^2(z) = \mu_j(z) \exp\left(\int_0^{z_1} \tilde{\gamma}_2(\varepsilon, z_2, \ldots, z_n)\, d\varepsilon \right), \quad 2 \le j \le n$$

for some appropriate functions $\mu_j(z) = \mu_j(z_2, \ldots, z_n)$ that are independent of the variable z_1 with $\mu_2(0) \neq 0$. The vector field $\mu(z) = 0 \cdot \partial_{z_1} + \mu_2(z)\partial_{z_2} + \cdots + \mu_n \partial_{z_n}$ of R^n is independent of the variable z_1. Thus by Theorem 3.1 we can find a change of coordinates $\varphi^2(z) = z + \varphi(z_2, \ldots, z_n)$ such that $\varphi^2_* \mu = \partial_{\tilde{z}_2}$. The same change of coordinates brings $\tilde{v}^2 = \tilde{v}_1^2 \partial_{z_1} + b(z)\mu(z)$, where $b(z) = \exp\left(\int_0^{z_1} \tilde{\gamma}_2(\varepsilon, z_2, \ldots, z_n)\, d\varepsilon\right)$, into $\varphi^2_* \tilde{v}_1^2 \partial_{\tilde{z}_1} + \partial_{\tilde{z}_2}$. Thus the change of coordinates $\tilde{z} = \varphi^2 \circ \varphi^1(x)$ transforms the distribution $\mathcal{D}(x) = \text{span}\left\{ v^1(x), v^2(x) \right\}$ into the straightened distribution $(\varphi^2 \circ \varphi^1)_* \mathcal{D}(\tilde{z}) = \text{span}\left\{ \partial_{\tilde{z}_1}, \partial_{\tilde{z}_2} \right\}$. □

5 Examples

We will illustrate the results in this section by considering several examples.

Example 5.1. (i) Consider the nonsingular vector field $v(x) = x_1 \partial_{x_1} + x_2 \partial_{x_2} + \partial_{x_3}$ on R^3. The change of coordinates $z = \varphi(x)$ should satisfy the system of partial differential equations

$$\begin{cases} x_1 \dfrac{\partial \varphi_1}{\partial x_1} + x_2 \dfrac{\partial \varphi_1}{\partial x_2} + \dfrac{\partial \varphi_1}{\partial x_3} & = & 0 \\ x_1 \dfrac{\partial \varphi_2}{\partial x_1} + x_2 \dfrac{\partial \varphi_2}{\partial x_2} + \dfrac{\partial \varphi_2}{\partial x_3} & = & 0 \\ x_1 \dfrac{\partial \varphi_3}{\partial x_1} + x_2 \dfrac{\partial \varphi_3}{\partial x_2} + \dfrac{\partial \varphi_3}{\partial x_3} & = & 1. \end{cases}$$

It is not difficult to guess or find a solution as given by $\varphi_1(x) = x_1 e^{-x_3}, \varphi_2(x) = x_2 e^{-x_3}, \varphi_3(x) = x_3$. Indeed, by the method of characteristics we would rewrite the system of partial differential equation as

$$\frac{dx_1}{x_1} = \frac{dx_2}{x_2} = \frac{dx_3}{1}$$

or equivalently $dx_1 - x_1 dt = 0, dx_2 - x_2 dt = 0, dx_3 - dt = 0$; which integrates easily as $x_3 = t = z_3, x_2 = z_2 e^{z_3}, x_1 = z_1 e^{z_3}$. This provides the inverse $x = \psi(z)$ from which the coordinates $z = \varphi(x)$ can be easily found. Since $v_3(x) = 1$, we have $\sigma_3(x) = 1$, and hence Theorem 2.1 gives

$$z = \varphi(x) \triangleq \begin{cases} z_1 = \varphi_1(x) & = & x_1 + \displaystyle\sum_{s=1}^{\infty} \frac{(-1)^s x_3^s}{s!} L_v^{s-1}(v_1)(x) \\ z_2 = \varphi_2(x) & = & x_2 + \displaystyle\sum_{s=1}^{\infty} \frac{(-1)^s x_3^s}{s!} L_v^{s-1}(v_2)(x) \\ z_3 = \varphi_3(x) & = & \displaystyle\sum_{s=1}^{\infty} \frac{(-1)^{s-1} x_3^s}{s!} L_v^{s-1}(1)(x). \end{cases}$$

A simple calculation shows that $L_\nu^{s-1}(\nu_1) = x_1$ and $L_\nu^{s-1}(\nu_2) = x_2$ for all s; which yields

$$z = \varphi(x) \triangleq \begin{cases} z_1 = \varphi_1(x) & = & x_1 + \displaystyle\sum_{s=1}^{\infty} \frac{(-1)^s x_3^s}{s!} x_1 = x_1 e^{-x_3} \\ z_2 = \varphi_2(x) & = & x_2 + \displaystyle\sum_{s=1}^{\infty} \frac{(-1)^s x_3^s}{s!} x_2 = x_2 e^{-x_3} \\ z_3 = \varphi_3(x) & = & \displaystyle\sum_{s=1}^{\infty} \frac{(-1)^{s-1} x_3^s}{s!} L_\nu^{s-1}(1)(x) = x_3. \end{cases}$$

Because $\partial_{z_3}^i \cdot L_\nu^{s-i-1}(\nu_j)(z) = 0$ for all $i \geq 1$, we have

$$\sum_{i=0}^{s-1} (-1)^i C_s^i \partial_{z_3}^i \cdot L_\nu^{s-i-1}(\nu_j)(z) = L_\nu^{s-1}(\nu_j)(z) = z_j.$$

Thus we can immediately verify that the inverse $x = \psi(z) = \mathbf{col}\,(\psi_1(z), \psi_2(z), \psi_3(z))$ is given by its components

$$x = \psi(z) \triangleq \begin{cases} x_1 = \psi_1(z) & = & z_1 + \displaystyle\sum_{s=1}^{\infty} \frac{z_3^s}{s!} z_1 = z_1 e^{z_3} \\ x_2 = \psi_2(z) & = & z_2 + \displaystyle\sum_{s=1}^{\infty} \frac{z_3^s}{s!} z_2 = z_2 e^{z_3} \\ x_3 = \psi_3(z) & = & \displaystyle\sum_{s=1}^{\infty} \frac{z_3^s}{s!} L_\nu^{s-1}(1)(z) = z_3. \end{cases}$$

(ii) Next, we consider the vector field $\nu(x) = x_1 x_2 \partial_{x_1} + x_2^2 \partial_{x_2} + \partial_{x_3}$ on R^3. The corresponding system of PDEs satisfied by φ is

$$\begin{cases} x_1 x_2 \dfrac{\partial \varphi_1}{\partial x_1} + x_2^2 \dfrac{\partial \varphi_1}{\partial x_2} + \dfrac{\partial \varphi_1}{\partial x_3} & = & 0 \\ x_1 x_2 \dfrac{\partial \varphi_2}{\partial x_1} + x_2^2 \dfrac{\partial \varphi_2}{\partial x_2} + \dfrac{\partial \varphi_2}{\partial x_3} & = & 0 \\ x_1 x_2 \dfrac{\partial \varphi_3}{\partial x_1} + x_2^2 \dfrac{\partial \varphi_3}{\partial x_2} + \dfrac{\partial \varphi_3}{\partial x_3} & = & 1. \end{cases}$$

It is clear that $\varphi_3(x) = x_3$ solves the last partial differential equation which is also given by the formula (4.1) with $n = 3$ and $\sigma_3 = 1$. Now, it is less easier to guess for a solution. However, we can apply (4.1) with $\nu_1(x) = x_1 x_2$, $\nu_2(x) = x_2^2$. It can easily be seen that $L_\nu^{s-1}(\nu_1) = s! x_1 x_2^s$ and also $L_\nu^{s-1}(\nu_2) = s! x_2^{s+1}$. Thus

$$z = \varphi(x) \triangleq \begin{cases} \varphi_1(x) & = & x_1 + \displaystyle\sum_{s=1}^{\infty} \frac{(-1)^s x_3^s}{s!} L_\nu^{s-1}(\nu_1)(x) \\ & = & x_1 + \displaystyle\sum_{s=1}^{\infty} (-1)^s x_3^s x_1 x_2^s = \frac{x_1}{1 + x_2 x_3} \\ \varphi_2(x) & = & x_2 + \displaystyle\sum_{s=1}^{\infty} \frac{(-1)^s x_3^s}{s!} L_\nu^{s-1}(\nu_2)(x) \\ & = & x_2 + \displaystyle\sum_{s=1}^{\infty} (-1)^s x_3^s x_2^{s+1} = \frac{x_2}{1 + x_2 x_3} \\ \varphi_3(x) & = & x_3. \end{cases}$$

Though an inverse can be found directly from the above change of coordinates, we want verify the formulas given by Theorem 3.1 (ii). Using the previous argument, i.e., $\partial^i_{z_3} \cdot L_\nu^{s-i-1}(\nu_j)(z) = 0$ for all $i \geq 1$, we get

$$\sum_{i=0}^{s-1} (-1)^i C_s^i \partial^i_{z_3} \cdot L_\nu^{s-i-1}(\nu_j)(z) = L_\nu^{s-1}(\nu_j)(z) = s! z_j z_2^s.$$

We then compute the inverse via Theorem 3.1 (ii) as

$$x = \psi(z) = \begin{cases} x_1 = \psi_1(z) & = & z_1 + \sum_{s=1}^{\infty} \frac{z_3^s}{s!} s! z_1 z_2^s = \dfrac{z_1}{1 - z_2 z_3} \\ x_2 = \psi_2(z) & = & z_2 + \sum_{s=1}^{\infty} \frac{z_3^s}{s!} s! z_2^{s+1} = \dfrac{z_2}{1 - z_2 z_3} \\ x_3 = \psi_3(z) & = & \sum_{s=1}^{\infty} \frac{z_3^s}{s!} L_\nu^{s-1}(1)(z) = z_3. \end{cases}$$

We can notice that although the vector field is defined globally, both the diffeomorphism and its inverse are only obtained locally. They are here defined inside the cylinder

$$\mathcal{C} = \left\{ (x_1, x_2, x_3) \in R^3 : |x_2 x_3| < 1 \right\}.$$

In the previous two examples, the vector field $\nu(x) \in R^3$ does not depend on the variable x_3. In the next example we will consider a case where it does.

(iii) Consider the vector field $\nu(x) = x_3 \partial_{x_1} + (x_2 + x_3)\partial_{x_2} + \partial_{x_3}$ on R^3. In this example $L_\nu(\nu_1) = 1$ and $L_\nu^{s-1}(\nu_1) = 0$ for $s \geq 3$. In the other hand we can see that $L_\nu^{s-1}(\nu_2) = x_2 + x_3 + 1$ for all $s \geq 2$. It thus follows that

$$\varphi_1(x) = x_1 + \sum_{s=1}^{\infty} \frac{(-1)^s x_3^s}{s!} L_\nu^{s-1}(\nu_1)(x) = x_1 - x_3 \nu_1(x) + \frac{x_3^2}{2!} L_\nu(\nu_1)(x) = x_1 - \frac{1}{2} x_3^2$$

and

$$\begin{aligned} \varphi_2(x) &= x_2 + \sum_{s=1}^{\infty} \frac{(-1)^s x_3^s}{s!} L_\nu^{s-1}(\nu_2)(x) = x_2 - x_3 \nu_2(x) + \sum_{s=2}^{\infty} \frac{(-1)^s x_3^s}{s!} (x_2 + x_3 + 1) \\ &= x_2 - x_3(x_2 + x_3) + (e^{-x_3} + x_3 - 1)(x_2 + x_3 + 1) = (x_2 + x_3 + 1)e^{-x_3} - 1. \end{aligned}$$

Once again we can compute the inverse directly or via Theorem 3.1 (ii). To find the inverse, first notice that $\partial^i_{z_3} \cdot L_\nu^{s-i-1}(\nu_1)(z) = 0$ if $(i, s) \neq (0, 1)$; which yields

$$\psi_1(z) = z_1 + \sum_{s=1}^{\infty} \frac{z_3^s}{s!} \left(\sum_{i=0}^{s-1} (-1)^i C_s^i \partial^i_{z_n} \cdot L_\nu^{s-i-1}(\nu_1)(z) \right) = z_1 + \frac{z_3^2}{2!} \nu_1(z) = z_1 + \frac{1}{2} z_3^2.$$

We also have $\partial^i_{z_3} \cdot L_\nu^{s-i-1}(\nu_2)(z) = 0$ for all $i \geq 2$, from which we deduce

$$\sum_{i=0}^{s-1} (-1)^i C_s^i \partial^i_{z_3} \cdot L_\nu^{s-i-1}(\nu_2)(z) = L_\nu^{s-1}(\nu_2)(z) - s\partial_{z_3} \cdot L_\nu^{s-2}(\nu_2)(z) = z_2 + z_3 + 1 - s.$$

By Theorem 3.1 (ii), we get the 2nd component of $\psi(z)$ as

$$\begin{aligned}
\psi_2(z) &= z_2+\sum_{s=1}^{\infty}\tfrac{z_3^s}{s!}\left(\sum_{i=0}^{s-1}(-1)^i C_s^i \partial_{z_3}^i \cdot L_\nu^{s-i-1}(\nu_2)(z)\right) = z_2+\sum_{s=1}^{\infty}\tfrac{z_3^s}{s!}(z_2+z_3+1-s)\\
&= z_2+\sum_{s=1}^{\infty}\tfrac{z_3^s}{s!}(z_2+z_3+1)-\sum_{s=1}^{\infty}\tfrac{z_3^s}{s!}s = z_2+(e^{z_3}-1)(z_2+z_3+1)-z_3e^{z_3}\\
&= (z_2+1)e^{z_3}-z_3-1.
\end{aligned}$$

It is straightforward to verify that

$$x=\psi(z)\triangleq\begin{cases} x_1=\psi_1(z) &= z_1+\dfrac{1}{2}z_3^2\\ x_2=\psi_2(z) &= (z_2+1)e^{z_3}-z_3-1\\ x_3=\psi_3(z) &= z_3.\end{cases}$$

is an inverse of $z=\varphi(x)$. ▷

The next example illustrates the fact that the series given by Theorem 3.1 are not Taylor series at the origin or in the variable x_k.

Example 5.2. Consider the nonsingular vector field $\nu(x)=\lambda(x_3)\partial_{x_1}+\partial_{x_3}$ in R^3, where λ is a flat function, that is, λ and all its derivatives are zero at $x_3=0$. An example is the well-known function given by

$$\lambda(x_3)=\begin{cases}\exp(-1/x_3^2) & \text{if } x_3\neq 0\\ 0 & \text{if } x_3=0.\end{cases}$$

It is straightforward to check that $L_\nu^{s-1}(\nu_1)(x)=\lambda^{(s-1)}(x_3)$ for all $s\geq 1$, where $\lambda^{(k)}(x_3)$ is the kth derivative of λ. Should the formula (3.1) have been a series expansion around 0 or at $x_k=0$, the straightening diffeomorphism would have been given by

$$z=\varphi(x)\triangleq\begin{cases}\varphi_1(x) &= x_1+\displaystyle\sum_{s=1}^{\infty}\frac{(-1)^s x_3^s}{s!}L_\nu^{s-1}(\nu_1)(0)=x_1\\ \varphi_2(x) &= x_2+\displaystyle\sum_{s=1}^{\infty}\frac{(-1)^s x_3^s}{s!}L_\nu^{s-1}(\nu_2)(0)=x_2\\ \varphi_3(x) &= \displaystyle\sum_{s=1}^{\infty}\frac{(-1)^{s-1} x_3^s}{s!}L_\nu^{s-1}(1)(0)=x_3;\end{cases}$$

which is impossible. However we can verify easily that $\varphi_1(x)=x_1-\int_0^{x_3}\lambda(\varepsilon)\,\mathrm{d}\varepsilon$; which coincides with

$$\varphi_1(x)=x_1+\sum_{s=1}^{\infty}\frac{(-1)^s x_3^s}{s!}\lambda^{(s-1)}(x_3).$$

Indeed we can show that

$$\int_0^{x_3}\lambda(\varepsilon)\,\mathrm{d}\varepsilon=-\sum_{s=1}^{\infty}\frac{(-1)^s x_3^s}{s!}\lambda^{(s-1)}(x_3).$$

The two functions coincide when $x_3 = 0$, and it is enough to verify that their derivatives are also equal. The derivative of the right hand side is

$$-\sum_{s=1}^{\infty} \frac{(-1)^s x_3^{s-1}}{(s-1)!} \lambda^{(s-1)}(x_3) - \sum_{s=1}^{\infty} \frac{(-1)^s x_3^s}{s!} \lambda^{(s)}(x_3) = \lambda(x_3)$$

after simplification (which equals the derivative of the left).

Now, to find the inverse of the normalizing coordinates, let us apply Theorem 3.1 (ii) with $n = 3$ and $k = 3$. First, we have $L_\nu^s \nu = \lambda^{(s)}(x_3)\partial_{x_1}$ for all $s \geq 1$. We thus obtain

$$\begin{aligned}
\psi(z) &= z + \sum_{s=1}^{\infty} \tfrac{z_3^s}{s!} \left(\sum_{i=0}^{s-1} (-1)^i C_s^i \partial_{z_3}^i \cdot L_\nu^{s-i-1}(\nu)(z) \right) \\
&= z + \sum_{s=1}^{\infty} \tfrac{z_3^s}{s!} \left(\sum_{i=0}^{s-1} (-1)^i C_s^i \partial_{z_3}^i \cdot \lambda^{(s-i-1)}(z_3)\partial_{z_1} \right) \\
&= z + \sum_{s=1}^{\infty} \tfrac{z_3^s}{s!} \left(\sum_{i=0}^{s-1} (-1)^i C_s^i \lambda^{(s-1)}(z_3)\partial_{z_1} \right) \\
&= z + \sum_{s=1}^{\infty} \tfrac{z_3^s}{s!} \left(\sum_{i=0}^{s-1} (-1)^i C_s^i \right) \lambda^{(s-1)}(z_3)\partial_{z_1} \\
&= z - \sum_{s=1}^{\infty} \frac{(-1)^s z_3^s}{s!} \lambda^{(s-1)}(z_3)\partial_{z_1} \\
&= \mathbf{col}\left(z_1 - \sum_{s=1}^{\infty} \frac{(-1)^s z_3^s}{s!} \lambda^{(s-1)}(z_3), z_2, z_3 \right)
\end{aligned}$$

It thus clearly follows that

$$\psi(z) = \mathbf{col}\left(z_1 + \int_0^{z_3} \lambda(\varepsilon)\, d\varepsilon, z_2, z_3 \right)$$

which was predictable directly from the change of coordinates $z = \varphi(x)$. ▷

6 Applications.

This section deals with few applications of the flow-box theorem among them the linearization of control systems [22], [23] (see also [24]), and integration of ordinary differential equations.

We use the rectifying theorem for vector fields to construct linearizing coordinates for control systems for which this is possible. For further details we refer to our papers [22], [23], [24]. The method can also be applied to nonlinearizable control systems to construct normal forms or simplify their dynamics as well as for an analysis of symmetries of vector fields. We will only illustrate here with few examples.

Example 6.1. Consider a single-input control system

$$\Sigma : \dot{x} = f(x) + g(x)u \triangleq \begin{cases} \dot{x}_1 &= x_2 - 2x_2x_3 + x_3^2 + 4x_2x_3u \\ \dot{x}_2 &= x_3 - 2x_3u \\ \dot{x}_3 &= u \end{cases}$$

with $f(x) = \mathbf{col}\,(x_2 - 2x_2x_3 + x_3^2, x_3, 0)$ and $g(x) = \mathbf{col}\,(4x_2x_3, -2x_3, 1)$. We start by rectifying the vector field $g(x)$. Denote $\nu(x) = g(x)$ and apply Theorem 2.1 with $n = 3$, and $\sigma_3(x) = 1$. Since

$$L_\nu(\nu_1) = -8x_3^2 + 4x_2, L_\nu^2(\nu_1) = -16x_3 - 8x_3 = -24x_3, L_\nu^3(\nu_1) = -24, L_\nu^{s-1}(\nu_1) = 0,\ s \geq 5,$$

we have

$$\begin{aligned} y_1 = \varphi_1(x) &= x_1 + \sum_{s=1}^{\infty}(-1)^s \tfrac{x_3^s}{s!} L_\nu^{s-1}(\nu_1)(x) \\ &= x_1 - x_3(4x_2x_3) + \tfrac{x_3^2}{2}(-8x_3^2 + 4x_2) - \tfrac{x_3^3}{6}(-24x_3) + \tfrac{x_3^4}{24}(-24) \\ &= x_1 - 4x_2x_3^2 - 4x_3^4 + 2x_2x_3^2 + 4x_3^4 - x_3^4 = x_1 - 2x_2x_3^2 - x_3^4. \end{aligned}$$

Similarly, we have $L_\nu(\nu_2) = -2$, and $L_\nu^{s-1}(\nu_2) = 0$ for $s \geq 3$; which yields

$$\begin{aligned} y_2 = \varphi_2(x) &= x_2 + \sum_{s=1}^{\infty}(-1)^s \tfrac{x_3^s}{s!} L_\nu^{s-1}(\nu_2)(x) \\ &= x_2 - x_3(-2x_3) + \tfrac{x_3^2}{2}(-2) = x_2 + 2x_3^2 - x_3^2 = x_2 + x_3^2. \end{aligned}$$

We apply the change of coordinates $y = \mathbf{col}\,(x_1 - 2x_2x_3^2 - x_3^4, x_2 + x_3^2, x_3)$ to transform the original system into

$$\tilde{\Sigma} : \dot{y} = \tilde{f}(y) + \tilde{g}(y)u \triangleq \begin{cases} \dot{y}_1 &= y_2 - 2y_2y_3 \\ \dot{y}_2 &= y_3 \\ \dot{y}_3 &= u \end{cases}$$

where $\tilde{g}(y) = \mathbf{col}\,(0, 0, 1)$ and $\tilde{f}(y) = \mathbf{col}\,(y_2 - 2y_2y_3, y_3, 0)$. The vector field $\tilde{f}(y)$ decomposes uniquely as $\tilde{f}(y) = \mathbf{col}\,(y_2, 0, 0) + y_3\mathbf{col}\,(-2y_2, 1, 0)$. The next step is to rectify the vector field $\nu(y) = \mathbf{col}\,(-2y_2, 1, 0)$. Theorem 3.1 with $k = 2$, and $\sigma_2(y) = 1$, yields

$$z = \tilde{\varphi}(y) \triangleq \begin{cases} z_1 &= y_1 + \sum_{s=1}^{\infty}(-1)^s \tfrac{y_2^s}{s!} L_\nu^{s-1}(\nu_1)(y) = y_1 - y_2(-2y_2) + \tfrac{y_2^2}{2}(-2) = y_1 + y_2^2 \\ z_2 &= y_2 \\ z_3 &= y_3. \end{cases}$$

The system is then transformed, via $z = \tilde{\varphi}(y)$, to the linear Brunovský form

$$\Lambda_{Br} : \dot{z} = Az + bu \triangleq \begin{cases} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= u. \end{cases}$$

The linearizing coordinates for the original system are thus obtained as a composition of the two-step changes of coordinates

$$z = \tilde{\varphi} \circ \varphi(x) \triangleq \begin{cases} z_1 &= x_1 - 2x_2x_3^2 - x_3^4 + (x_2 + x_3^2)^2 = x_1 + x_2^2 \\ z_2 &= x_2 + x_3^2 \\ z_3 &= x_3. \end{cases}$$

Of course, this linearizing coordinates could have been obtained directly or by other methods. What we want emphasize here is that the method works and is applicable for any linearizable system.

Example 6.2. We consider the following example

$$\begin{cases} \dot{x}_1 = x_2 + \left(\frac{1}{2}x_2 - \frac{1}{12}x_3x_4\right)u & \dot{x}_3 = x_4 + x_4u \\ \dot{x}_2 = x_3 + \frac{1}{2}x_3u & \dot{x}_4 = u. \end{cases}$$

Because of the strict feedforward structure, we showed in [18] (using a 4-step algorithm) that the change of coordinates

$$z = \varphi(z) \triangleq \begin{cases} z_1 = x_1 - \frac{1}{24}\left(12x_2x_4 - 4x_3x_4^2 + x_4^4\right) \\ z_2 = x_2 - \frac{1}{2}\left(x_3x_4 - \frac{1}{3}x_4^3\right) \\ z_3 = x_3 - \frac{1}{2}x_4^2 \\ z_4 = x_4 \end{cases} \tag{6.1}$$

linearizes the system. We can recover such coordinates directly by applying the algorithm given in the proof. Denote by $f(x) = \mathbf{col}\,(x_2, x_3, x_4, 0)$ and

$$\nu(x) \triangleq g(x) = \mathbf{col}\left(\frac{1}{2}x_2 - \frac{1}{12}x_3x_4,\, \frac{1}{2}x_3,\, x_4,\, 1\right).$$

The first step consists of rectifying the control vector field via Theorem 3.1. Since $\nu_4 = 1$, hence $\sigma_4 = 1$, and we have

$$L_\nu(\nu_1) = \frac{1}{2}\left(\frac{1}{2}x_3\right) - \frac{1}{12}\left(x_4^2 + x_3\right) = \frac{1}{6}x_3 - \frac{1}{12}x_4^2,$$

and $L_\nu^2(\nu_1) = \frac{1}{6}x_4 - \frac{1}{6}x_4 = 0$ (thus $L_\nu^{s-1}(\nu_1) = 0, \forall s \geq 3$). It follows that

$$\varphi_1(x) = x_1 - x_4\nu_1(x) + \frac{1}{2}x_4^2 L_\nu(\nu_1) = x_1 - \frac{1}{2}x_2x_4 + \frac{1}{6}x_3x_4^2 - \frac{1}{24}x_4^3.$$

We can also verify easily that $L_\nu(\nu_2) = \frac{1}{2}x_4$, $L_\nu^2(\nu_2) = \frac{1}{2}$, and $L_\nu^{s-1}(\nu_2) = 0$ for all $s \geq 3$. Thus we get

$$\begin{aligned} \varphi_2(x) &= x_2 - x_4\nu_2(x) + \frac{1}{2}x_4^2 L_\nu(\nu_2) - \frac{1}{6}x_4^3 L_\nu^2(\nu_2) \\ &= x_2 - \frac{1}{2}x_3x_4 + \frac{1}{4}x_4^3 - \frac{1}{12}x_4^3 = x_2 - \frac{1}{2}x_3x_4 + \frac{1}{6}x_4^3. \end{aligned}$$

Similarly, $L_\nu(\nu_3) = 1$, and $L_\nu^{s-1}(\nu_3) = 0$ for all $s \geq 3$; which implies

$$\varphi_3(x) = x_3 - x_4\nu_3(x) + \frac{1}{2}x_4^2 L_\nu(\nu_2) = x_3 - x_4^2 + \frac{1}{2}x_4^2 = x_3 - \frac{1}{2}x_4^2.$$

Because $\nu_4(x) = 1$, we get $\varphi_4(x) = x_4$, and the change of coordinates (6.1) rectifies the control vector field g and linearizes the system at the same time.

Notice that the algorithm described in [18] allowed only to find such linearizing coordinates by computing a component at a time (holding other components identity) starting from φ_3 then φ_2 and finally φ_1 and updating the system at each step. A composition of different coordinates changes gave (6.1). However, Theorem 3.1 allows to compute those components independently to each other. ▷

Example 6.3. Consider the following example [14] motivated by a mixed-culture bioreactor

$$\left\{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ (1-\ln x_3)x_2 \\ -px_1x_3 \end{pmatrix} - \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2 \right. \tag{6.2}$$

with $x_1 > 0, x_2 > 0, x_3 > 0$. An equilibrium point for this system is $x_3 = 1, u_1 = 1, u_2 = 1$ with x_1 and x_2 arbitrary. Define the new coordinates $\mathbf{x}_i = x_i - 1$, $\mathbf{u}_j = u_j - 1$ so the system above can be written in the form

$$\Sigma : \dot{\mathbf{x}} = f(\mathbf{x}) + g_1(\mathbf{x})\mathbf{u}_1 + g_2(\mathbf{x})\mathbf{u}_2$$

where

$$f(\mathbf{x}) = \begin{pmatrix} 0 \\ -(1+\mathbf{x}_2)\ln(1+\mathbf{x}_3) \\ -p(1+\mathbf{x}_1)(1+\mathbf{x}_3) - \mathbf{x}_3 \end{pmatrix}, \quad g_1(\mathbf{x}) = -\begin{pmatrix} 1+\mathbf{x}_1 \\ 1+\mathbf{x}_2 \\ 1+\mathbf{x}_3 \end{pmatrix} \text{ and } g_2(\mathbf{x}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We look for a change of coordinates $y = \varphi(\mathbf{x})$ that rectifies the distribution $\mathcal{D} = \operatorname{span}\{g_1, g_2\}$, i.e., such that $(\varphi)_*\mathcal{D} = \beta(y)\{\partial_{y_2}, \partial_{y_3}\}$ or equivalently $(\varphi)_* g_2 = \partial_{y_3}$ and $(\varphi)_* g_1 = \partial_{y_2} + \beta_{12}(y)\partial_{y_3}$. Apply Theorem 3.1 with $\nu = g_1$ and $\sigma_2 = -(1+\mathbf{x}_2)^{-1}$. Because

$$\begin{aligned} L_{\sigma_2\nu}(\sigma_2\nu_1) &= \frac{1}{1+\mathbf{x}_2} \times \frac{1+\mathbf{x}_1}{1+\mathbf{x}_2} - \frac{1+\mathbf{x}_1}{(1+\mathbf{x}_2)^2} \times 1 = 0 \\ L_{\sigma_2\nu}(\sigma_2) &= -\frac{1}{(1+\mathbf{x}_2)^2} \times 1 = -\frac{1}{(1+\mathbf{x}_2)^2} \end{aligned}$$

it follows that $L^{s-1}_{\sigma_2\nu}(\sigma_2\nu_1) = 0$ and $L^{s-1}_{\sigma_2\nu}(\sigma_2) = \frac{(-1)^{s-1}(s-1)!}{(1+\mathbf{x}_2)^s}$, $s \geq 1$. Thus the change of coordinates

$$y = \varphi(\mathbf{x}) \triangleq \left\{ \begin{aligned} y_1 &= \mathbf{x}_1 + \sum_{s=1}^{\infty} \frac{(-1)^s \mathbf{x}_2^s}{s!} L^{s-1}_{\sigma_2\nu}(\sigma_2\nu_1)(\mathbf{x}) \\ &= \mathbf{x}_1 - \mathbf{x}_2(\sigma_2\nu_1)(\mathbf{x}) = \frac{\mathbf{x}_1 - \mathbf{x}_2}{1+\mathbf{x}_2} \\ y_2 &= \sum_{s=1}^{\infty} \frac{(-1)^s \mathbf{x}_2^s}{s!} L^{s-1}_{\sigma_2\nu}(\sigma_2)(\mathbf{x}) \\ &= \sum_{s=1}^{\infty} \frac{(-1)^s \mathbf{x}_2^s}{s!} \frac{(-1)^{s-1}(s-1)!}{(1+\mathbf{x}_2)^s} \\ &= -\sum_{s=1}^{\infty} \frac{\mathbf{x}_2^s (1+\mathbf{x}_2)^{-s}}{s} = -\ln(1+\mathbf{x}_2) \\ y_3 &= \mathbf{x}_3 \end{aligned} \right.$$

transforms Σ into

$$\breve{\Sigma} : \dot{y} = \breve{f}(y) + \breve{g}_1(y)\breve{u}_1 + \breve{g}_2(y)\breve{u}_2$$

where

$$\breve{f}(y) = \begin{pmatrix} (1+y_1)\ln(1+y_3) \\ \ln(1+y_3) \\ \lambda(y_1,y_2,y_3) \end{pmatrix}, \quad \breve{g}_1(y) = \begin{pmatrix} 0 \\ 1 \\ -1-y_3 \end{pmatrix} \text{ and } \breve{g}_2(y) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

where

$$\lambda = -pe^{-y_2}(1+y_1)(1+y_3) - y_3$$

This system is in feedback form and the following change of coordinates

$$z = \breve{\varphi}_1(y) \triangleq \begin{cases} z_1 &= \breve{\phi}_1(y) = y_1 \\ z_2 &= \breve{\phi}_2(y) = (1+y_1)\ln(1+y_3) \\ z_3 &= \breve{\phi}_3(y) = y_3 \end{cases}$$

with feedback $\mathbf{v}_1 = L_{\breve{f}}\breve{\phi}_2(y), \mathbf{v}_2 = \breve{u}_2 - (1+y_3)\breve{u}_1 + \lambda(y_1,y_2,y_3)$ brings the latter system into the linear form

$$\Sigma : \dot{z} = f(z) + g_1(z)\mathbf{v}_1 + g_2(z)\mathbf{v}_2 \triangleq \begin{cases} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \mathbf{v}_1 \\ \dot{z}_3 &= \mathbf{v}_2. \end{cases}$$

We deduce that (6.2) can be linearized by the composition of coordinates changes $z_1 = \frac{x_1-x_2}{x_2}, z_2 = \frac{x_1}{x_2}\ln x_3, z_3 = x_3 - 1$ with the relations between old and new control inputs given by $x_1x_3(\ln x_3)^2 - px_1^2x_3 + x_1x_3u_1 + x_1u_2 = x_2x_3\mathbf{v}_1$ and $-px_1x_3 - x_2u_1 + u_2 = \mathbf{v}_2$. Notice that the coordinates transformations are not unique and the following change of coordinates $z_1 = \ln(x_1/x_2), z_2 = \ln x_3, z_3 = \ln x_1$ with appropriate feedback has been proposed in [14] to linearize the system. We recovered this latter change of coordinates and feedback in [25] using the linearizing approach for strict feedforward systems.

Example 6.4. Consider the system $\Sigma : \dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2$ described in the coordinates $x = (x_1,\ldots,x_5)^{\mathrm{T}} \in R^5$ by

$$f(x) = \begin{pmatrix} x_2(1+x_3) \\ x_3(1+x_1) \\ x_1 + x_5 + x_1^2 \\ x_5 + x_1^2 \\ 0 \end{pmatrix}, \quad g_1(x) = \begin{pmatrix} 0 \\ -x_2 \\ 1+x_3 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and } g_2(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

We rectify the vector field $g_1(x)$. Put $\nu = g_1(x)$ and apply Theorem II.2 with $n = 5$ and $\sigma_3 = (1+x_3)^{-1}$, thus $\sigma_3\nu = -x_2(1+x_3)^{-1}\partial_{x_2} + \partial_{x_3}$. Since $\nu_1 = \nu_4 = \nu_5 = 0$ we have $\varphi_1(x) = x_1, \varphi_4(x) = x_4$ and $\varphi_5(x) = x_5$. On the other side $\nu_2(x) = -x_2$ implies

$$L_{\sigma_3\nu}(\sigma_3\nu_2) = 2x_2(1+x_3)^{-2}, L^2_{\sigma_3\nu}(\sigma_3\nu_2) = -6x_2(1+x_3)^{-3}$$

which recurrently gives

$$L^{s-1}_{\sigma_3\nu}(\sigma_3\nu_2) = (-1)^s s! x_2(1+x_3)^{-s}.$$

It follows that

$$\varphi_2(x)=x_2+\sum_{s=1}^{\infty}\frac{(-1)^s x_3^s}{s!}L_{\sigma_3\nu}^{s-1}(\sigma_3\nu_2)(x)=x_2(1+x_3).$$

To calculate $\varphi_3(x)$, notice that $L_{\sigma_3\nu}(\sigma_3)=-(1+x_3)^{-2}$ and $L_{\sigma_3\nu}^2(\sigma_3)=2(1+x_3)^{-3}$. Thus a simple recurrence shows that $L_{\sigma_3\nu}^{s-1}\sigma_3=(-1)^{s-1}(s-1)!(1+x_3)^{-s}$, $s\geq 1$, which implies

$$\begin{aligned}
y_3=\varphi_3(x) &= \sum_{s=1}^{\infty}\frac{(-1)^{s+1}x_3^s}{s!}L_{\sigma_3\nu}^{s-1}(\sigma_3)(x)\\
&= \sum_{s=1}^{\infty}\frac{1}{s}\left(\frac{x_3}{1+x_3}\right)^s\\
&= \sum_{s=1}^{\infty}\int\left(\frac{x_3}{1+x_3}\right)^{s-1}\left(\frac{x_3}{1+x_3}\right)'\mathrm{d}x_3\\
&= \int\frac{1}{1+x_3}\mathrm{d}x_3=\ln(1+x_3).
\end{aligned}$$

We apply the change of coordinates

$$y=\varphi(x)\triangleq\begin{cases} y_1=x_1\\ y_2=x_2(1+x_3)\\ y_3=\ln(1+x_3)\\ y_4=x_4\\ y_5=x_5\end{cases}$$

whose inverse is given by (Theorem 3.1 (ii) can be used)

$$x=\varphi^{-1}(y)\triangleq\begin{cases} x_1=y_1\\ x_2=y_2e^{-y_3}\\ x_3=e^{y_3}-1\\ x_4=y_4\\ x_5=y_5\end{cases}$$

to transform the original system into

$$\breve{\Sigma}\triangleq\begin{cases} \dot{y}_1 &= y_2\\ \dot{y}_2 &= (1+y_1)e^{y_3}(e^{y_3}-1)+y_2e^{-y_3}(y_1+y_5+y_1^2)\\ \dot{y}_3 &= e^{-y_3}(y_1+y_5+y_1^2)+u_1\\ \dot{y}_4 &= y_5+y_1^2\\ \dot{y}_5 &= u_2.\end{cases}$$

The system is in feedback form and can be put into the linear Brunovský form via

$$z=\breve{\varphi}(y)\triangleq\begin{cases} z_1 &= \breve{\phi}_1(y)=y_1\\ z_2 &= \breve{\phi}_2(y)=y_2\\ z_3 &= \breve{\phi}_3(y)=(1+y_1)e^{y_3}(e^{y_3}-1)+y_2e^{-y_3}(y_1+y_5+y_1^2)\\ x_4 &= \breve{\phi}_4(y)=y_4\\ x_5 &= \breve{\phi}_5(y)=y_5+y_1^2\end{cases}$$

with $v_1 = \frac{\partial\varphi_3}{\partial y_1}\dot{y}_1 + \cdots + \frac{\partial\varphi_3}{\partial y_5}\dot{y}_5$ and $v_2 = \frac{\partial\varphi_5}{\partial y_1}\dot{y}_1 + \cdots + \frac{\partial\varphi_5}{\partial y_5}\dot{y}_5$. The composition $z = \breve{\varphi} \circ \varphi(x)$ gives the linearizing coordinates

$$\begin{cases} x_1 &= x_1 \\ z_2 &= x_2(1+x_3) \\ z_3 &= (1+x_{\mathbf{3}1})x_3(1+x_3) + x_2(x_1+x_5+x_1^2) \\ z_4 &= x_4 \\ z_5 &= x_5 + x_1^2 \end{cases}$$

with appropriate feedback.

Example 6.5. Consider a simplified model of a VTOL with dynamics [17] (see Fig. 1.)

$$\begin{cases} \ddot{x} = -\sin(\theta)\frac{T}{M} + \cos(\theta)\frac{2\sin\alpha}{M}F \\ \ddot{y} = -\cos(\theta)\frac{T}{M} + \sin(\theta)\frac{2\sin\alpha}{M}F - \mathfrak{g} \\ \ddot{\theta} = \frac{2l}{J}\cos\alpha F \end{cases} \tag{6.3}$$

where M, J, l and $\mathfrak{g}$ denote the mass, moment of inertia, distance between wingtips and gravitational acceleration. The control inputs are the thrust T, and the rolling moment due to the torque F, whose direction forms a fixed angle α with the horizontal body axis. The position of center mass and the roll angle with respect to the horizon are (x,y), and θ while $(\dot{x},\dot{y})$ and $\dot{\theta}$ stand for their respective velocities. Let $\mathbf{x}_1 = x, \mathbf{x}_2 = \dot{x}, \mathbf{x}_3 = \theta, \mathbf{x}_4 = \dot{\theta}, \mathbf{x}_5 = y, \mathbf{x}_6 = \dot{y}$ with control inputs $u_1 = \frac{2l}{J}\cos\alpha F$ and $u_2 = -\cos(\theta)\frac{T}{M} + \sin(\theta)\frac{2\sin\alpha}{M}F - \mathfrak{g}$. The system rewrites in the form

$$\Sigma : \dot{\mathbf{x}} = f(\mathbf{x}) + g_1(\mathbf{x})u_1 + g_2(\mathbf{x})u_2, \mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_6) \in R^6$$

with

$$f(\mathbf{x}) = \begin{pmatrix} \mathbf{x}_2 \\ \mathfrak{g}\tan\mathbf{x}_3 \\ \mathbf{x}_4 \\ 0 \\ \mathbf{x}_6 \\ 0 \end{pmatrix}, \quad g_1(\mathbf{x}) = \begin{pmatrix} 0 \\ \eta(\mathbf{x}_3) \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } g_2(\mathbf{x}) = \begin{pmatrix} 0 \\ \tan\mathbf{x}_3 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

where $\eta(\mathbf{x}_3) = \frac{J\tan\alpha}{Ml}\left(\frac{\cos^2\mathbf{x}_3 - \sin^2\mathbf{x}_3}{\cos\mathbf{x}_3}\right)$. Put $\nu^1 = g_1$ and $\nu^2 = g_2$. We look for $z = \varphi(\mathbf{x})$ that rectifies the distribution $\mathcal{D} = \operatorname{span}\{\nu^1, \nu^2\}$, i.e., such that $(\varphi)_*\mathcal{D} = \beta(z)\{\partial_{z_4}, \partial_{z_6}\}$. Apply Theorem 3.1 first to $\nu^1 = g_1(\mathbf{x})$ with $n = 6$ and $\sigma_4 = 1$. Since $L_{\nu^1}^{s-1}(\nu_2^1) = 0$ for all $s \geq 2$ it follows that

$$\mathbf{y} = \varphi(\mathbf{x}) \triangleq \begin{cases} \mathbf{y}_1 = \mathbf{x}_1 \\ \mathbf{y}_2 = \mathbf{x}_2 + \displaystyle\sum_{s=1}^{\infty} \frac{(-1)^s \mathbf{x}_4^s}{s!} L_{\sigma_4\nu^1}^{s-1}(\sigma_4\nu_2^1)(\mathbf{x}) \\ \quad = \mathbf{x}_2 - \mathbf{x}_4\nu_1^1(\mathbf{x}) = \mathbf{x}_2 - \mathbf{x}_4\eta(\mathbf{x}_3) \\ \mathbf{y}_3 = \mathbf{x}_3 \\ \mathbf{y}_4 = \mathbf{x}_4 \\ \mathbf{y}_5 = \mathbf{x}_5 \\ \mathbf{y}_6 = \mathbf{x}_6. \end{cases}$$

The distribution $\mathcal{D}$ is transformed into

$$(\varphi)_*\mathcal{D} = \text{span}\left\{\check{v}^1, \check{v}^2\right\} = \text{span}\left\{\begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\ \tan\mathbf{y}_3\\0\\0\\0\\1\end{pmatrix}\right\}.$$

Now we look for $z = \check{\phi}(\mathbf{y})$ such that $(\check{\phi})_*\check{v}^2 = \partial_{z_6}$. Similar to the steps above we obtain

$$z = \check{\phi}(\mathbf{y}) \triangleq \begin{cases} z_1 = \mathbf{y}_1 \\ z_2 = \mathbf{y}_2 + \displaystyle\sum_{s=1}^{\infty} \frac{(-1)^s \mathbf{y}_6^s}{s!} L^{s-1}_{\sigma_6 \tilde{v}^2}(\sigma_6 \tilde{v}_2^2)(\mathbf{y}) \\ \quad = \mathbf{y}_2 - \mathbf{y}_6 \tilde{v}_2^2(\mathbf{y}) = \mathbf{y}_2 - \mathbf{y}_6 \tan(\mathbf{y}_3) \\ z_3 = \mathbf{y}_3 \\ z_4 = \mathbf{y}_4 \\ z_5 = \mathbf{y}_5 \\ z_6 = \mathbf{y}_6. \end{cases}$$

Hence the previous distribution is straightened by latest change of coordinates. Thus, the change of coordinates $z = \check{\phi} \circ \varphi(\mathbf{x})$

$$z = \check{\phi} \circ \varphi(\mathbf{x}) \triangleq \begin{cases} z_1 = \mathbf{x}_1 \\ z_2 = \mathbf{x}_2 - \mathbf{x}_4 \eta(\mathbf{x}_3) - \mathbf{x}_6 \tan(\mathbf{x}_3) \\ z_3 = \mathbf{x}_3 \\ z_4 = \mathbf{x}_4 \\ z_5 = \mathbf{x}_5 \\ z_6 = \mathbf{x}_6 \end{cases}$$

takes the original system Σ into $\Sigma : \dot{z} = f(z) + b_1 u_1 + b_2 u_2, z \in R^6$ with

$$f(z) = \begin{pmatrix} z_2 + z_4 \eta(z_3) + z_6 \tan(z_3) \\ \mathrm{g} \tan z_3 - \eta'(z_3) z_4^2 - z_6 z_4 \sec^2(z_3) \\ z_4 \\ 0 \\ z_6 \\ 0 \end{pmatrix}$$

Since $\frac{\partial^2 f}{\partial z_4^2} \neq 0$, the integrability condition fails to be satisfied and the system is not feedback linearizable. The algorithm stops. What is worth noticing here is, though the system is not feedback linearizable, this method provides an easy way of verifying that fact. Indeed, the classical method would require, in general, to verify that the distribution

$$\mathcal{D}^{n-2} = \text{span}\left\{g_i, ad_f g_i, \ldots, ad_f^{n-2} g_i, i = 1, \ldots, m\right\}$$

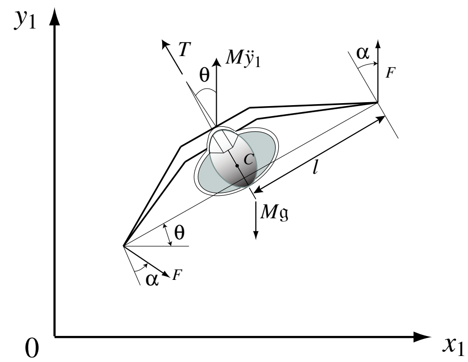

Figure 1. Forces acting on the aircraft.

is involutive by calculating all Lie brackets. Even once we check that all distributions are involutive, finding linearizing change of coordinates and feedback is another challenge.

Other applications of the flow box theorem include solving explicitly systems of ordinary differential equations and finding symmetries of vector fields. We refer to our work in progress for further details on those issues. We want just illustrate here with one example.

Example 6.6. Consider the simplest system of ordinary differential equations

$$\left\{\begin{array}{rcl}\dot{x}_1 & = & x_2\\ \dot{x}_2 & = & -x_1\end{array}\right.$$

with initial condition $x_1(0) = a_1, x_2(0) = a_2$. Obviously, a solution is obtained directly by integrating the second order system associated $\ddot{x}_1 + x_1 = 0$ and $\ddot{x}_2 + x_2 = 0$. We get $x_1(t) = a_1 \cos t + a_2 \sin t$ and $x_2(t) = -a_1 \sin t + a_2 \cos t$. We will recover this solution by extended the system as follows

$$\left\{\begin{array}{rcl}\dot{x}_1 & = & x_2\\ \dot{x}_2 & = & -x_1\\ \dot{x}_3 & = & 1\end{array}\right.$$

with initial condition $x_1(0) = a_1, x_2(0) = a_2, x_3(0) = 0$. Let $\nu(x) = \mathbf{col}(x_2, -x_1, 1)$ be the vector field associated with the extended system. It is straightforward to verify that $L_\nu^{2s}(\nu_1) = (-1)^s x_2$ and $L_\nu^{2s-1}(\nu_1) = (-1)^s x_1$ on one side and $L_\nu^{2s}(\nu_2) = (-1)^{s-1} x_1$ and $L_\nu^{2s-1}(\nu_1) = (-1)^s x_2$ on the other side. Since $\partial_{x_3}^i \cdot L_\nu^{s-i-1}(\nu_1) = 0$ and $\partial_{x_3}^i \cdot L_\nu^{s-i-1}(\nu_2) = 0$ for all $i \geq 1$, we deduce from Theorem 3.1 (ii) that the change of coordinates $x=\psi(z)$ is given by

$$x=\psi(z)\triangleq\left\{\begin{array}{rcl} x_1 &=& z_1+\sum\limits_{s=1}^{\infty}\frac{z_3^s}{s!}\left(\sum\limits_{i=0}^{s-1}(-1)^iC_s^i\partial_{z_3}^i\cdot L_\nu^{s-i-1}(\nu_1)(z)\right)=z_1+\sum\limits_{s=1}^{\infty}\frac{z_3^s}{s!}L_\nu^{s-1}(\nu_1)(z)\\ &=& z_1+\sum\limits_{s=1}^{\infty}\frac{z_3^{2s}}{(2s)!}(-1)^sz_1+\sum\limits_{s=0}^{\infty}\frac{z_3^{2s+1}}{(2s+1)!}(-1)^sz_2=z_1\cos z_3+z_2\sin z_3\\ x_2 &=& z_2+\sum\limits_{s=1}^{\infty}\frac{z_3^s}{s!}\left(\sum\limits_{i=0}^{s-1}(-1)^iC_s^i\partial_{z_3}^i\cdot L_\nu^{s-i-1}(\nu_2)(z)\right)=z_2+\sum\limits_{s=1}^{\infty}\frac{z_3^s}{s!}L_\nu^{s-1}(\nu_2)(z)\\ &=& z_2+\sum\limits_{s=1}^{\infty}\frac{z_3^{2s}}{(2s)!}(-1)^sz_2-\sum\limits_{s=0}^{\infty}\frac{z_3^{2s+1}}{(2s+1)!}(-1)^sz_1=-z_1\sin z_3+z_2\cos z_3\\ x_3 &=& z_3\end{array}\right.$$

Since $x_3=t$ ($\dot{x}_3=1$ and $x_3(0)=0$) we then have $x_1(t)=z_1\cos t+z_2\sin t, x_2(t)=-z_1\sin t+z_2\cos t$ and using the initial conditions we arrive to the solution above.

Conclusion

In this paper we have provided explicit formulas for finding a diffeomorphism rectifying a non singular vector field as well as its inverse in terms of power series of functions that are Lie derivatives of the components of the vector field along itself. We have established the convergence of those series and extended the results to the Frobënius case, and we have also provided several examples as well as an application to the linearization of control systems.

References

[1] V. I. Arnold (1988), *Geometrical Methods in the Theory of Ordinary Differential Equations*, 2nd Edition, Springer-Verlag.

[2] R. W. Brockett, *Feedback invariants for nonlinear systems*, in Proc. IFAC Congress, Helsinski, 1978.

[3] E. Cartan, *Leçons sur les invariants intégraux*, Hermann (1922).

[4] C. Chevalley, *Theory of Lie Groups*, Princeton University Press, 1946

[5] A. R. Forsyth, *Theory of Differential Equations*, Dover Publications, Inc.

[6] L. R. Hunt and R. Su, Linear equivalents of nonlinear time varying systems, in *Proceedings of Mathematical Theory of Networks* & *Systems*, Santa Monica, CA, USA, (1981), pp. 119-123.

[7] A. Isidori (1995), *Nonlinear Control Systems*, 3rd ed., Springer, London.

[8] B. Jakubczyk and W. Respondek, On linearization of control systems, *Bull. Acad. Polon. Sci. Ser. Math.*, 28, (1980), pp. 517-522.

[9] T. Kailath (1980), *Linear Systems*, Prentice Hall Information and System Sciences Series, USA.

[10] M. Krstic, Feedback linearizability and explicit integrator forwarding controllers for classes of feedforward systems, in *IEEE Trans. Automat. Control*, 49, (2004), pp. 1668-1682.

[11] M. Krstic, Explicit Forwarding Controllers-Beyond Linearizable Class, in *Proceedings of the 2005 American Control Conference*, Portland, Oregon, USA, (2005), pp. 3556-3561.

[12] M. Krstic, Feedforward systems linearizable by coordinate change, in *Proceedings of the 2004 American Control Conference*, Boston, Massachussetts, USA, (2004), pp. 4348-4353.

[13] Ph. Mullhaupt, Quotient submanifolds for static feedback linearization in *Systems & Control Letters*, 55(2006), pp. 549-557.

[14] H. Nijmeijer and A. J. van der Schaft (1990), *Nonlinear Dynamical Control Systems* Springer-Verlag, New York.

[15] P. J. Olver (1986), *Applications of Lie Groups to Differential Equations*, Graduate Texts in Mathematics 107, Springer-Verlag.

[16] P. J. Olver (1995), *Equivalence, Invariants, and Symmetry*, Cambridge Press University, ISBN 0-521-47811-1.

[17] A. Serrani, A. Isidori, C. I. Byrnes, and L. Marconi (2000), Recent advances in output regulation of nonlinear systems, *Nonlinear Control in the Year 2000*, A. Isidori, F. Lamnabhi-Lagarrigue and W. Respondek (eds), LNCIS vol. 259, 2, pp. 409-419.

[18] I.A. Tall and W. Respondek, On Linearizability of Strict Feedforward Systems, in *Proceedings of the 2008 American Control Conference*, Seattle, Washington, USA, pp. 1929-1934.

[19] I.A. Tall and W. Respondek, Feedback Linearizable Strict Feedforward Systems, in *Proceedings of the 47*th *IEEE Conference on Decision and Control*, (2008), Cancún, Mexico, pp. 2499-2504.

[20] I.A. Tall, Linearizable Feedforward Systems: A Special Class, in *Proceedings of the 17*th *IEEE Conference on Control and Applications Multi-conference on Systems and Control*, San Antonio, TX (2008), pp. 1201-1206.

[21] I.A. Tall, (Feedback) Linearizable Feedforward Systems: A Special Class, *IEEE Transactions on Automatic Control*, 55:7, (2010), pp 1736-1742.

[22] I.A. Tall, State Linearization of Nonlinear Control Systems: An Explicit Algorithm *48th IEEE Conference on Decison and Control*, China, (2009), pp. 7448-7453.

[23] I.A. Tall, Explicit Feedback Linearization of Nonlinear Control Systems, in *48th IEEE Conference on Decison and Control*, China, (2009), pp. 7454-7459.

[24] I. A. Tall, State and Feedback Linearizations of Single-Input Control Systems, *Systems & Control Letters*, 59(2010), pp.429-441.

[25] I.A. Tall and W. Respondek, Analytic Normal Forms and Symmetries of Strict Feedforward Control Systems, in *International Journal of Robust and Nonlinear Control*, 20(2010) pp. 1431-1454.