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EXPONENTIAL STABILITY OF LINEAR NONAUTONOMOUS SYSTEMS

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Abstract

In this paper, we study the exponential stability of linear nonautonomous systems with multiple delays. Using Lyapunov-like function, we find sufficient conditions for the exponential stability in terms of the solution of a Riccati differential equation. Our results are illustrated with numerical examples.

AMS Subject Classification: 34K20.

Keywords: Exponential stability, time-delay, Lyapunov function, Riccati equation.

1 Introduction

The topic of Lyapunov stability of linear systems has been an interesting research area in the past decades. An integral part of the stability analysis of differential equations is the existence of inherent time delays. Time delays are frequently encountered in many physical and chemical processes as well as in the models of hereditary systems, Lotka-Volterra systems, control of the growth of global economy, control of epidemics, etc. Therefore, the stability problem of time-delay systems has been received considerable attention from many researchers (see; e.g. [5, 6, 10, 12, 14] and references therein). One of the extended stability properties is the concept of the α-stability, which relates to the exponential stability with a convergent rate $\alpha > 0$. Namely, a retarded system

$$\dot{x} = f(t, x(t), x(t-h)), \quad t \geq 0,$$
$$x(t) = \phi(t), \quad t \in [-h, 0],$$

is α-stable, with $\alpha > 0$, if there is a function $\xi(.)$ such that for each $\phi(.)$, the solution $x(t,\phi)$ of the system satisfies

$$\|x(t,\phi)\| \leq \xi(\|\phi\|)e^{-\alpha t}, \quad \forall t \geq 0,$$

where $\|\phi\| = \max\{\|\phi(t)\| : t \in [-h, 0]\}$. This implies that for $\alpha > 0$, the system can be made exponentially stable with the convergent rate α. It is well known that there are many different methods to study the stability problem of time-delay linear autonomous systems. The

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widely used method is the approach of Lyapunov functions with Razumikhin techniques and the asymptotic stability conditions are presented in terms of the solution of either linear matrix inequalities or Riccati equations [2, 7, 8]. By using both the time-domain and the frequency-domain techniques, the paper [15] derived sufficient conditions for the asymptotic stability of a linear autonomous system with multiple time delays of the form

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{m} A_i x(t-h_i), \quad t \geq 0, \\ x(t) &= \phi(t), \quad t \in [-h, 0], \end{aligned} \tag{1.1}$$

where A_i are given constant matrices, $h = \max\{h_i : i = 1, 2, \ldots, m\}$. These conditions depend only on the eigenvalues of A_0 and the norm values of A_i of the system. For studying the α-stability problem, based on the asymptotic stability of the linear undelayed part, i.e. A_0 is a Hurwitz matrix, the papers [13, 14] proposed sufficient conditions for the α-stability of system (1.1) in terms of the solution of a scalar inequality involving the eigenvalues, the matrix measures and the spectral radius of the system matrices. It is worth noticing that although the approach used in these papers allows us to derive the less conservative stability conditions, but it can not be applied to non-autonomous delay systems. The reason is that, the assumption $A_0(t)$ to be a Hurwitz matrix for each $t \geq 0$, i.e. $\mathrm{Re}\,\lambda(A(t)) < 0$, for each t, does not implies the exponential stability of the linear non-autonomous system $\dot{x} = A_0(t)x$. It is the purpose of this paper to search sufficient conditions for the α-stability of non-autonomous delay systems. Using the Lyapunov-like function method, we develop the results obtained in [3, 14] to the non-autonomous systems with multiple delays. Do not using any Lyapunov stability theorem, we establish sufficient conditions for the α-stability of system (2.1), which are given in terms of the solution of a Riccati differential equation (RDE). These conditions do not involve any stability property of the system matrix $A_0(t)$. Although the problem of solving of RDEs is in general still not easy, various effective approaches for finding the solutions of RDEs can be found in [1, 4, 9, 16].

The paper is organized as follows. Section 2 presents notations, mathematical definitions and an auxiliary lemma used in the next section. The sufficient conditions for the α-stability are presented in Section 3. Numerical examples illustrated the obtained result are also given in Section 3. The paper ends with cited references.

2 Preliminaries

The following notations will be used for the remaining this paper.
$\mathbb{R}^+$ denotes the set of all real non-negative numbers; $\mathbb{R}^n$ denotes the n-dimensional space with the scalar product $\langle .,. \rangle$ and the vector norm $\|.\|$;
$\mathbb{R}^{n \times r}$ denotes the space of all matrices of dimension $(n \times r)$. A^T denotes the transpose of the vector/matrix A; a matrix A is symmetric if $A = A^T$; I denotes the identity matrix;
$\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\max}(A) = \max\{\mathrm{Re}\,\lambda : \lambda \in \lambda(A)\}$;
$\|A\|$ denotes the spectral norm of the matrix defined by

$$\|A\| = \sqrt{\lambda_{\max}(A^T A)};$$

$\eta(A)$ denotes the matrix measure of the matrix A given by

$$\eta(A) = \frac{1}{2}\lambda_{\max}(A + A^T).$$

$C([a,b],\mathbb{R}^n)$ denotes the set of all $\mathbb{R}^n$-valued continuous functions on $[a,b]$;
Matrix A is called semi-positive definite ($A \geq 0$) if $\langle Ax, x\rangle \geq 0$, for all $x \in \mathbb{R}^n$;A is positive definite ($A > 0$) if $\langle Ax, x\rangle > 0$ for all $x \neq 0$;

In the sequel, sometimes for the sake of brevity, we will omit the arguments of matrix-valued functions, if it does not cause any confusion.

Let us consider the following linear non-autonomous system with multiple delays

$$\begin{aligned} \dot{x}(t) &= A_0(t)x(t) + \sum_{i=1}^{m} A_i(t)x(t-h_i), \quad t \geq 0, \\ x(t) &= \phi(t), \quad t \in [-h, 0], \end{aligned} \tag{2.1}$$

where $h = \max\{h_i : i = 1,2,\ldots,m\}, A_i(t), i = 0,1,\ldots,m$, are given matrix functions and $\phi(t) \in C([-h,0],\mathbb{R}^n)$.

Definition

The system (2.1) is said to be α-stable, if there is a function $\xi(.) : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $\phi(t) \in C([-h,0],\mathbb{R}^n)$, the solution $x(t,\phi)$ of the system satisfies

$$\|x(t,\phi)\| \leq \xi(\|\phi\|)e^{-\alpha t}, \quad \forall t \in \mathbb{R}^+.$$

The following well-known lemma, which is derived from completing the square, will be used in the proof of our main result.

Lemma 2.1. *Assume that $S \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix. Then for every $P, Q \in \mathbb{R}^{n\times n}$,*

$$\langle Px, x\rangle + 2\langle Qy, x\rangle - \langle Sy, y\rangle \leq \langle (P + QS^{-1}Q^T)x, x\rangle, \quad \forall x, y \in \mathbb{R}^n.$$

3 Main results

Consider the linear non-autonomous delay system (2.1), where the matrix functions $A_i(t)$, $i = 0,1,\ldots,m$, are continuous on $\mathbb{R}^+$. Let us set

$$A_{0,\alpha}(t) = A_0(t) + \alpha I, \quad A_{i,\alpha}(t) = e^{\alpha h_i}A_i(t), i = 1,2,\ldots,m.$$

Theorem 3.1. *The linear non-autonomous system* (2.1) *is α-stable if there is a symmetric semi-positive definite matrix $P(t)$, $t \in \mathbb{R}^+$ such that*

$$\begin{aligned} &\dot{P}(t) + A_{0,\alpha}^T(t)[P(t)+I] + [P(t)+I]A_{0,\alpha}(t) \\ &+ \sum_{i=1}^{m}[P(t)+I]A_{i,\alpha}(t)A_{i,\alpha}^T(t)[P(t)+I] + mI = 0. \end{aligned} \tag{3.1}$$

Proof. Let $P(t) \geq 0$, $t \in \mathbb{R}^+$ be a solution of the RDE (3.1). We take the following change of the state variable

$$y(t) = e^{\alpha t}x(t), \quad t \in \mathbb{R}^+,$$

then the linear delay system (2.1) is transformed to the delay system

$$\begin{aligned} \dot{y}(t) &= A_{0,\alpha}(t)y(t) + \sum_{i=1}^{m} A_{i,\alpha}(t)y(t-h_i), \\ y(t) &= e^{\alpha t}\phi(t), \quad t \in [-h, 0], \end{aligned} \tag{3.2}$$

Consider the following time-varying Lyapunov-like function

$$V(t, y(t)) = \langle P(t)y(t), y(t)\rangle + \|y(t)\|^2 + \sum_{i=1}^{m} \int_{t-h_i}^{t} \|y(s)\|^2 ds.$$

Taking the derivative of $V(.)$ in t along the solution of $y(t)$ of system (3.2) and using the RDE (3.1), we have

$$\begin{aligned} &\dot{V}(t, y(t)) \\ &= \langle \dot{P}(t)y(t), y(t)\rangle + 2\langle P(t)\dot{y}(t), y(t)\rangle + 2\langle \dot{y}(t), y(t)\rangle + m\|y(t)\|^2 - \sum_{i=1}^{m} \|y(t-h_i)\|^2, \\ &= \langle \dot{P}(t)y(t), y(t)\rangle + 2\langle P(t)A_{0,\alpha}(t)y(t), y(t)\rangle + 2\sum_{i=1}^{m} \langle P(t)A_{i,\alpha}(t)y(t-h_i), y(t)\rangle \\ &\quad + 2\langle A_{0,\alpha}(t)y(t), y(t)\rangle + 2\sum_{i=1}^{m} \langle A_{i,\alpha}(t)y(t-h_i), y(t)\rangle \\ &\quad + m\|y(t)\|^2 - \sum_{i=1}^{m} \|y(t-h_i)\|^2 \\ &= \langle \dot{P}(t)y(t), y(t)\rangle + 2\langle (P(t)+I)A_{0,\alpha}(t)y(t), y(t)\rangle \\ &\quad + 2\sum_{i=1}^{m} \langle (P(t)+I)A_{i,\alpha}(t)y(t-h_i), y(t)\rangle + m\|y(t)\|^2 - \sum_{i=1}^{m} \|y(t-h_i)\|^2, \\ &= -\sum_{i=1}^{m} \langle [P(t)+I]A_{i,\alpha}(t)A_{i,\alpha}^T(t)[P(t)+I]y(t), y(t)\rangle \\ &\quad + 2\sum_{i=1}^{m} \langle [P(t)+I]A_{i,\alpha}(t)y(t-h_i), y(t)\rangle - \sum_{i=1}^{m} \langle y(t-h_i), y(t-h_i)\rangle \\ &= \sum_{i=1}^{m} \{-\langle [P(t)+I]A_{i,\alpha}(t)A_{i,\alpha}^T(t)[P(t)+I]y(t), y(t)\rangle \\ &\quad + 2\langle [P(t)+I]A_{i,\alpha}(t)y(t-h_i), y(t)\rangle - \langle y(t-h_i), y(t-h_i)\rangle\}. \end{aligned} \tag{3.3}$$

Applying Lemma 2.1 to the above equality, we have

$$\dot{V}(t, y(t)) \leq 0, \quad \forall t \in \mathbb{R}^+.$$

Integrating both sides of this inequality from 0 to t, we find

$$V(t, y(t)) - V(0, y(0)) \leq 0, \quad \forall t \in \mathbb{R}^+,$$

and hence

$$\langle P(t)y(t),y(t)\rangle + \|y(t)\|^2 + \sum_{i=1}^{m}\int_{t-h_i}^{t}\|y(s)\|^2 ds$$
$$\le \langle P_0 y(0),y(0)\rangle + \|y(0)\|^2 + \sum_{i=1}^{m}\int_{-h_i}^{0}\|y(s)\|^2 ds,$$

where $P_0 = P(0) \ge 0$ is any initial condition. Since

$$\langle P(t)y,y\rangle \ge 0, \quad \int_{t-h_i}^{t}\|y(s)\|^2 ds \ge 0,$$
$$\int_{-h_i}^{0}\|y(s)\|^2 ds \le \|\phi\|\int_{-h_i}^{0} e^{\alpha s}ds = \frac{1}{\alpha}(1-e^{-\alpha h_i})\|\phi\|,$$

it follows that

$$\|y(t)\|^2 \le \langle P_0 y(0),y(0)\rangle + \|y(0)\|^2 + \frac{1}{\alpha}\sum_{i=1}^{m}(1-e^{-\alpha h_i})\|\phi\|.$$

Therefore, the solution $y(t,\phi)$ of the system (3.2) is bounded. Returning to the solution $x(t,\phi)$ of system (2.1) and noting that

$$\|y(0)\| = \|x(0)\| = \phi(0) \le \|\phi\|,$$

we have $\|x(t,\phi)\| \le \xi(\|\phi\|)e^{-\alpha t}$ for all $t \in \mathbb{R}^+$, where

$$\xi(\|\phi\|) := \{\|P_0\|\|\phi\|^2 + \|\phi\|^2 + \frac{1}{\alpha}\sum_{i=1}^{m}(1-e^{-\alpha h_i})\|\phi\|\}^{\frac{1}{2}}.$$

This implies system (2.1) begin α-stable and completes the proof. □

Remark

Note that the existence of a semi-positive definite matrix solution $P(t)$ of RDE (3.1) guarantees the boundedness of the solution of transformed system (3.2), and hence the exponential stability of the linear non-autonomous delay system (2.1). Also, the stability of $A(t)$ is not assumed.

Example 3.2. Consider the following linear non-autonomous delay system in $\mathbb{R}^2$:

$$\dot{x} = A_0(t)x + A_1(t)x(t-0.5) + A_2(t)x(t-1), \quad t \in \mathbb{R}^+,$$

with any initial function $\phi(t) \in C([-1,0],\mathbb{R}^2)$ and

$$A_0(t) = \begin{pmatrix} a_0(t) & 0 \\ 0 & -7.5 \end{pmatrix}, \quad A_1(t) = \begin{pmatrix} e^{-0.5}a_1(t) & 0 \\ 0 & e^{-0.5}\sqrt{3} \end{pmatrix},$$
$$A_2(t) = \begin{pmatrix} e^{-1}a_1(t) & 0 \\ 0 & e^{-1}\sqrt{3} \end{pmatrix},$$

where

$$a_0(t) = \frac{7e^{-9t} - 5}{2(1+e^{-9t})}, \quad a_1(t) = \frac{1}{\sqrt{2}(1+e^{-9t})}.$$

We have $h_1 = 0.5$, $h_2 = 1$, $m = 2$ and the matrix $A_0(t)$ is not asymptotically stable, since $\mathrm{Re}\,\lambda(A(0)) = 0.5 > 0$. Taking $\alpha = 1$, we have

$$A_{0,\alpha}(t) = \begin{pmatrix} a_0(t)+1 & 0 \\ 0 & -6.5 \end{pmatrix}, \quad A_{1,\alpha}(t) = A_{2,\alpha}(t) = \begin{pmatrix} a_1(t) & 0 \\ 0 & \sqrt{3} \end{pmatrix}.$$

The solution of RDE (3.1) is

$$P(t) = \begin{pmatrix} e^{-9t} & 0 \\ 0 & 1 \end{pmatrix} \geq 0, \quad \forall t \in \mathbb{R}^+.$$

Therefore, the system is 1-stable.

For the autonomous delay systems, we have the following α-stability condition as a consequence.

Corollary 3.3. *The linear delay system* (2.1)*, where A_i are constant matrices, is α-stable if there is a symmetric semi-positive definite matrix $P \in \mathbb{R}^{n\times n}$, which is a solution of the algebraic Riccati equation*

$$A_{0,\alpha}^T[P+I] + [P+I]A_{0,\alpha} + \sum_{i=1}^{m}[P+I]A_{i,\alpha}A_{i,\alpha}^T[P+I] + mI = 0. \tag{3.4}$$

Example 3.4. Consider the linear autonomous delay system

$$\dot{x}(t) = A_0x(t) + A_1x(t-2) + A_2x(t-4), \quad t \in \mathbb{R}^+,$$

with any initial function $\phi(t) \in C([-4,0],\mathbb{R}^2)$ and

$$A_0 = \begin{pmatrix} -\frac{17}{6} & 0 \\ \frac{4}{3} & -3.5 \end{pmatrix}, \quad A_1 = \begin{pmatrix} e^{-1} & 0 \\ 0 & e^{-1} \end{pmatrix}, \quad A_2 = \begin{pmatrix} e^{-2} & 0 \\ 0 & e^{-2} \end{pmatrix}.$$

In this case, we have $m = 2$, $h_1 = 2$, $h_2 = 4$. Taking $\alpha = 0.5$, we find

$$A_{0,\alpha}(t) = \begin{pmatrix} -\frac{7}{3} & 0 \\ \frac{4}{3} & -3 \end{pmatrix}, \quad A_{1,\alpha}(t) = A_{2,\alpha}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and the solution of algebraic Riccati equation (3.4) is

$$P = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \geq 0.$$

Therefore, the system is 0.5-stable.

Remark

Note that we can estimate the value of $V(t,y)$ as follows. Since

$$2(P+I)A_{0,\alpha} = A_0^T P + PA_0 + A_0 + A_0^T + 2\alpha(P+I),$$

from (3.3) it follows that

$$\begin{aligned}\dot{V}(t,y(t)) &= \langle [\dot{P}(t) + A_0^T(t)P(t) + P(t)A_0(t) + mI]y(t), y(t)\rangle \\ &\quad + \langle [A_0(t) + A_0^T(t)]y(t), y(t)\rangle + 2\alpha\langle (P(t)+I)y(t), y(t)\rangle \\ &\quad + \sum_{i=1}^{m} \Big\{ 2\langle [P(t)+I]A_{i,\alpha}(t)y(t-h_i), y(t)\rangle - \|y(t-h_i)\|^2 \Big\}.\end{aligned}$$

Using Lemma 2.1, we have

$$\begin{aligned}&\sum_{i=1}^{m} \Big\{ 2\langle [P+I]A_{i,\alpha}y(t-h_i), y(t)\rangle - \|y(t-h_i)\|^2 \Big\} \\ &\le \sum_{i=1}^{m} \langle [P+I]A_{i,\alpha}A_{i,\alpha}^T[P+I]y(t), y(t)\rangle.\end{aligned}$$

On the other hand, since

$$\sum_{i=1}^{m} \langle [P(t)+I]A_{i,\alpha}(t)A_{i,\alpha}^T(t)[P(t)+I]y(t), y(t)\rangle \le m\|P(t)+I\|^2 e^{2\alpha h}\|A(t)\|^2\|y(t)\|^2,$$

with $h = \max\{h_1, h_2, \dots, h_m\}$, $\|A(t)\|^2 = \max\{\|A_1(t)\|^2, \|A_2(t)\|^2, \dots, \|A_m(t)\|^2\}$, we obtain

$$\begin{aligned}\dot{V}(t,y(t)) &\le \langle [\dot{P}(t) + A_0^T(t)P(t) + P(t)A_0(t) + mI]y(t), y(t)\rangle \\ &\quad + \Big[2\eta(A_0(t)) + 2\alpha\|P(t)+I\| + m\|P(t)+I\|^2 e^{2\alpha h}\|A(t)\|^2\Big]\|y(t)\|^2.\end{aligned}$$

Therefore, the α-stability condition of Theorem 3.1 can be given in terms of the solution of the following Lyapunov equation, which does not involve α:

$$\dot{P}(t) + A_0^T(t)P(t) + P(t)A_0(t) + mI = 0. \tag{3.5}$$

In this case, if we assume that $P(t)$, $A_i(t)$ are bounded on $\mathbb{R}^+$ and

$$\eta(A_0) := \sup_{t\in\mathbb{R}^+} \eta(A_0(t)) < +\infty, \tag{3.6}$$

then the rate of convergence $\alpha > 0$ can be defined as a solution of the scalar inequality

$$\eta(A_0) + \alpha\|P_I\| + \frac{m}{2}e^{2\alpha h}\|P_I\|^2\|A\|^2 \le 0, \tag{3.7}$$

where

$$P_I = \sup_{t\in\mathbb{R}^+} \|P(t)+I\|, \quad \|A\|^2 = \sup_{t\in\mathbb{R}^+} \|A(t)\|^2.$$

Therefore, we have the following α-stability condition.

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