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ON THE INDEPENDENT DOMINATION NUMBER OF THE GENERALIZED PETERSEN GRAPHS

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Abstract

Here we consider an infinite sub-family of the generalized Petersen graphs $P(n,k)$ for $n = 2k+1 \geq 3$, and using the two algorithms that A. Behzad et al presented in [1], we determine an upper bound and a lower bound for the independent domination numbers of these graphs.

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1 Introduction

For a graph $G = (V,E)$, with vertex set V and edge set E, a subset $S \subseteq V$ is said to *dominates* V if each vertex of $V - S$ is adjacent to some vertex of S. The set V itself has this property and, for a finite graph G, the minimum cardinality of subsets S that dominate V is called the *domination number* of G, and is denoted by $\gamma(G)$. Domination numbers for graphs and associated concepts have been studied for many years and there is an extensive literature on the subject, see [3]. In general, determining the domination number is an NP-complete problem. In fact, the book [3] contains a chapter, entitled *Domination*, *complexity and algorithms*, devoted to this broad subject. Also a subset $S \subseteq V$ is said to be an *independent dominating set* if S is both a dominating set and an independent set, that is, S is a dominating set which its no two vertices are adjacent. Also the minimum cardinality of independent dominating sets S of V is called the *independent domination number* of G, and is denoted by $i(G)$.

The *open neighborhood* of a vertex $v \in V$ is $N_G(v) = \{u \mid uv \in E\}$ and its *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$. Let $S \subseteq V$, and $v \in S$. Vertex u is called a *private neighbour* of v with respect to S (denoted by u is an S-pn of v) if $u \in N_G(v) - N_G[S - v]$. The set $pn(v;S) = N_G(v) - N_G[S - v]$ of all S-pn's of v is called the *private neighbourhood set* of v with respect to S (see [2, 3]).

In [5], Watkins introduced the notion of generalized Petersen graph (GPG for short) as follows: for any integer $n \geq 3$ let Z_n be additive group on $\{1,2,...,n\}$ and $k \in Z_n - \{0\}$, the

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graph $P(n,k)$ is defined on the set $\{O_i, I_i \mid i \in Z_n\}$ of $2n$ vertices, and with the adjacencies given by O_iO_{i+1}, O_iI_i, I_iI_{i+k} for all i. If $k = n/2$, then every vertex I_i has degree 2 and every vertex O_i has degree 3, otherwise $P(n,k)$ is 3-regular. Here the subscripts are reduced modulo n. In this notation, the classical Petersen graph is $P(5,2)$. A. Behzad et al in [1] considered an infinite sub-family of the generalized Petersen graphs $P(2k+1,k)$, for $k \geq 1$, and they presented two algorithms which between them lead to the determination of upper and lower bounds on the domination number of these graphs and then proved that for each odd integer $n = 2k+1 \geq 3$, $\gamma(P(n,k)) \leq \lceil 3n/5 \rceil$, and moreover $\gamma(P(n,k)) \leq \gamma(P(n+2,k+1)) \leq \gamma(P(n,k)) + 2$.

Let here $n = 2k+1 \geq 3$, $G(n) := P(n,k)$, and $V(G(n)) = O^p \cup I^p$, where $O^p = \{O_i | 1 \leq i \leq n\}$ and $I^p = \{I_i | 1 \leq i \leq n\}$. We note that $G(n)$ is obtained by the union of two cycles with length n, $C_I : I_1, I_{k+1}, I_n, I_k, I_{2k}, I_{k-1}, I_{2k-1}, I_{k-2}, \ldots, I_3, I_{k+3}, I_2, I_{k+2}$, and $C_O : O_1, O_2, O_3, \ldots, O_n$ which every vertex I_i of C_I is adjacent to vertex O_i of C_O.

Here similar to [1] we show for each odd integer $n \geq 5$, $i(G(n)) \leq i(G(n+2)) \leq i(G(n)) + 2$.

2 Algorithms

In this section, we give two algorithms of [1] which state how we can obtain $G(n)$ from $G(n+2)$ or $G(n+2)$ from $G(n)$.

2.1 Integration algorithm

INPUT: the graph $G(n) = (O^p \cup I^p, E_1 \cup E_2 \cup E_3)$ with $n = 2k+1 \geq 7$.
OUTPUT: a graph G'' with $2(n-2)$ vertices.

STEP 1.
Choose i such that $1 \leq i \leq k$, remove the four pairs of vertices
$\{O_i, O_{i+1}\}$, $\{I_i, I_{i+1}\}$, $\{O_{i+k}, O_{i+k+1}\}$ and $\{I_{i+k}, I_{i+k+1}\}$,
along with their 15 incident edges, and denote the resulting graph by G'.

STEP 2.
Add four new vertices O'_i, I'_i, O'_{i+k-1}, I'_{i+k-1},
and define the graph G'' to have vertex set
$V(G'') = V(G') \cup \{O'_i, I'_i, O'_{i+k-1}, I'_{i+k-1}\}$
and edge set
$E(G'') = E(G') \cup \{O_{i-1}O'_i, O'_iO_{i+2}, O'_iI'_i, I'_iI'_{i+k-1}, I'_iI_{i+k+2},$
$I_{i-1}I'_{i+k-1}, O_{i+k+2}O'_{i+k-1}, O'^{>}_{i+k-1}O_{i+k-1}, O'_{i+k-1}I'_{i+k-1}\}$.
Return G''.

Lemma 2.2 of [1] says that the above graph G'' is isomorphic to $G(n-2)$.

2.2 Disintegration algorithm

INPUT: the graph $G(n) = (O^p \cup I^p, E_1 \cup E_2 \cup E_3)$ with $n = 2k+1 \geq 5$.
OUTPUT: a graph G'' with $2(n+2)$ vertices.

STEP 1.
Choose i such that $2 \leq i \leq k+1$, remove the four pairs of vertices

O_i, I_i, O_{i+k}, and I_{i+k},
along with their 9 incident edges, and denote the resulting graph by G' .
STEP 2.
Add eight new vertices
$V'' = \{O'_{i-1}, O'_i, I'_{i-1}, I'_i, O'_{i+k}, O'_{i+k+1}, I'_{i+k}, I'_{i+k+1}\}$,
and define the graph G'' to have vertex set $V(G'') = V(G') \cup V''$
and edge set $E(G'') = E(G') \cup$
$\{O_{i-1}O'_{i-1}, O'_{i-1}O'_i, O'_iO_{i+1}, O'_{i-1}I'_{i-1}, O'_iI'_i,$
$O_{i+k-1}O'_{i+k}, O'_{i+k}O'_{i+k+1}, O'_{i+k+1}O_{i+k+1}, O'_{i+k}I'_{i+k},$
$O'_{i+k+1}I'_{i+k+1}, I'_{i-1}I'_{i+k}, I'_{i+k}I_{i-1}, I'_{i+k+1}I'_{i-1}, I'_{i+k+1}I'_i, I'{}_iI_{i+k+1}\}$.
Return G''.

Lemma 2.4 [1] says that the above graph G'' is isomorphic to $G(n+2)$.

3 Main Result

Lemma 3.1. *Let n be an odd integer such that $n = 2k+1 \geq 5$. Then*

$$i(G(n)) \leq i(G(n+2)).$$

Proof. To keep the notation in line with that of algorithm 2.1, we assume that $n \geq 7$, and prove that $i(G(n-2)) \leq i(G(n))$. Let $G = G(n)$, and let $S \subseteq V(G)$ be an independent dominating set for $V(G)$ of minimum cardinality. Trivially at least one element of I^p, say I_1, must lie in S. Let G'' be the graph returned by algorithm 2.1 with the index $i = 1$. By Lemma 2.2 of [1], $G'' \cong G(n-2)$. We will identify $V(G(n-2))$ with $V(G'')$ so that $V(G(n-2)) = (O^p \cup I^p \backslash T) \cup T'$, where $T' = \{O'_1, I'_1, O'_k, I'_k\}$ and $T = \{O_1, O_2, I_1, I_2, O_{k+1}, O_{k+2}, I_{k+1}, I_{k+2}\}$. Let G' be the subgraph of G spanned by $V(G) \backslash T$, so that G' is also a subgraph of $G(n-2)$. Then the independent subset $S' := S \cap V(G')$ dominates all vertices in $V(G')$, except possibly vertices in $R := \{O_3, O_n, O_k, O_{k+3}, I_n, I_{k+3}\}$. Since $I_1 \in S$ it follows that $\{O_1, I_{k+1}, I_{k+2}\} \cap S = \emptyset$. So $1 \leq |S \cap T| \leq 3$. If $|S \cap T| = 2$, then $S \cap T$ is one of the four sets $\{I_1, I_2\}$, $\{I_1, O_{k+1}\}$, $\{I_1, O_{k+2}\}$, $\{I_1, O_2\}$. For $|S \cap T| = 3$, $S \cap T$ is also one of the four sets $\{I_1, I_2, O_{k+1}\}$, $\{I_1, I_2, O_{k+2}\}$, $\{I_1, O_2, O_{k+1}\}$, $\{I_1, O_2, O_{k+2}\}$.

In the follow, for each of the eight cases, we present an independent dominating set S'' with cardinality at most $|S|$ such that dominates $V(G'')$.

Case 1. $S \cap T = \{I_1, I_2\}$.
Hence $I_{k+3} \notin S$, and so S' dominates all $V(G')$ except possibly $\{I_{k+3}\}$. If $\{O_k, O_{k+3}\} \cap S = \emptyset$, then we choose $S'' = (S - (S \cap T)) \cup \{I'_1, O'_k\}$, and on the otherwise $S'' = S' \cup \{I'_1\}$.

Case 2. $S \cap T = \{I_1, O_{k+1}\}$.
Hence $I_{k+3}, O_3 \in S$, and $O_{k+3}, O_k \notin S$. So S' dominates all $V(G')$ except possibly $\{O_k\}$. Now we choose $S'' = S' \cup \{O'_k\}$.

Case 3. $S \cap T = \{I_1, O_{k+2}\}$.
Then, since $\{O_2, I_2\} \cap S = \emptyset$, must O_3 and I_{k+3} lie in S. Hence $O_{k+3} \notin S$ and so S' dominates all $V(G')$. We choose $S'' = S' \cup \{O'_k\}$, $S'' = S'$, and $S'' = S' \cup \{I'_k\}$, respectively, for three cases $O_k \notin S$; $O_k, I_n \in S$; and $O_k \in S$, $I_n \notin S$.

Case 4. $S \cap T = \{I_1, O_2\}$.

Hence $O_{k+3}, O_k \in S$. Because $\{O_{k+1}, O_{k+2}, I_{k+1}, I_{k+2}\} \cap S = \emptyset$. We also know S' dominates all $V(G')$ except possibly $\{O_3\}$. Since $O_3 \neq O_k$, then $k \geq 4$. For $k = 4$, since O_k is O_4 and dominates O_3, let $S'' = S'$. Therefore let $k \geq 5$. If O_3 is dominated by S', then we choose $S'' = S' \cup \{I'_1\}$, and if no, we choose $S'' = S' \cup \{I'_1, O_3\}$.
Case 5. $S \cap T = \{I_1, I_2, O_{k+1}\}$.
Hence $I_{k+3}, O_k \notin S$ and S' dominates all $V(G')$ except possibly $\{I_{k+3}, O_k\}$. Choose $S'' = S' \cup \{I'_1, O'_k\}$, $S'' = S' \cup \{I'_1\}$, and $S'' = S' \cup \{I'_1, O_k\}$, respectively, for three cases $O_{k+3} \notin S$; $O_{k+3}, O_{k-1} \in S$; and $O_{k+3} \in S$, $O_{k-1} \notin S$.
Case 6. $S \cap T = \{I_1, I_2, O_{k+2}\}$.
Hence $I_{k+3}, O_{k+3} \notin S$ and S' dominates all $V(G')$ except possibly $\{O_{k+3}, I_{k+3}\}$. If $O_k \notin S$, then choose $S'' = S' \cup \{I'_1, O'_k\}$. Let $O_k \in S$. If S' dominates O_{k+3}, then let $S'' = S' \cup \{I'_1\}$, and if S' does not dominate O_{k+3}, then set $S'' = S' \cup \{I'_1, O_{k+3}\}$.
Case 7. $S \cap T = \{I_1, O_2, O_{k+1}\}$.
Hence $O_k, O_3 \notin S$ and S' dominates all $V(G')$ except possibly $\{O_k, O_3\}$. Let $O_{k+3} \notin S$. Then $S' \cup \{I'_1, O'_k\}$ dominates all vertices except possibly the vertex O_3. So in this case, we add O_3 to it. But if $O_{k+3} \in S$, the set $S' \cup \{I'_1\}$ dominates all vertices except possibly the vertices O_3 and O_k. Then we add those vertex or vertices to $S' \cup \{I'_1\}$ which are not dominated by it.
Case 8. $S \cap T = \{I_1, O_2, O_{k+2}\}$.
Hence $O_{k+3}, O_3 \notin S$ and S' dominates all $V(G')$ except possibly $\{O_{k+3}, O_3\}$. Let $O_k \notin S$. Then $S' \cup \{I'_1, O'_k\}$ dominates all vertices except possibly the vertex O_3 that in this case we add O_3 to $S' \cup \{I'_1, O'_k\}$. For the case $O_k \in S$, the set $S' \cup \{I'_1\}$ dominates all vertices except possibly the vertices O_3 and O_{k+3}. Then we add those vertex or the vertices to $S' \cup \{I'_1\}$, which are not dominated by it.

In the all cases, the subset S'' has size at most $|S|$, and so $i(G(n)) \leq i(G(n+2))$, where $n \geq 5$ and is odd.

□

Lemma 3.2. *Let n be an odd integer such that* $n = 2k+1 \geq 3$. *Then*

$$i(G(n+2)) \leq i(G(n)) + 2.$$

Proof. Let $G = G(n)$, and $S \subseteq V(G)$ be an independent dominating set with minimum cardinality of G. Here we may assume $I_2 \in S$. By Lemma 2.4 of [1], $G(n+2)$ is isomorphic to the graph G'' returned by algorithm 2.2 with the index $i = 2$ at STEP 1. Moreover, we may assume that the graph G' constructed in STEP 1 of algorithm 2.2 is the subgraph of G spanned by $V(G) \setminus T$, where $T = \{O_2, I_2, O_{k+2}, I_{k+2}\}$. The subset $S' := S \cap V(G')$ is independent and dominates all vertices in $V(G')$, except possibly vertices in $R := \{O_1, O_3, O_{k+1}, O_{k+3}, I_1, I_{k+3}\}$.

We will show that $V(G'')$ contains an independent subset S'' such that $S' \subseteq S''$, S'' dominates $V(G'')$, and $|S''| \leq |S| + 2$. Then $i(G(n+2)) = i(G'') \leq |S| + 2 = i(G(n)) + 2$, and this completes the proof. To produce such a set S'', we add to the set S' the appropriate number of vertices of $V(G'')$ so that the set $T' := \{O'_1, O'_2, I'_1, I'_2, O'_{k+2}, O'_{k+3}, I'_{k+2}, I'_{k+3}\} \cup R$ is dominated. Note that $I_2 \in T \cap S$ and $1 \leq |S \cap T| \leq 2$. To continuing the proof, we consider two following cases.
Case 1. $|S \cap T| = 1$.

Then $O_2, I_{k+2}, I_{k+3} \notin S$. We first see $S \cap \{O_{k+1}, O_{k+3}\} \neq \emptyset$. Since otherwise, for dominating vertex O_{k+2} by S, must $I_{k+2} \in S$. But it is a contradiction to independence of S . In the three cases $O_{k+1}, O_{k+3} \in S$; $O_{k+1} \in S$, and $O_{k+3} \notin S$; $O_{k+3} \in S$, and $O_{k+1} \notin S$ each of the three respective sets $S'' = S' \cup \{I'_1, I'_2\}$, $S'' = S' \cup \{I'_1, I'_2, O'_{k+3}\}$, and $S'' = S' \cup \{I'_1, I'_2, O'_{k+2}\}$ dominates $V(G'')$.

Case 2. $|\, S \cap T \,| = 2$.

Then $S \cap T = \{I_2, O_{k+2}\}$, and so $\{O_{k+1}, O_{k+3}, I_{k+2}, I_{k+3}\} \cap S = \emptyset$. In the three cases $O_k \in S$; $O_{k+4} \in S$, and $O_{k+4}, O_k \notin S$, each of the three respective sets $S'' = S' \cup \{I'_1, I'_2, O'_{k+3}\}$, $S'' = S' \cup \{I'_1, I'_2, O'_{k+2}\}$, and $S'' = S' \cup \{I'_1, I'_2, O'_{k+2}, O_{k+3}\}$ dominates $V(G'')$.

In the all cases, S'' has size at most $|\, S \,| + 2$, and so $i(G(n+2)) \leq i(G(n)) + 2$, where $n \geq 3$, and is odd. □

Thus far we have proved that:

Theorem 3.3. *For each odd integer $n \geq 5$,*

$$i(G(n)) \leq i(G(n+2)) \leq i(G(n)) + 2.$$

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