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MUTATIONS AND POINTED BRAUER TREES

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Abstract

We will give a sequence of irreducible mutations converting Brauer star algebra to any two-restricted star-to-tree complex.

1. Introduction

Brauer tree algebras are important objects in the modular representation theory because blocks with cyclic defect groups are Brauer tree algebras. Also it is known that for a block of G with cyclic defect group P, its Brauer correspondent with respect to $N_G(P)$ is a Brauer star algebra. In [7] Rickard showed that two Brauer tree algebras are derived equivalent if and only if their Brauer trees have the same numbers of edges and exceptional multiplicities. This implies that Broué's abelian defect conjecture holds in the case of cyclic defect groups. For the proof, for any Brauer tree algebra, Rickard constructed a tilting complex over the Brauer tree algebra which induces a derived equivalence from the Brauer tree algebra to the Brauer star algebra. Later, in [9], for any Brauer tree algebra by using pointing of the Brauer tree, Schaps-Zakay constructed a lot of tilting complexes over Brauer star algebra which induce equivalence opposite to the one constructed by Rickard, and the class of the complexes includes the tilting complex which induces the inverse equivalence to the one induced by Rickard tree-to-star complex. Also the class of Schaps-Zakay star-to-tree complexes is all of the two-restricted star-to-tree complexes (Definition 2.13). In [8] by using pointings of Brauer trees, Rickard-Schaps constructed tree-to-star complexes giving inverse equivalences to those induced by the Schaps-Zakay star-to-tree complexes.

On the other hand, nowadays silting mutations are studied by many people, in particular, since mutations for tilting complexes over symmetric algebras produce various tilting complexes, we are interested in the mutations in case of symmetric algebras. As one of the results of the mutations, in [2] Aihara showed that if a symmetric algebra is of finite representation type, then the algebra is tilting-connected (in fact, tilting-discrete). Hence all tilting complexes over a Brauer tree algebra are controlled by mutations since Brauer tree algebras are of finite representation type. Also, for a Brauer tree algebra A_G associated to a Brauer tree G and for the Brauer star algebra B derived equivalent to A_G, in [10] Schaps-Zvi gave a sequence of the irreducible mutations which converts the stalk complex B to the Schaps-Zakay star-to-tree complex corresponding to the *reverse pointing* or the *left alternating pointing* of the Brauer tree G.

Our aim is to generalize their result and show that for any Brauer tree G and for *any point-*

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ing of G, we can find a sequence of irreducible mutations which converts the stalk complex of the Brauer star algebra to the Schaps-Zakay star-to-tree complex corresponding to the pointing. Since each Schaps-Zakay star-to-tree complex induces an inverse equivalence to the one induced by the Rickard-Schaps tree-to-star complex for the same pointing, giving a sequence of mutations which converts the stalk complex of the Brauer star algebra to the Schaps-Zakay star-to-tree complex is equivalent to giving one to convert the stalk complex of the Brauer tree algebra to the Rickard-Schaps tree-to-star complex. Therefore for any Brauer tree G and any pointing of G, we give, in Algorithm 3.2, a sequence of irreducible mutations which converts the stalk complex A_G to the Rickard-Schaps tree-to-star complex T corresponding to the pointing (Theorem 4.1).

Also, when considering mutations of Brauer tree algebras, the Kauer move (Definition 2.7) is important. The Kauer move tells us how the Brauer tree mutates when we apply an irreducible mutation to Brauer tree algebra, that is, we can determine easily and explicitly the Brauer tree of the endomorphism algebra of the tilting complex obtained by the irreducible mutation. In that way, we want to determine easily and explicitly how to convert the Schaps-Zakay star-to-tree complex by mutations. There, we introduce the Kauer move for pointed Brauer trees, which is a local move for pointed Brauer tree $G(p)$ converting it to another pointed Brauer tree $\mu_i^{\epsilon}(G(p))$, where $\epsilon \in \{+, -\}$, satisfying the following property, which tells us how the two-restricted star-to-tree complex mutates by irreducible mutations (Theorem 4.3): Let G be a Brauer tree, $G(p)$ a pointed Brauer tree of G and $T(G(p))$ a Schaps-Zakay star-to-tree complex, then $\mu_i^{\epsilon}(T(G(p)) \cong T(\mu_i^{\epsilon}(G(p))$.

Throughout this paper, algebra means finite dimensional basic algebra. For an algebra Γ, Γ-modules means finitely generated left Γ-modules, and we denote by k an algebraically closed field, by $A = A_G$ a Brauer tree algebra associated to a Brauer tree G with e edges numbered as $1, 2, \ldots, e$ and by B a Brauer star algebra derived equivalent to A. Moreover, for an algebra Γ, we denote by $D^b(\Gamma)$ the derived category of the complexes of finite generated Γ-modules.

We now describe the organization of this paper.

In Sect. 2, we recall some facts on Brauer tree algebras, mutations, and pointed Brauer trees.

In Sect. 3, we introduce a Kauer move for pointed Brauer trees, and give an algorithm which define a sequence of the irreducible mutations from the pointed Brauer tree.

In Sect. 4, we prove the sequence of irreducible mutations obtained in Sect. 3 converts the stalk complex to the Rickard-Schaps tree-to-star complex corresponding to the pointed Brauer tree.

In Sect. 5, for a concrete pointed Brauer tree, we explain how to get the sequence of irreducible mutations, and confirm that the sequence converts the stalk complex to the Rickard-Schaps tree-to-star complex.

2. Preliminaries

2.1. Derived equivalences for Brauer tree algebras. We recall first the definition of Brauer tree algebras and their properties.

Let G be a finite connected tree. We say that G is a Brauer tree of type (e, m) (or simply say Brauer tree) if G with e edges has the following additional structures:

- a positive integer m attached to one vertex (we call the integer the exceptional multiplicity and the vertex the exceptional vertex).
- cyclic orderings of the edges incident to a given vertex (when we embed the tree in the plane, this ordering is usually described by the counter-clockwise ordering or the clockwise ordering around each vertex).

Throughout this paper, the cyclic orderings around the vertices will be counter-clockwise orderings around vertices when we embed trees in a plane. Brauer tree algebras are defined by using Brauer trees.

Definition 2.1. Given a Brauer tree G of type (e, m), we say that a k-algebra A is a Brauer tree algebra associated to G if

- there is a one-to-one correspondence between the edges of G and the isomorphism classes of simple A-modules.
- for every indecomposable projective module P, $\operatorname{top} P$ is isomorphic to $\operatorname{soc} P$.
- for each indecomposable projective module P,

$$\operatorname{rad} P/\operatorname{soc} P \cong U(v_1) \oplus U(v_2),$$

  where v_1 and v_2 are the vertices adjacent to the edge corresponding to the simple module $S := \operatorname{top} P$ and where $U(v_i)$ is a uniserial A-module (possibly zero) with the following composition factors in the following order for $i = 1, 2$:
  (1) $T_1, T_2, \ldots, T_r$ in the case that v_i is a non-exceptional vertex (where the sequence $S = T_0, T_1, T_2, \ldots, T_r, T_{r+1} = S$ is given by counter-clockwise counting the edges emanating from a vertex v_i for $i = 1, 2$).
  (2) $T_1, T_2, \ldots, T_r, S, T_1, T_2, \ldots, T_r, S, T_1, T_2, \ldots, T_r$, where S occurs $m - 1$ times and each T_j occurs m times for each $1 \leq j \leq r$ in the case that v_i is the exceptional vertex with exceptional multiplicity m (where the sequence $S = T_0, T_1, T_2, \ldots, T_r, T_{r+1} = S$ is given by counter-clockwise counting the edges emanating from a vertex v_i for $i = 1, 2$).

The following proposition is easily checked but very essential.

Proposition 2.2 (see [7])**.** *Let a Brauer tree algebra A associated to a Brauer tree G of type (e, m). For non-isomorphic simple A-modules S and T,*

$$\dim \operatorname{Hom}_A(P(S), P(T)) = \begin{cases} 0 & \text{if the edges } S \text{ and } T \text{ have no vertex in common,} \\ 1 & \text{if the edges } S \text{ and } T \text{ have a non-exceptional vertex in common,} \\ m & \text{if the edges } S \text{ and } T \text{ have the exceptional vertex in common.} \end{cases}$$

There is a very important Brauer tree, which we call the Brauer star. The Brauer tree G of type (e, m) is said to be Brauer star of type (e, m) if all e edges are adjacent to the exceptional vertex. In particular, we call the Brauer tree algebra associated to the Brauer star a Brauer star algebra.

We give a definition of a tilting complex before giving the statement on derived equivalence of Brauer tree algebras.

Definition 2.3. Let Γ be a Noetherian ring. Let T be a bounded complex of projective Γ-modules. We call T a tilting complex if T satisfies the following two conditions.

(1) $\mathrm{Hom}_{D^b(\Gamma)}(T, T[n]) = 0$ for all $n \in \mathbb{Z} - \{0\}$.
(2) The smallest triangulated full subcategory of $D^b(\Gamma)$ containing all direct summands of finite direct sums of T is $K^b(\Gamma\text{-proj})$.

In [6], Rickard showed that two Noetherian rings Γ and Λ are derived equivalent if and only if there exists a tilting complex over Γ with $\mathrm{End}_{D^b(\Gamma)}(T) \cong \Lambda^{op}$. By using this result, Rickard gave in [7] the following result.

Theorem 2.4 ([7])**.** *Let A_i be a Brauer tree algebra associated to a Brauer tree G_i of type (e_i, m_i) for $i = 1, 2$. Then A_1 and A_2 are derived equivalent if and only if $e_1 = e_2$ and $m_1 = m_2$.*

For the proof of this result, for any Brauer tree algebra A of type (e, m), Rickard constructed a tilting complex T over A whose opposite algebra of the endomorphism algebra of T is isomorphic to the Brauer star algebra of type (e, m).

Definition 2.5. Let A be a Brauer tree algebra associated to a Brauer tree G of type (e, m), and B a Brauer star algebra of type (e, m). A tilting complex T over A is a tree-to-star complex if $\mathrm{End}_{D^b(A)}(T) \cong B^{op}$. A tilting complex $\hat{T}$ over B is a star-to-tree complex for A if $\mathrm{End}_{D^b(B)}(\hat{T}) \cong A^{op}$.

2.2. Mutations for Brauer tree algebras. In this section, let Γ be a basic finite dimensional symmetric algebra.

Definition-Theorem 2.6 ([3]). Let T be a tilting complex of Γ-modules. For a decomposition $T = M \oplus X$, we take a triangle

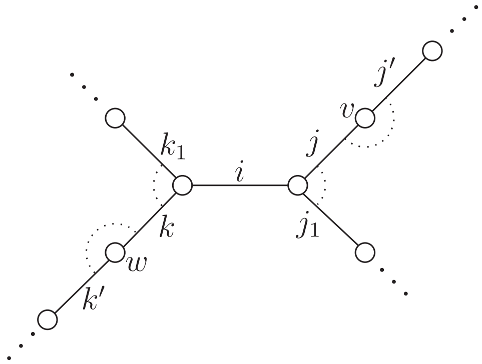
$$X \xrightarrow{f} M' \to Cone(f) \to X[1]$$

with a minimal left add M-approximation $f : X \to M'$ of X, where $M' \in \mathrm{add}\, M$. Then $\mu_X^-(T) := M \oplus Cone(f)$ is a tilting complex again. We call it a left mutation of T with respect to X. Dually we define a right mutation $\mu_X^+(T)$ of T with respect to X. For a left or right mutation $\mu_X^\epsilon(T)$, where $\epsilon \in \{+, -\}$, we call the mutation irreducible if X is an indecomposable complex. Also we let mutation means both left mutation and right mutation.

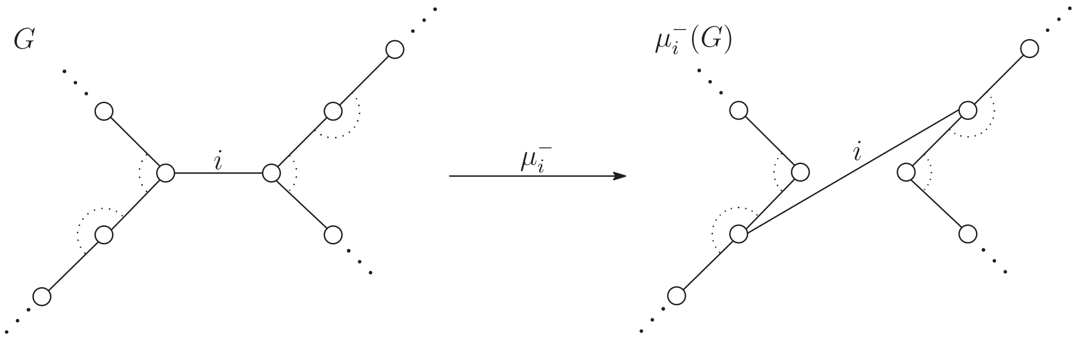
Let A be a Brauer tree algebra associated to the Brauer tree G_A. When considering the opposite algebra of the endomorphism algebra of $\mu_i^-(A)$, the Kauer move plays an important role. We recall the definition of the Kauer move.

Definition 2.7 (see [5, 1]). Let G be a Brauer tree. For G and an edge i of G, we call a local move as in (i) or (ii) below a Kauer move at i:

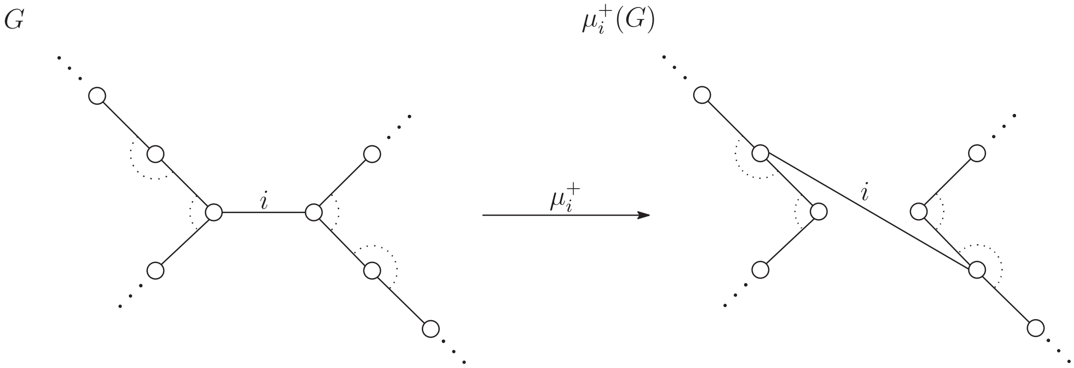
(i) For an edge i of G, let $(j_1, \ldots, j_n = j, i, j_1)$ and $(k_1, \ldots, k_m = k, i, k_1)$ be cyclic orderings of the two vertices adjacent to the edge i (possibly the edge i is an external edge, that is, $m = 1$ and $k_1 = i$). Let v, w be the vertices of the edges j, k, respectively, which are not adjacent to the edge i. Let j', k' be the next edges before j, k in the cyclic orderings at v, w, respectively.



We define $\mu_i^-(G)$ as follows. Detach i from the two vertices adjacent to the edge i, and attach the edge to v and w so that the cyclic orderings at v and w are $(i, j, \dots, j', i)$ and $(i, k, \dots, k', i)$ respectively.



(ii) Dually, we define $\mu_i^+(G)$.



Next result, following from [5, 1], tells us the structure of the opposite algebra of the endomorphism algebra of $\mu_i^-(A)$.

Proposition 2.8 ([5, 1])**.** *LetA_G be a Brauer tree algebra associated to a Brauer tree G. For any i and* $\epsilon \in \{+, -\}$, *we have* $\mathrm{End}_{D^b(A_G)}(\mu_i^\epsilon(A_G)) \cong A^{op}_{\mu_i^\epsilon(G)}$.

Next, we consider the tilting connectedness for Brauer tree algebras.

Definition 2.9. Let T_1 and T_2 be basic tilting complexes in $K^b(\Gamma\text{-proj})$. We say that T_1 and T_2 are connected if T_1 can be obtained from T_2 by iterated irreducible mutations. Also $K^b(\Gamma\text{-proj})$ is called tilting-connected if all basic tilting complexes in $K^b(\Gamma\text{-proj})$ are connected to each other.

Theorem 2.10 ([2]). *Let Γ be a finite dimensional symmetric algebra of finite representation type. Then $K^b(\Gamma\text{-proj})$ is tilting-connected.*

In particular, since Brauer tree algebras are symmetric algebras of finite representation type, the homotopy category $K^b(A\text{-proj})$ of a Brauer tree algebra A is tilting connected. Hence, for tilting complex T over the Brauer tree algebra A, there is a sequence of mutations $(\mu_{i_1}^{\epsilon_1}, \mu_{i_2}^{\epsilon_2}, \dots, \mu_{i_n}^{\epsilon_n})$ such that $(\mu_{i_n}^{\epsilon_n} \cdots \mu_{i_2}^{\epsilon_2}\mu_{i_1}^{\epsilon_1})(A) \cong T$. Our goal is to give such decompositions for Rickard-Schaps tree-to-star complexes, a class of tree-to-star complexes, (the definition of the Rickard-Schaps complexes can be seen in Section 2.3). For this problem, two types of particular cases are done in [10]. To explain further, they gave sequences of mutations with the required properties for the pointed Brauer trees with left alternating pointings or reversed pointings (for the definitions of left alternating pointing and reversed pointing, please refer to [10, Definition 2.18]). Our aim is that for any pointed Brauer tree $G(p)$ and the Rickard-Schaps tree-to-star complex given by the pointed Brauer tree $G(p)$, we give a required sequence of mutations from the pointed Brauer tree $G(p)$. We will give the solution of this problem in Theorem 4.1.

2.3. Tilting complexes given by pointed Brauer trees. In [9], it was shown that there is a one-to-one correspondence between the set of multiplicity-free two-restricted tilting complexes for the Brauer star algebra of type (e, m) and the set of pointed Brauer trees of type (e, m).

First we give the definition of the pointings and the pointed Brauer trees.

Definition 2.11 ([9]). A pointing of a Brauer tree consists of the choice of one sector at each non-exceptional vertex. Then we give a point in that sector for indication. We call the resulting tree with this additional structure a pointed Brauer tree.

We denote a pointed Brauer tree of a Brauer tree G by $G(p)$. Also for a pointed Brauer tree $G(p)$ we denote the Brauer tree which forsakes the pointing of $G(p)$ by G.

Remark 2.12. For the pointed Brauer tree $G(p)$, we give the one-to-one correspondence among the set of the points, the set of non-exceptional vertices, and the set of the edges as follows. For a point of $G(p)$, let the corresponding non-exceptional vertex be the non-exceptional vertex which the point is on. For an edge of $G(p)$, let the corresponding non-exceptional vertex be the farther from the exceptional vertex of the non-exceptional vertices on the end of the edge. We easily see that these correspondences give one-to-one correspondence among the three sets.

Next we recall the definition of two-restricted tilting complexes.

Definition 2.13 ([9]). Let $\hat{T}$ be a tilting complex over a Brauer star algebra B. We call $\hat{T}$ a two-restricted tilting complex if any indecomposable direct summand of $\hat{T}$ is a shift of the following elementary complex, where the leftmost nonzero term is in degree 0.

(1) $S_i : 0 \to Q_i \to 0$,
(2) $T_{jk} : 0 \to Q_j \xrightarrow{h_{jk}} Q_k \to 0$,
where the map h_{jk} has maximal rank among homomorphisms from Q_j to Q_k.

In [9], it was shown that there is a one-to-one correspondence between the set of multiplicity-free two-restricted tilting complexes for the Brauer star algebra of type (e, m) and the set of pointed Brauer trees of type (e, m). We give the construction of the two-restricted tilting complexes for the Brauer star algebra based on [9]. To give the construction, we first give the definition of the vertex numbering.

Definition 2.14 ([9, Definition 4.1]). Let $G(p)$ be a pointed Brauer tree. Then we number each edge in the following way. We call the resulting numbering for the all edges the vertex numbering.

(1) Pick an arbitrary branch at the exceptional vertex as a starting point, and we number the exceptional vertex 0.
(2) Taking Green's walk defined in [4] around the tree in the cyclic ordering, we assign a number to each vertex whenever the corresponding point is reached.
(3) We number each edge the same number as the corresponding vertex (see Remark 2.12).

In [9], the authors introduced an algorithm constructing two-restricted tilting complexes over Brauer star algebras from pointed Brauer trees by using vertex numberings. We explain the algorithm based on [9].

Algorithm 2.15 ([9]). Let $G(p)$ be a pointed Brauer tree of type (e, m). We define a complex $\hat{T}_i$ inductively on the distance from the exceptional vertex as follows, and put $\hat{T} = \bigoplus_{i=1}^{e} \hat{T}_i$. Then $\hat{T}$ is a two-restricted tilting complex over a Brauer star algebra B of type (e, m) with endomorphism algebra the Brauer tree algebra associated to the Brauer tree G.

(1) For an edge i adjacent to the exceptional vertex, let $\hat{T}_i$ be the stalk complex $0 \to Q_i \to 0$, where Q_i is in degree 0 and where $B = \bigoplus_{i=1}^{e} Q_i$.
(2) For an edge i not adjacent to the exceptional vertex, let $i_1, i_2, \ldots, i_{n-1}, i_n = i$ be the minimal path from the exceptional vertex to the edge i, and assume that we have $\hat{T}_{i_{n-1}}$. Let $f(i_j)$ be the vertex numbering of i_j for each j. Then we distinguish two cases.
  (2.a) If $f(i_{n-1}) > f(i)$, we set $\hat{T}_i = (0 \to Q_{i_{n-1}} \to Q_i \to 0)[l_n]$, where $[l_n]$ is the shift required to ensure that $Q_{i_{n-1}}$ is in the same degree in $\hat{T}_{i_{n-1}}$ and $\hat{T}_i$.
  (2.b) If $f(i_{n-1}) < f(i)$, we set $\hat{T}_i = (0 \to Q_i \to Q_{i_{n-1}} \to 0)[l_n]$, where again $[l_n]$ is the shift required to ensure that $Q_{i_{n-1}}$ is in the same degree in $\hat{T}_{i_{n-1}}$ and $\hat{T}_i$.

Given a pointed Brauer tree, in [9], a two-restricted star-to-tree tilting complex $\hat{T}$ is given, which is unique up to cyclic permutations of the Brauer star. For this two-restricted star-to-tree tilting complex $\hat{T}$ corresponding to the pointed Brauer tree, in [8], Rickard-Schaps give the construction of a folded tree-to-star complex T such that T induces an equivalence inverse to the one induced by $\hat{T}$. The Rickard-Schaps tree-to-star complexes are given as follows.

Algorithm 2.16 ([8]). Let $G(p)$ be a pointed Brauer tree of type (e, m) and let $A = A_G$ be a Brauer tree algebra associated to G. We define a complex T_i over A inductively on the distance from the exceptional vertex as follows, and put $T = \bigoplus_{i=1}^{e} T_i$. Then T is a tree-to-star complex which induces an inverse equivalence of the one induced by the two-restricted star-to-tree complex corresponding to the pointed Brauer tree $G(p)$.

(1) For an edge i adjacent to the exceptional vertex, let T_i be the stalk complex $0 \to P_i \to 0$, where P_i is in degree 0, where $A = \bigoplus_{i=1}^{e} P_i$.

(2) For an edge i not adjacent to the exceptional vertex, let $i_1, i_2, \ldots, i_{n-1}, i_n = i$ be the minimal path from the exceptional vertex to the edge i, and let $f(i_j)$ be the vertex numbering of i_j for each j. Then we distinguish two cases.

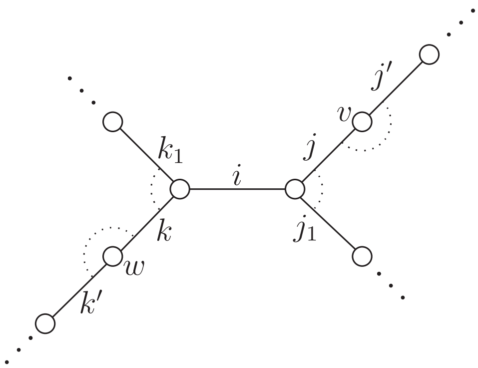
(2.a) If $f(i_{n-1}) > f(i)$, we set $T_i = (0 \to P_i \to T_{i_{n-1}} \to 0)$, where the map is induced by a nonzero homomorphism from $P_i \to P_{i_{n-1}}$ (the map is unique up to a scalar by Proposition 2.2 since the vertex which the edges i_{n-1} and i have in common is not a exceptional vertex).

(2.b) If $f(i_{j-1}) < f(i_j)$, we set $T_i = (0 \to T_{i_{n-1}} \to P_i \to 0)$, where the map is induced by a nonzero homomorphism from $P_{i_{n-1}} \to P_i$ (the map is again unique up to a scalar by Proposition 2.2 since the vertex which the edges i_{n-1} and i have in common is not a exceptional vertex).

3. Algorithm for a sequence of mutations

3.1. Kauer move for a pointed Brauer tree. In this section, we introduce a *Kauer move for a pointed Brauer tree* to give a sequence of mutations converting a Brauer tree algebra to the Rickard-Schaps tree-to-star complex.

Definition 3.1. We consider the following situation. Let $G(p)$ be a pointed Brauer tree of a Brauer tree G. For an edge i of $G(p)$, let $(j_1, \ldots, j_n = j, i, j_1)$ and $(k_1, \ldots, k_m = k, i, k_1)$ be cyclic orderings of the two vertices adjacent to the edge i. Let v, w be the vertices of the edges j, k, respectively, which are not adjacent to the edge i. Let j', k' be the next edge before j, k in the cyclic orderings at v, w, respectively.



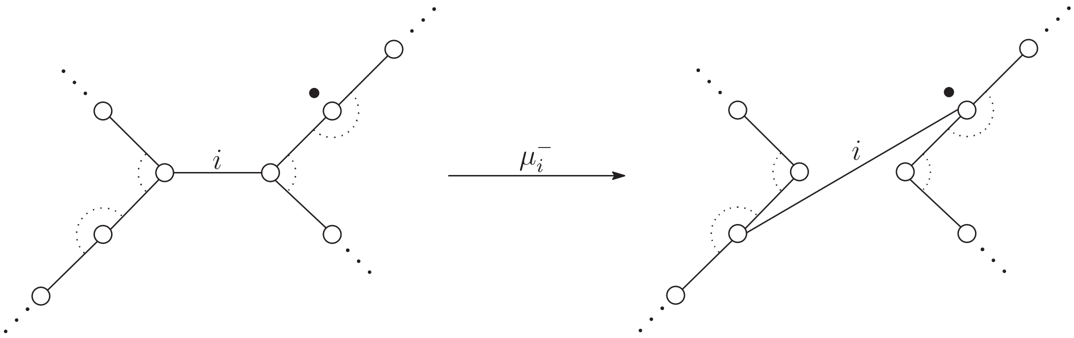
Then we define a new pointed Brauer tree $\mu_i^-(G(p))$ with the following properties.

(1) As a Brauer tree without pointing, $\mu_i^-(G(p)) = \mu_i^-(G)$.

(2) (a) Let $r(v)$ be a point on the vertex v.

(i) If $r(v)$ is in the sector (j, j') in $G(p)$, then $r(v)$ is in the same sector in $\mu_i^-(G(p))$.

(ii) If $r(v)$ is in the sector between (j', j) in $G(p)$, then the point $r(v)$ in $\mu_i^-(G(p))$ is in the sector (j', i).



(b) Let $r(w)$ be a point on the vertex w. We put the point $r(w)$ in $\mu_i^-(G(p))$ in the same way as we put $r(v)$ in $\mu_i^-(G(p))$.

(c) Any other point in $\mu_i^-(G(p))$ is in the same sector as the point in $G(p)$.

We call this local move a Kauer move for the pointed Brauer tree at i.

Let $A = A_G$ be a Brauer tree algebra associated to a Brauer tree G. For a pointed Brauer tree $G(p)$ of the Brauer tree G, we get the Rickard-Schaps tree-to-star complex T from the pointed Brauer tree $G(p)$. Since the homotopy category of the Brauer tree algebra A is tilting-connected by Theorem 2.10, there exists a sequence of mutations $(\mu_{i_n}^{\epsilon_n}, \dots, \mu_{i_2}^{\epsilon_2}, \mu_{i_1}^{\epsilon_1})$ satisfying that $(\mu_{i_n}^{\epsilon_n} \cdots \mu_{i_2}^{\epsilon_2}\mu_{i_1}^{\epsilon_1})(A) \cong T$, where ϵ_l means $+$ or $-$ for each l. To find this sequence of mutations, we give a following algorithm which will give us such a sequence of mutations for the pointed Brauer tree $G(p)$.

ALGORITHM 3.2. Let G be a Brauer tree, and $G(p)$ a pointed Brauer tree of the Brauer tree G.

1. Fix an arbitrary branch and denote all edges to which belong the branch by $S_1, \dots, S_k$.
2. Take a Green's walk around $G(p)$ so that it meets another edge after meeting all $S_1, \dots, S_k$. Then we have the first vertex that one would meet the walk of the all vertices whose corresponding edges belong to branches which are not leaves. If the edge corresponding to the vertex is not adjacent to the exceptional vertex, we let j_1 be the edge and let $\epsilon(j_1)$ be $-$. If the edge is adjacent to the exceptional vertex, taking a *reverse* Green's walk around $G(p)$ so that it meets another edge after meeting all $S_1, \dots, S_k$, we have the edge corresponding to the first vertex with the same property and we let j_1 the edge and let $\epsilon(j_1)$ be $+$.
3. Assume we have a sequence of mutations $(\mu_{j_{l-1}}^{\epsilon(j_{l-1})}, \dots, \mu_{j_2}^{\epsilon(j_2)}, \mu_{j_1}^{\epsilon(j_1)})$. Then we take the same process as 2 for the pointed Brauer tree $(\mu_{j_{l-1}}^{\epsilon(j_{l-1})} \cdots \mu_{j_2}^{\epsilon(j_2)}\mu_{j_1}^{\epsilon(j_1)})(G(p))$ which is defined in Definition 3.1, and we get the edge j_l and the sign $\epsilon(j_l)$.
4. We repeat the process 3 until $(\mu_{j_n}^{\epsilon(j_n)} \cdots \mu_{j_2}^{\epsilon(j_2)}\mu_{j_1}^{\epsilon(j_1)})(G(p))$ gets to the Brauer star.
5. Putting $i_k := j_{n+1-k}$ and $\epsilon_k := \epsilon(j_{n+1-k})$, we get a sequence $(\mu_{i_n}^{\epsilon_n}, \dots, \mu_{i_2}^{\epsilon_2}, \mu_{i_1}^{\epsilon_1})$ of irreducible mutations with $(\mu_{i_1}^{\epsilon_1}\mu_{i_2}^{\epsilon_2} \cdots \mu_{i_n}^{\epsilon_n})(G(p))$ a Brauer star.

On a sequences of mutations obtained from Algorithm 3.2, we prepare a permutation σ_l associated to $\mu_{i_l}^{\epsilon_l}$ and Lemma 3.4 about this permutation which will be helpful in proving Theorem 4.1.

Notation 3.3. Let $A = A_n$ be a Brauer tree algebra associated to a Brauer tree G and let $G(p)$ be a pointed Brauer tree of the Brauer tree G and let $(\mu_{i_1}^{\epsilon_1}, \mu_{i_2}^{\epsilon_2}, \dots, \mu_{i_n}^{\epsilon_n})$ be a sequence of mutations obtained from $G(p)$ by Algorithm 3.2 and let A_k be the opposite algebra of endomorphism algebra of the tilting complex $(\mu_{k+1}^{\epsilon_{k+1}}\mu_k^{\epsilon_k}\cdots\mu_n^{\epsilon_n})(A_n)$ for $0 \leq k \leq n-1$:

$$A_n \xrightarrow{\mu_{i_n}^{\epsilon_n}} A_{n-1} \to \cdots \to A_1 \xrightarrow{\mu_{i_1}^{\epsilon_1}} A_0.$$

Then, for each $1 \leq l \leq n$ and for the mutation $A_l \xrightarrow{\mu_{i_l}^{\epsilon_l}} A_{l-1}$ we define a permutation σ_l associated to $\mu_{i_l}^{\epsilon_l}$ of the indices $\{1, 2, \dots, e\}$ of A_l as follows:

(1) If i_l is a leaf, we let σ_l be the identity map.

(2) If i_l is not a leaf and $\epsilon_l = -$, then we put $\sigma_l = (i_l\ \ y)$, where y is the edge which is next before i_l in the cyclic ordering at the vertex corresponding to i_l in the Brauer tree of A_{i_l}

(3) If i_l is not a leaf and $\epsilon_l = +$, then we put $\sigma_l = (i_l\ \ y)$, where y is the edge which is next after to i_l in cyclic ordering at the vertex corresponding to i_l in the Brauer tree of A_{i_l}

Lemma 3.4. *Let A be a Brauer tree algebra associated to a Brauer tree G and let $G(p)$ be a pointed Brauer tree of the Brauer tree G. For the pointed Brauer tree $G(p)$, let $(\mu_{i_1}^{\epsilon_1}, \mu_{i_2}^{\epsilon_2}, \dots, \mu_{i_n}^{\epsilon_n})$ be a sequence of mutations obtained from* Algorithm 3.2*:*

$$A = A_n \xrightarrow{\mu_{i_n}^{\epsilon_n}} A_{n-1} \to \cdots \to A_2 \xrightarrow{\mu_{i_2}^{\epsilon_2}} A_1 \xrightarrow{\mu_{i_1}^{\epsilon_1}} A_0.$$

Moreover let $f : \{S_1, S_2, \dots, S_e\} \to \{1, 2, \dots, e\}$ be the bijection mapping each index of an edge to the edge numbering for the pointed Brauer tree $G(p)$ (see Definition 2.14). *Let σ_k be a permutation associated to $\mu_{i_k}^{\epsilon_k}$, and put $\sigma^{(n)} := \sigma_1\sigma_2\cdots\sigma_n$. Then the cyclic ordering of the Brauer star $(\mu_{i_1}^{\epsilon_1}\mu_{i_2}^{\epsilon_2}\cdots\mu_{i_n}^{\epsilon_n})(G(p))$ is given by $< \sigma^{(n)}f^{-1}(1), \sigma^{(n)}f^{-1}(2), \dots, \sigma^{(n)}f^{-1}(e) >$.*

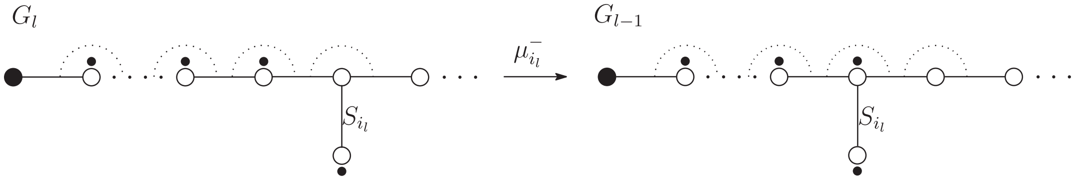
Proof. For $0 \leq l \leq n-1$, let $f_l : \{S_1, S_2, \dots, S_e\} \to \{1, 2, \dots, e\}$ be the bijection mapping each index of an edge of $(\mu_{i_{l+1}}^{\epsilon_{l+1}}\mu_{i_l}^{\epsilon_l}\cdots\mu_{i_n}^{\epsilon_n}(G(p)))$ to the vertex numbering of the pointed Brauer tree in Definition 2.14, and put $f_n = f$, where we determine each f_l by the following Green's walk (recall the vertex numbering in Definition 2.14 depends on the choice of the branch Green's walk starts at):

- The case $G(p)$ has at least two branches:
  - We define each function f_l by the Green's walk with the same property as the one in the processes 2 and 3 of Algorithm 3.2.
- The case $G(p)$ has only one branch:
  - If $(\mu_{i_k}^{\epsilon_k}\mu_{i_{k-1}}^{\epsilon_{k-1}}\cdots\mu_{i_n}^{\epsilon_n}(G(p)))$ has only one branch and $(\mu_{i_{k+1}}^{\epsilon_{k+1}}\mu_{i_k}^{\epsilon_k}\cdots\mu_{i_n}^{\epsilon_n}(G(p)))$ has two branches, then we define the function f_k by the Green's walk around $(\mu_{i_{k+1}}^{\epsilon_{k+1}}\mu_{i_k}^{\epsilon_k}\cdots\mu_{i_n}^{\epsilon_n}(G(p)))$ starting at the branch to which the edge i_{k+1} belong. Then we denote all edges which belong to the branch by $S_{r_1}, \dots, S_{r_k}$
  - If both $(\mu_{i_l}^{\epsilon_l}\mu_{i_{l-1}}^{\epsilon_{l-1}}\cdots\mu_{i_n}^{\epsilon_n}(G(p)))$ and $(\mu_{i_{l+1}}^{\epsilon_{l+1}}\mu_{i_l}^{\epsilon_l}\cdots\mu_{i_n}^{\epsilon_n}(G(p)))$ have at least two branches, we define the function f_l by the Green's walk around $(\mu_{i_{l+1}}^{\epsilon_{l+1}}\mu_{i_l}^{\epsilon_l}\cdots\mu_{i_n}^{\epsilon_n}(G(p)))$ such that it meets another edge after meeting all $S_{r_1}, \dots, S_{r_k}$.

In order to prove the lemma, we show that $f_{l-1}^{-1} = \sigma_l f_l^{-1}$ for each $1 \leq l \leq n$.

Case 1. i_l is a leaf.

We assume $\epsilon_l = -$.

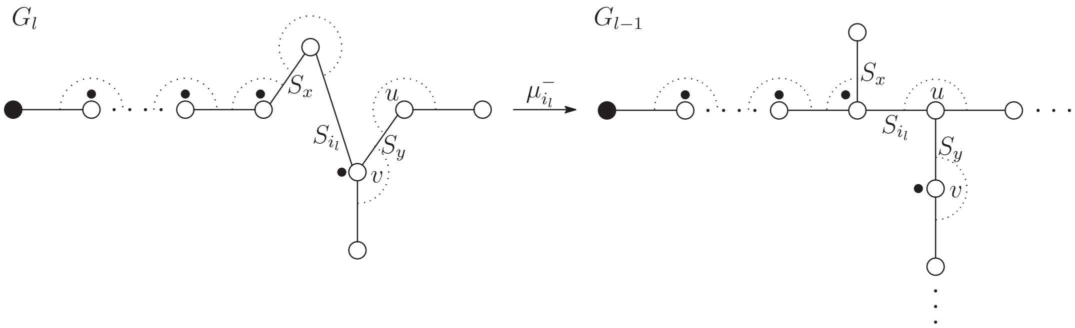


In this case, we have σ_l is the identity map by the definition of σ_l (see Notation 3.3). Also by the definition of the construction of the sequence of mutations, in the minimal paths from the exceptional vertex to the vertex corresponding to i_l in both G_l and G_{l-1}, all the points are on the left on the minimal path because we choose i_l to correspond to the first point on the right in the Green's walk.

Hence we have $f_l(S_{i_l}) = f_{l-1}(S_{i_l})$. Also, for any edge S_j except S_{i_l}, we clearly have $f_l(S_j) = f_{l-1}(S_j)$. Hence we have $f_l = f_{l-1}$ which implies $f_{l-1}^{-1} = id\, f_l^{-1} = \sigma_l f_l^{-1}$ as claimed. The dual argument shows that the statement for $\epsilon_l = +$ holds.

Case 2. i_l is not a leaf.

We assume $\epsilon_l = -$.



Let v be the vertex corresponding to the edge S_{i_l} in G_l, and let S_y be the edge which is next before S_{i_l} in the cyclic ordering at v in G_l, and let u be the vertex corresponding to the edge S_y in G_l. Then we have $\sigma_l = (i_l\ \ y)$ by the definition. Also by the definition of the construction of the sequence of mutations, in the minimal paths from the exceptional vertex to v in both G_l and G_{l-1}, all the points are on the left. Hence the edge numberings of S_{i_l} in G_l and of S_y in G_{l-1} are the minimal numbers in the branches, which implies that $f_l(S_{i_l}) = f_{l-1}(S_y)$. Also, we clearly have both numbers corresponding to u in G_l and in G_{l-1} are equal in whichever sector the point corresponding to the vertex u are. Hence we have $f_l(S_y) = f_{l-1}(S_{i_l})$. For another vertex we have the numbers of the vertex in both G_l and G_{l-1} are equal, hence for any vertex except v and u, we clearly have the edges corresponding to the vertex in both G_l and G_{l-1} have equal edge numberings. Hence we have $f_l(S_m) = f_{l-1}(S_m)$ for any edge S_m except S_{i_l} and S_y.

Thus we have that $f_l(S_y) = f_{l-1}(S_{i_l})$, $f_l(S_{i_l}) = f_{l-1}(S_y)$ and $f_l(S_m) = f_{l-1}(S_m)$ for any edge S_m except S_y and S_{i_l}. Hence we have $f_l = f_{l-1}\sigma_l$, which implies that $f_l^{-1} = \sigma_l^{-1} f_{l-1}^{-1} = \sigma_l f_{l-1}^{-1}$, as claimed.

The dual argument shows that the statement for $\epsilon_l = +$ holds.

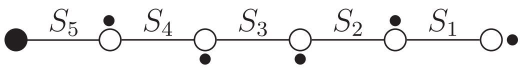
Now we have proven that $f_{l-1}^{-1} = \sigma_l f_l^{-1}$ for each $1 \leq l \leq n$. On the other hand, for $0 \leq k \leq n-1$, putting $G_k := (\mu_{i_{k+1}}^{\epsilon_{k+1}} \mu_{i_k}^{\epsilon_k} \cdots \mu_{i_n}^{\epsilon_n})(G)$ which is the Brauer tree of A_k and putting $G_n = G$, we have

$$\begin{aligned}
\text{the cyclic ordering of } G_0 &= \text{the cyclic ordering of } \mu_{i_1}^{\epsilon_1}(G_1)\\
&= \text{the cyclic ordering of } \mu_{i_1}^{\epsilon_1}\mu_{i_2}^{\epsilon_2}(G_2)\\
&= \cdots\\
&= \text{the cyclic ordering of } \mu_{i_1}^{\epsilon_1}\mu_{i_2}^{\epsilon_2}\cdots\mu_{i_n}^{\epsilon_n}(G_n).
\end{aligned}$$

Therefore we have that

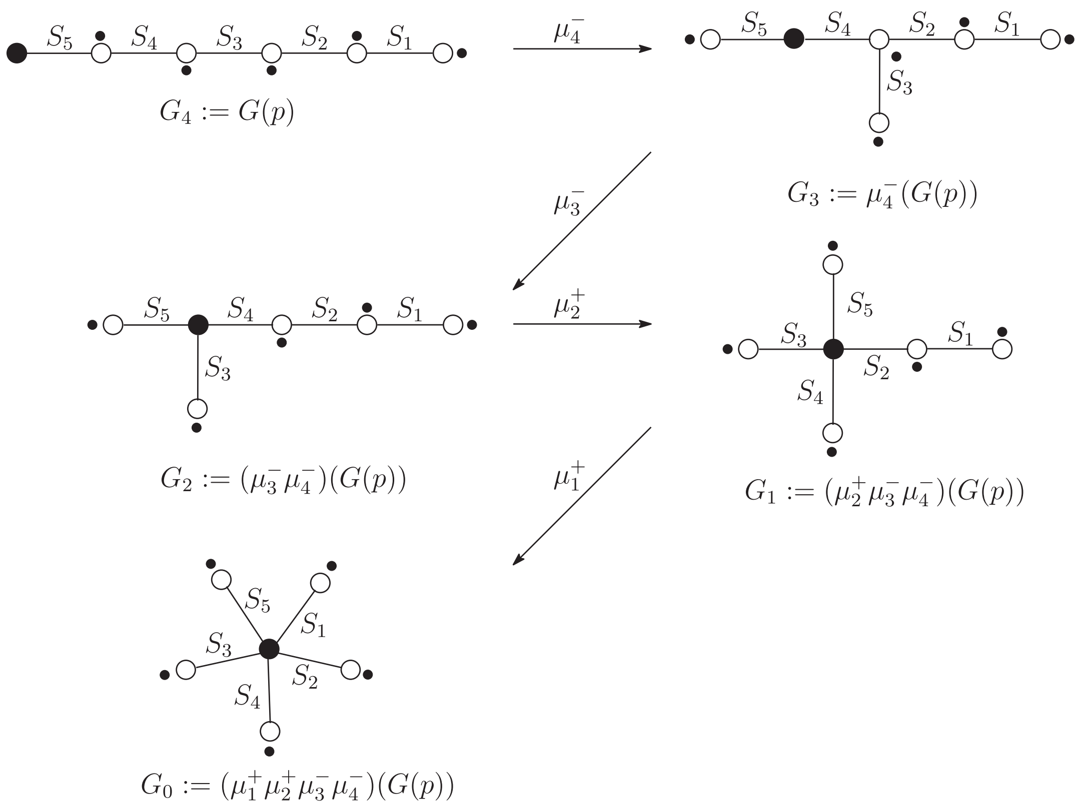
$$\begin{aligned}
\text{the cyclic ordering of } G_0 &=< f_0^{-1}(1), f_0^{-1}(2), \ldots, f_0^{-1}(e) >\\
&=< \sigma_1 f_1^{-1}(1), \sigma_1 f_1^{-1}(2), \ldots, \sigma_1 f_1^{-1}(e) >\\
&=< \sigma_1\sigma_2 f_2^{-1}(1), \sigma_1\sigma_2 f_2^{-1}(2), \ldots, \sigma_1\sigma_2 f_2^{-1}(e) >\\
&= \cdots\\
&=< \sigma_1\sigma_2\cdots\sigma_n f_n^{-1}(1), \sigma_1\sigma_2\cdots\sigma_n f_n^{-1}(2), \ldots, \sigma_1\sigma_2\cdots\sigma_n f_n^{-1}(e) > . \qquad \square
\end{aligned}$$

Example 3.5. We give an example of Lemma 3.4. We consider the following pointed Brauer tree.



$G(p)$

By Algorithm 3.2, we have a sequence $(\mu_1^+, \mu_2^+, \mu_3^-, \mu_4^-)$ of mutations and the following pointed Brauer trees (the detailed calculation can be seen in Section 5):



In this setting, the function $f_i : \{S_1, \dots, S_5\} \to \{1, \dots, 5\}$ is defined as follows for each $0 \le i \le 4$:

$$f_4(S_1) = 3, f_4(S_2) = 4, f_4(S_3) = 2, f_4(S_4) = 1, f_4(S_5) = 5,$$

$$f_3(S_1) = 3, f_3(S_2) = 4, f_3(S_3) = 1, f_3(S_4) = 2, f_3(S_5) = 5,$$

$$f_2(S_1) = 3, f_2(S_2) = 4, f_2(S_3) = 1, f_2(S_4) = 2, f_2(S_5) = 5,$$

$$f_1(S_1) = 4, f_1(S_2) = 3, f_1(S_3) = 1, f_1(S_4) = 2, f_1(S_5) = 5,$$

$$f_0(S_1) = 4, f_0(S_2) = 3, f_0(S_3) = 1, f_0(S_4) = 2, f_0(S_5) = 5.$$

In particular, the fuction f in the notation of Lemma 3.4 is f_4 in this setting. Moreover the permutations corresponding $\mu_4^-, \mu_3^-, \mu_2^+, \mu_1^+$ are $\sigma_4 := (3\ \ 4), \sigma_3 := id, \sigma_2 := (1\ \ 2), \sigma_1 := id$ respectively (see Notation 3.3). The cyclic ordering of G_0 is given by $< S_3, S_4, S_2, S_1, S_5 >$. It is certain that this cyclic ordering is equal to

$$< (1\ 2)(3\ 4)f^{-1}(1), (1\ 2)(3\ 4)f^{-1}(2), (1\ 2)(3\ 4)f^{-1}(3), (1\ 2)(3\ 4)f^{-1}(4), (1\ 2)(3\ 4)f^{-1}(5) > .$$

4. Main Theorem

Theorem 4.1. *Let A be a Brauer tree algebra associated to a Brauer tree G, let $G(p)$ be a pointed Brauer tree of G and let T be a Rickard-Schaps tree-to-star complex obtained from* *the pointed Brauer tree $G(p)$. Then for a sequence of mutations $(\mu_{i_1}^{\epsilon_1}, \mu_{i_2}^{\epsilon_2}, \dots, \mu_{i_n}^{\epsilon_n})$ obtained by the* Algorithm 3.2*, we have $(\mu_{i_1}^{\epsilon_1} \cdots \mu_{i_{n-1}}^{\epsilon_{n-1}} \mu_{i_n}^{\epsilon_n})(A) \cong T$.*

Proof. We defined a Kauer move for a pointed Brauer tree (see Definition 3.1). By using this definition, we denote a pointed Brauer tree $(\mu_{i_{k+1}}^{\epsilon_{k+1}} \cdots \mu_{i_{n-1}}^{\epsilon_{n-1}} \mu_{i_n}^{\epsilon_n})(G(p))$ by $G_k(p)$, where $G_n(p) = G(p)$. Then we remark that $G_0(p)$ is the pointed Brauer tree of the Brauer star. Moreover we let A_k be a Brauer tree algebra associated to the Brauer tree G_k of which $G_k(p)$ is the pointed Brauer tree.

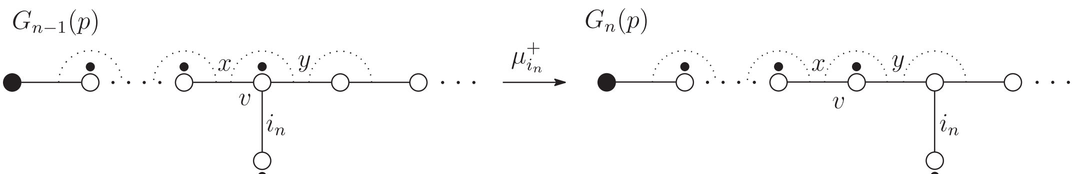
If G is a Brauer tree whose all edges except one edge i_1 are adjacent to the exceptional vertex, then the statement holds (see the proof of Theorem 4.1 in [10] and its dual argument). In this case, T is isomorphic to $\mu_{i_1}^{\epsilon_1}(A)$. We assume as our induction that the statement holds if the sequence of mutations is length $n-1$. Then for a sequence of mutations and Brauer tree algebras which reaches the Brauer star algebra A_0

$$A_{n-1} \xrightarrow{\mu_{i_{n-1}}^{\epsilon_{n-1}}} A_{n-2} \to \cdots \to A_1 \xrightarrow{\mu_{i_1}^{\epsilon_1}} A_0,$$

it holds that $(\mu_{i_1}^{\epsilon_1} \cdots \mu_{i_{n-1}}^{\epsilon_{n-1}})(A) \cong T^{(n-1)}$, where $T^{(n-1)}$ is the Rickard-Schaps tree-to-star complex obtained from the pointed Brauer tree $G_{n-1}(p)$. In particular, since $\mathrm{End}_{D^b(A_{n-1})}(T^{(n-1)}) \cong B^{op}$ where B is the basic Brauer star algebra Morita equivalent to A_0 with cyclic ordering of the exceptional vertex $< f_{n-1}^{-1}(1), f_{n-1}^{-1}(2), \dots, f_{n-1}^{-1}(e) >$, by Lemma 3.4, for each $1 \leq j \leq e$, the j-th component of $(\sigma_1\sigma_2 \cdots \sigma_{n-1})(\mu_{i_1}^{\epsilon_1} \cdots \mu_{i_{n-1}}^{\epsilon_{n-1}})(A_{n-1})$ is isomorphic to the j-th component of $T^{(n-1)}$, where each σ_j is a permutation defined in Notation 3.3. In other words, for the indecomposable summand $P_j^{(n-1)}$ of A_{n-1} and $T_j^{(n-1)}$ of $T^{(n-1)}$, it holds that $(\mu_{i_1}^{\epsilon_1} \cdots \mu_{i_{n-1}}^{\epsilon_{n-1}})(P_{\sigma_1\sigma_2\cdots\sigma_{n-1}(j)}^{(n-1)}) \cong T_j^{(n-1)}$. From the isomorphism $(\sigma_1\sigma_2 \cdots \sigma_{n-1})(\mu_{i_1}^{\epsilon_1} \cdots \mu_{i_{n-1}}^{\epsilon_{n-1}})(A_{n-1}) \cong T^{(n-1)}$, we have
$(\sigma_{n-1} \cdots \sigma_2\sigma_1)(\mu_{i_{n-1}}^{-\epsilon_{n-1}} \cdots \mu_{i_2}^{-\epsilon_2}\mu_{i_1}^{-\epsilon_1})(A_0) \cong \hat{T}^{(n-1)}$, where $\hat{T}^{(n-1)}$ is the Schaps-Zakay star-to-tree complex obtained from the pointed Brauer tree $G_{n-1}(p)$ which induces the inverse equivalence to the tilting by the Rickard-Schaps tree-to-star complex $T^{(n-1)}$ obtained from the pointed Brauer tree $G_{n-1}(p)$ (we remark that $\sigma_j^{-1} = \sigma_j$ for each j since σ_j is a transposition). Then to prove the statement for n it is enough to show that $(\sigma_n\sigma_{n-1} \cdots \sigma_2\sigma_1)(\mu_{i_n}^{-\epsilon_n}\mu_{i_{n-1}}^{-\epsilon_{n-1}} \cdots \mu_{i_2}^{-\epsilon_2}\mu_{i_1}^{-\epsilon_1})(A_0)$ is isomorphic to the star-to-tree complex $\hat{T}^{(n)}$ obtained from $G_n(p)$ which induces the inverse equivalence to the tilting by the tree-to-star complex $T^{(n)}$.
Case 1. i_n is a leaf.

We prove the statement for $-\epsilon_n = +$ or equivalently $\epsilon_n = -$. The dual argument shows that the statement for the other case holds.

In the pointed Brauer tree $G_{n-1}(p)$ of A_{n-1}, let v be the closer vertex on both ends of i_n from the exceptional vertex and let $< i_n, y, \dots, x >$ be the cyclic ordering around v in $G_{n-1}(p)$.



Since $\mathrm{End}_{D^b(A_n)}(\mu_{i_n}^{-}(A_n)) \cong A_{n-1}^{op}$, by the choice of i_n and the definition of the Kauer move for the pointed Brauer tree, in the pointed Brauer tree $G_n(p)$ corresponding to A_n, the point which one first meets on a Green's walk in the branch is the one corresponding to i_n. Hence the point corresponding to the edge x is in the sector (y, x) in both $G_{n-1}(p)$ and $G_n(p)$. The star-to-tree complex given by the pointed Brauer tree is determined by the minimal path with points from the exceptional vertex to each edge, but we clearly have, for any edge j except i_n, the minimal path with points from the exceptional vertex to the vertex corresponding to j in $G_{n-1}(p)$ coincide with the one in $G_n(p)$. Hence we have $\bigoplus_{j\neq i_n} \hat{T}_j^{(n-1)} \cong \bigoplus_{j\neq i_n} \hat{T}_j^{(n)}$.

We prove that $\mu_{i_n}^+(\hat{T}_{i_n}^{(n-1)}) \cong \hat{T}_{i_n}^{(n)}$. Now we put $A_{n-1} = \bigoplus_i Q_i^{(n-1)}$, where each $Q_i^{(n-1)}$ is an indecomposable projective A_{n-1}-module. Then we have

$$\mu_{i_n}^+(Q_{i_n}^{(n-1)}) = [Q_y^{(n-1)} \to Q_{i_n}^{(n-1)}] = Cone \begin{bmatrix} Q_y^{(n-1)} \\ \downarrow \\ Q_{i_n}^{(n-1)} \end{bmatrix} [-1],$$

where $Q_y^{(n-1)}$ is in degree 0. Also for a derived equivalence $F_{n-1} : D^b(A_{n-1}) \to D^b(B)$ induced by $T^{(n-1)}$, it holds that $\hat{T}_{i_n}^{(n-1)} = F_{n-1}(Q_{i_n}^{(n-1)})$. Also for the vertex numberings $f_{n-1}(x)$, $f_{n-1}(y)$ and $f_{n-1}(i_n)$ of x, y and i_n in the pointed Brauer tree of A_{n-1}, we have $f_{n-1}(x) > f_{n-1}(y)$ and $f_{n-1}(x) > f_{n-1}(i_n)$, so we have $F_{n-1}(Q_y^{(n-1)}) \cong [Q_x \to Q_y]$ and $F_{n-1}(Q_{i_n}) \cong [Q_x \to Q_{i_n}]$ where each Q_i is an indecomposable B-module with $B = \bigoplus_i Q_i$ and where, in the respective complexes, Q_y and Q_{i_n} are in the degree $d^{(n-1)}(y) - 1 = d^{(n-1)}(i_n) - 1$. Thus we have

$$\begin{aligned}
\mu_{i_n}^+(\hat{T}_{i_n}^{(n-1)}) &= \mu_{i_n}^+(F_{n-1}(Q_{i_n}^{(n-1)})) \\
&= F_{n-1}(\mu_{i_n}^+(Q_{i_n}^{(n-1)})) \\
&= Cone \begin{bmatrix} F_{n-1}(Q_y^{(n-1)}) \\ \downarrow \\ F_{n-1}(Q_{i_n}^{(n-1)}) \end{bmatrix} [-1] \\
&= Cone \begin{bmatrix} Q_x & \to & Q_y \\ \| & & \downarrow \\ Q_x & \to & Q_{i_n} \end{bmatrix} [-1] \\
&= Q_y \to Q_{i_n}
\end{aligned}$$

where in the last complex the degree of Q_y is $d^{(n-1)}(y) - 1 + 1 - 1 = d^{(n-1)}(y) - 1$.

On the other hand, by the construction from the pointed Brauer tree $G_n(p)$ of A_n, we have

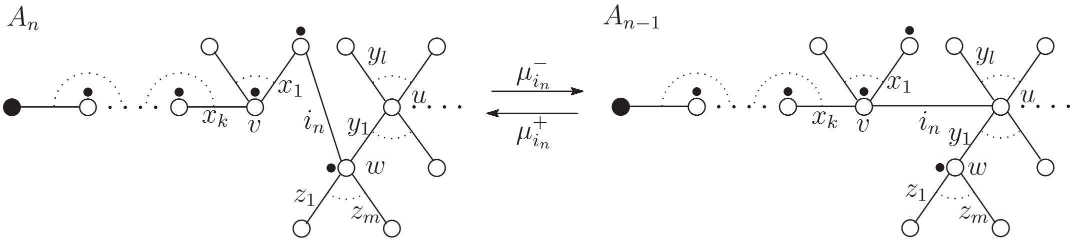
$$\begin{array}{ccccccc}
\hat{T}_y^{(n)} & = & Q_x & \to & Q_y & & \\
\hat{T}_{i_n}^{(n)} & = & & & Q_y & \to & Q_{i_n}.
\end{array}$$

where in $\hat{T}_{i_n}^{(n)}$, Q_y is in degree $d^{(n)}(y) - 1$. Now both minimal paths with points from the exceptional vertex to the vertex corresponding to y in A_{n-1} and A_n coincide, so we have $d^{(n)}(y) = d^{(n-1)}(y)$. Hence we have $\hat{T}_{i_n}^{(n)} \cong \mu_{i_n}^+(\hat{T}_{i_n}^{(n-1)})$. Therefore we conclude that $\hat{T}^{(n)} \cong \mu_{i_n}^+(\hat{T}^{(n-1)})$ in the case that i_n is a leaf.

Case 2. i_l is not a leaf.

We prove the statement for $-\epsilon_n = +$ or equivalently $\epsilon_n = -$. The dual argument shows that the statement for the other case holds.

First, for the proof, we fix the following notation for the pointed Brauer tree $G_{n-1}(p)$ of A_{n-1}. Of the two ends of the edge i_n in $G_{n-1}(p)$ we denote the vertex closer to the exceptional vertex by v. Let $< i_n, x_1, \dots, x_k >$ be the cyclic ordering at v in $G_{n-1}(p)$, where the edge corresponding to v is x_k. We denote the farther vertex from the exceptional vertex on both ends of the edge i_n in $G_{n-1}(p)$ by u. Let $< i_n, y_1, \dots, y_l >$ be the cyclic ordering at u. Let w be the vertex corresponding to y_1. Let $< y_1, z_1, \dots, z_m >$ be the cyclic ordering at w in $G_{n-1}(p)$.



Since $\mathrm{End}_{D^b(A_n)}(\mu^-_{i_n}(A_n)) \cong A^{op}_{n-1}$, by Definition 3.1 of a Kauer move for the pointed Brauer tree, the first point in the branch which we meet on a Green's walk around the pointed Brauer tree $G_n(p)$ is the point corresponding to i_n (see above figures). Also the point corresponding to x_k is in the sector (x_1, x_k) in both $G_n(p)$ and $G_{n-1}(p)$. By the Kauer move for the pointed Brauer tree, in the pointed Brauer tree $G_{n-1}(p)$, the point corresponding to y_1 is in the sector (y_1, z_1) and the point corresponding to i_n is in the sector (y_1, i_n). We remark that $\sigma_n = (i_n\ y_1)$ (see Notation 3.3). We prove that $\hat{T}^{(n)} \cong \mu^+_{i_n}(\mu^{-\epsilon_{n-1}}_{i_{n-1}} \cdots \mu^{-\epsilon_1}_{i_1}(A_0))$ under the assumption that $\mu^{-\epsilon_{n-1}}_{i_{n-1}} \cdots \mu^{-\epsilon_1}_{i_1}(A_0) \cong \hat{T}^{(n-1)}$.

By the construction of Schaps-Zakay star-to-tree complexes from pointed Brauer trees, $\hat{T}^{(n)}_{y_1} = [Q_{y_1} \to Q_{i_n}]$, where Q_{i_n} is in the degree $d^{(n)}(i_n) - 1$. Also by the construction, $\hat{T}^{(n-1)}_{y_1} = [Q_{i_n} \to Q_{y_1}]$, where Q_{y_1} is in degree $d^{(n-1)}(y) - 1 = d^{(n-1)}(i_n)$. Hence

$$\sigma_n(\hat{T}^{(n-1)}_{y_1}) = \sigma_n[Q_{i_n} \to Q_{y_1}] = [Q_{y_1} \to Q_{i_n}],$$

where in the last complex, Q_{i_n} is in degree $d^{(n-1)}(i_n)$. Since $d^{(n-1)}(i_n) = d^{(n)}(i_n) - 1$, we conclude $\hat{T}^{(n)}_{y_1} \cong \sigma_n(\hat{T}^{(n-1)}_{y_1})$.

Next we prove that $\hat{T}^{(n)}_{i_n} \cong \mu^+_{i_n}(\hat{T}^{(n-1)}_{i_n})$. By the construction of star-to-tree complex from the pointed Brauer tree $G_n(p)$, we have $\hat{T}^{(n)}_{i_n} = [Q_{x_1} \to Q_{i_n}]$, where Q_{x_1} is in degree $d^{(n)}(x_1) - 1$. We calculate $\mu^+_{i_n}(\hat{T}^{(n-1)}_{i_n})$. Put $A_{n-1} = \bigoplus Q^{(n-1)}_i$, where each $Q^{(n-1)}_i$ is an indecomposable projective A_{n-1}-module. Then we have $\mu^+_{i_n}(Q^{(n-1)}_{i_n}) = [Q^{(n-1)}_{x_1} \oplus Q^{(n-1)}_{y_1} \to Q^{(n-1)}_{i_n}]$, where $Q^{(n-1)}_{i_n}$ is in degree 1. Hence denoting a derived equivalence induced by $T^{(n-1)}$ by $F_{n-1} : D^b(A_{n-1}) \to D^b(A_0)$, since the image of $Q^{(n-1)}_{i_n}$ under the equivalence is $\hat{T}^{(n-1)}_{i_n}$, we have

$$\begin{aligned}
\mu^+_{i_n}(\hat{T}^{(n-1)}_{i_n}) &\cong \mu^+_{i_n}(F_{n-1}(Q^{(n-1)}_{i_n})) \\
&\cong F_{n-1}(\mu^+_{i_n}(Q^{(n-1)}_{i_n})) \\
&\cong F_{n-1}(Q^{(n-1)}_{x_1} \oplus Q^{(n-1)}_{y_1} \to Q^{(n-1)}_{i_n}) \\
&\cong F_{n-1}(Cone \begin{bmatrix} Q^{(n-1)}_{x_1} \oplus Q^{(n-1)}_{y_1} \\ \downarrow \\ Q^{(n-1)}_{i_n} \end{bmatrix})[-1]
\end{aligned}$$

$$\cong Cone(\left[\begin{array}{c} F_{n-1}(Q_{x_1}^{(n-1)}) \oplus F_{n-1}(Q_{y_1}^{(n-1)}) \\ \downarrow \\ F_{n-1}(Q_{i_n}^{(n-1)}) \end{array}\right])[-1]$$

Here, since $F_{n-1}(Q_{x_1}^{(n-1)}), F_{n-1}(Q_{y_1}^{(n-1)})$ and $F_{n-1}(Q_{i_n}^{(n-1)})$ are $\hat{T}_{x_1}^{(n-1)}, \hat{T}_{y_1}^{(n-1)}$ and $\hat{T}_{i_n}^{(n-1)}$ respectively, by the construction of Schaps-Zakay star-to-tree complexes from pointed Brauer trees, we have

$$\begin{array}{lllllll} F_{n-1}(Q_{x_1}^{(n-1)}) & = [Q_{x_k} & \to & \underline{Q_{x_1}} & \to & 0 & \to & 0] \\ F_{n-1}(Q_{y_1}^{(n-1)}) & = [0 & \to & \underline{Q_{i_n}} & \to & Q_{y_1} & \to & 0] \\ F_{n-1}(Q_{i_n}^{(n-1)}) & = [Q_{x_k} & \to & \underline{Q_{i_n}} & \to & 0 & \to & 0], \end{array}$$

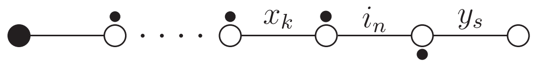
where the all underlined terms are in degree $d^{(n-1)}(x_1) - 1$. Hence we have

$$Cone(\left[\begin{array}{c} F_{n-1}(Q_{x_1}^{(n-1)}) \oplus F_{n-1}(Q_{y_1}^{(n-1)}) \\ \downarrow \\ F_{n-1}(Q_{i_n}^{(n-1)}) \end{array}\right])[-1] \cong [Q_{x_1} \to Q_{y_1}],$$

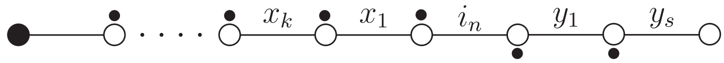
where Q_{x_1} is in the degree $d^{(n-1)}(x_1) - 1$. Thus we conclude that $\sigma_n(\mu_{i_n}^+(\hat{T}_{i_n}^{(n-1)})) \cong \hat{T}_{i_n}^{(n)}$.

Next we prove $\hat{T}_{y_s}^{(n)} \cong \sigma_n(\mu_{i_n}^+(\hat{T}_{y_s}^{(n-1)}))$ for $1 < s \leq l$. Since $\mu_{i_n}^+$ mutates only the indecomposable summand $\hat{T}_{i_n}$, we have only to show that $\hat{T}_{y_s}^{(n)} \cong \sigma_n(\hat{T}_{y_s}^{(n-1)})$. Suppose the point corresponding to y_1 is in the sector (y_1, y_s) in the pointed Brauer tree $G_n(p)$. Then the minimal paths from the exceptional vertex to the vertex corresponding to y_s in G_n and G_{n-1} are as follows.

$G_{n-1}(p)$



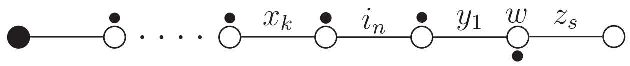
$G_n(p)$



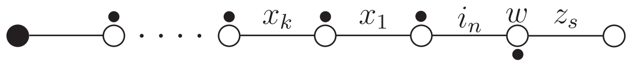
Then by the construction of star-to-tree complex from the pointed Brauer tree $G_{n-1}(p)$, we have that $\hat{T}_{y_s}^{(n-1)} = [Q_{y_s} \to Q_{i_n}]$, where Q_{i_n} is in the degree $d^{(n-1)}(i_n) - 1$, and that $\hat{T}_{y_s}^{(n)} = [Q_{y_s} \to Q_{y_1}]$, where Q_{y_1} is in the degree $d^{(n)}(i_n) - 2$. Here, since $d^{(n)}(i_n) = d^{(n-1)}(i_n) + 1$, we have $d^{(n)}(i_n) - 2 = d^{(n-1)}(i_n) - 1$. Also, since $\sigma_n = (i_n \ y_1)$, we have $\sigma_n(\hat{T}_{y_s}^{(n-1)}) \cong \hat{T}_{y_s}^{(n)}$. Similar argument shows the statement of the case the point corresponding to y_1 is in the sector (y_s, y_1) in the pointed Brauer tree $G_n(p)$ (in the case, the rightmost points in $G_{n-1}(p)$ and $G_n(p)$ in the above figure will be reversed).

Next we prove $\hat{T}_{z_s}^{(n)} \cong \sigma_n(\mu_{i_n}^+(\hat{T}_{z_s}^{(n-1)}))$ for $1 \leq s \leq m$. Similar to above argument, we prove that $\hat{T}_{z_s}^{(n)} \cong \sigma_n(\hat{T}_{z_s}^{(n-1)})$. Then the minimal paths from the exceptional vertex to the vertex corresponding to z_s in G_n and G_{n-1} are as follows.

G_{n-1}



G_n



Then by the construction of star-to-tree complex from the pointed Brauer trees $G_{n-1}(p)$ and $G_n(p)$, we have that $\hat{T}^{(n-1)}_{z_s} = [Q_{z_s} \to Q_{y_1}]$, where Q_{y_1} is in degree $d^{(n-1)}(y_1) - 1$, and that $\hat{T}^{(n)}_{z_s} = [Q_{z_s} \to Q_{i_n}]$, where Q_{i_n} is the degree $d^{(n)}(i_n) - 1$. Also since $d^{(n)}(i_n) = d^{(n-1)}(y_1)$, we have $\sigma_n(\hat{T}^{(n-1)}_{z_s}) \cong \hat{T}^{(n)}_{z_s}$.

Finally, we can easily see that the statement for another edge holds by the construction of star-to-tree complexes. □

For the pointed Brauer tree $G(p)$ of a Brauer tree G and Brauer tree algebra $A = A_G$, Rickard-Schaps tree-to-star complex of A-modules obtained from $G(p)$ induces an inverse equivalence to the one by the Schaps-Zakay star-to-tree complex obtained by $G(p)$. Hence we have the following by Theorem 4.1.

Corollary 4.2. *Let A be a Brauer tree algebra associated to a Brauer tree G and let $G(p)$ be a pointed Brauer tree of G and let $\hat{T}$ be Schaps-Zakay star-to-tree complex obtained from the pointed Brauer tree $G(p)$. Then for a sequence of mutations $(\mu^{\epsilon_1}_{i_1}, \mu^{\epsilon_2}_{i_2}, \dots, \mu^{\epsilon_n}_{i_n})$ obtained by the* Algorithm 3.2*, we have $(\mu^{-\epsilon_n}_{i_n} \cdots \mu^{-\epsilon_2}_{i_2} \mu^{-\epsilon_1}_{i_1})(B) \cong \hat{T}$ where B is the Brauer star algebra derived equivalent to A.*

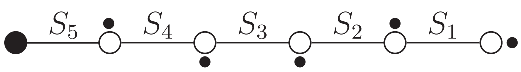
By the proof of Theorem 4.1, we get the following theorem.

Theorem 4.3. *Let G be a Brauer tree, $G(p)$ a pointed Brauer tree of G, $\mu^{\epsilon}_i(G(p))$ a pointed Brauer tree obtained by applying the Kauer move for pointed Brauer trees* (Definition 3.1)*, where $\epsilon \in \{+, -\}$, and $\hat{T}(G(p))$ a two-restricted star-to-tree complex corresponding to $G(p)$. Assume that the sum of all distance of the edges of $\mu^{\epsilon}_i(G(p))$ from the exceptional vertex is strictly smaller than that of $G(p)$. Then the star-to-tree complex obtained by applying the mutation μ^{ϵ}_i to $\hat{T}(G(p))$ is isomorphic to the star-to-tree complex corresponding to $\mu^{\epsilon}_i(G(p))$. In other words we get the following isomorphism:*

$$\mu^{\epsilon}_i(\hat{T}(G(p))) \cong \hat{T}(\mu^{\epsilon}_i(G(p))).$$

5. Example

We shall give an example. We shall use the following notation. Let G be the underlying Brauer tree of the following pointed Brauer tree $G(p)$, and let $A = A_G$ be the Brauer tree algebra associated to the Brauer tree G.



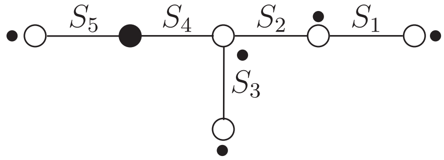
$G(p)$

For simplicity, we denote the projective cover of the simple A-module S_i by P_i for each $1 \leq i \leq 5$. By using this notation, let $A = \bigoplus_{i=1}^{5} P_i$ be a decomposition of A as a direct sum of indecomposable projective modules. The Rickard-Schaps tree-to-star complex T given by the pointed Brauer tree is as follows:

$$\begin{array}{ccccc}
P_4 & \to & P_1 \oplus P_3 \oplus P_5 & \to & P_2 \\
 & & \oplus & & \\
P_4 & \to & P_3 \oplus P_5 & \to & P_2 \\
 & & \oplus & & \\
P_4 & \to & P_3 \oplus P_5 & & \\
 & & \oplus & & \\
P_4 & \to & P_5 & & \\
 & & \oplus & & \\
 & & P_5 & &
\end{array}$$

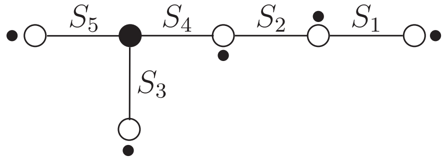
We shall give a sequence of mutations converting A to T.

To find such a sequence of mutations, we apply Algorithm 3.2. The first point the Green's walk around the Brauer tree $G(p)$ from the exceptional vertex would meet is the point corresponding to the edge S_4. Hence the first mutation of the required sequence is μ_4^-. The Kauer move for the pointed Brauer tree μ_4^- converts $G(p)$ to the following pointed Brauer tree (Definition 3.1):



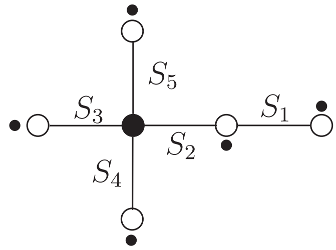
$\mu_4^-(G(p))$

Similarly, the first point the Green's walk around the Brauer tree $\mu_4^-(G(p))$ from the exceptional vertex would meet is the point corresponding to the edge S_3. Hence the second mutation of the required sequence is μ_3^-. The Kauer move for the pointed Brauer tree μ_3^- converts $\mu_4^-(G(p))$ to the following pointed Brauer tree:



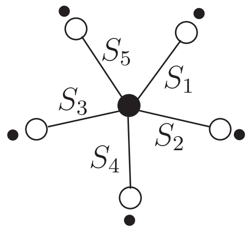
$(\mu_3^- \mu_4^-)(G(p))$

Next, we consider the third mutation of the required sequence. The first point the Green's walk around the Brauer tree $(\mu_3^-\mu_4^-)(G(p))$ from the exceptional vertex would meet is the point corresponding to the edge S_4. However the edge S_4 is adjacent to the exceptional vertex. Hence we take the reverse Green's walk instead of the Green's walk. The point the reverse Green's walk around the Brauer tree $(\mu_3^-\mu_4^-)(G(p))$ from the exceptional vertex would meet first is the point corresponding to S_2. Hence the third mutation is μ_2^+, and the mutation converts the pointed Brauer tree $(\mu_3^-\mu_4^-)(G(p))$ to the following pointed Brauer tree:



$$(\mu_2^+\mu_3^-\mu_4^-)(G(p))$$

Similarly we get the fourth mutation μ_1^+ converting the pointed Brauer tree $(\mu_2^+\mu_3^-\mu_4^-)(G(p))$ to the following pointed Brauer tree:



$$(\mu_1^+\mu_2^+\mu_3^-\mu_4^-)(G(p))$$

Hence by Theorem 4.1, we get $(\mu_1^+\mu_2^+\mu_3^-\mu_4^-)(A) \cong T$. In fact, we can check the following calculation process:

$$\begin{bmatrix} P_1 \\ \oplus \\ P_2 \\ \oplus \\ P_3 \\ \oplus \\ P_4 \\ \oplus \\ P_5 \end{bmatrix} \xrightarrow{\mu_4^-} \begin{bmatrix} & & P_1 \\ & & \oplus \\ & & P_2 \\ & & \oplus \\ & & P_3 \\ & & \oplus \\ P_4 & \to & P_3 \oplus P_5 \\ & & \oplus \\ & & P_5 \end{bmatrix} \xrightarrow{\mu_3^-} \begin{bmatrix} & & P_1 \\ & & \oplus \\ & & P_2 \\ & & \oplus \\ P_4 & \to & P_5 \\ & & \oplus \\ P_4 & \to & P_3 \oplus P_5 \\ & & \oplus \\ & & P_5 \end{bmatrix} \xrightarrow{\mu_2^+}$$

$$\begin{bmatrix} & & P_1 & & \\ & & \oplus & & \\ P_4 & \to & P_1 \oplus P_3 \oplus P_5 & \to & P_2 \\ & & \oplus & & \\ P_4 & \to & P_5 & & \\ & & \oplus & & \\ P_4 & \to & P_3 \oplus P_5 & & \\ & & \oplus & & \\ & & P_5 & & \end{bmatrix} \xrightarrow{\mu_1^+} \begin{bmatrix} P_4 & \to & P_3 \oplus P_5 & \to & P_2 \\ & & \oplus & & \\ P_4 & \to & P_1 \oplus P_3 \oplus P_5 & \to & P_2 \\ & & \oplus & & \\ P_4 & \to & P_5 & & \\ & & \oplus & & \\ P_4 & \to & P_3 \oplus P_5 & & \\ & & \oplus & & \\ & & P_5 & & \end{bmatrix}$$

We remark that the resulting tree-to-star complex $(\mu_1^+\mu_2^+\mu_3^-\mu_4^-)(A)$ coincides with the Rickard-Schaps tree-to-star complex T when ignoring the indices of direct summands of the complexes. We shall find a permutation to make the indices of $(\mu_1^+\mu_2^+\mu_3^-\mu_4^-)(A)$ coincide with that of T. By the proof of Theorem 4.1, we know that the permutation σ obtained by composing the permutations corresponding to $\mu_4^-, \mu_3^-, \mu_2^+$ and μ_1^+ (see Notation 3.3) converts the index of each projective module of $(\mu_1^+\mu_2^+\mu_3^-\mu_4^-)^{-1}(A_0) = (\mu_4^+\mu_3^+\mu_2^-\mu_1^-)(A_0)$ to that of $\hat{T}$, where $\hat{T}$ is the star-to-tree complex obtained from the pointed Brauer tree $G(p)$ which induces an inverse equivalence to the one by Rickard-Schaps tree-to-star complex T, and where A_0 is the Brauer star algebra associated to the Brauer star $\mu_1^+\mu_2^+\mu_3^-\mu_4^-(G)$. Since the indices of the projective modules of the Brauer star algebra correspond the indices of direct summands of tree-to-star complex, the inverse σ^{-1} of the permutation converts the indices of direct summands of $(\mu_1^+\mu_2^+\mu_3^-\mu_4^-)(A)$ to that of T. The permutations corresponding $\mu_4^-, \mu_3^-, \mu_2^+, \mu_1^+$ are $(3\ \ 4), id, (1\ \ 2), id$ respectively. Hence the permutation σ converting the indices of $(\mu_4^+\mu_3^+\mu_2^-\mu_1^-)(A_0)$ to those of $\hat{T}$ is $(3\ \ 4)(1\ \ 2)$. Hence the required permutation is $\sigma^{-1} = (1\ \ 2)(3\ \ 4)$. Thus we get that $(1\ \ 2)(3\ \ 4)(\mu_1^+\mu_2^+\mu_3^-\mu_4^-)(A)$ coincides with T completely. Indeed, putting $\mu := \mu_1^+\mu_2^+\mu_3^-\mu_4^-$, for the Rickard-Schaps tree-to-star complex $T = \bigoplus_{i=1}^5 T_i$ and for the complex $\bigoplus_{i=1}^5 \mu(P_i)$ it holds that $T_i \cong \mu(P_{(3\ 4)(1\ 2)(i)})$ for each $1 \leq i \leq 5$:

$$\left[\begin{array}{cccccc} T_1 = & P_4 & \to & P_1 \oplus P_3 \oplus P_5 & \to & P_2 \\ & & & \oplus & & \\ T_2 = & P_4 & \to & P_3 \oplus P_5 & \to & P_2 \\ & & & \oplus & & \\ T_3 = & P_4 & \to & P_3 \oplus P_5 & & \\ & & & \oplus & & \\ T_4 = & P_4 & \to & P_5 & & \\ & & & \oplus & & \\ T_5 = & & & P_5 & & \end{array}\right] \left[\begin{array}{cccccc} \mu(P_1) = & P_4 & \to & P_3 \oplus P_5 & \to & P_2 \\ & & & \oplus & & \\ \mu(P_2) = & P_4 & \to & P_1 \oplus P_3 \oplus P_5 & \to & P_2 \\ & & & \oplus & & \\ \mu(P_3) = & P_4 & \to & P_5 & & \\ & & & \oplus & & \\ \mu(P_4) = & P_4 & \to & P_3 \oplus P_5 & & \\ & & & \oplus & & \\ \mu(P_5) = & & & P_5 & & \end{array}\right].$$

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