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CONFORMAL CHANGES OF ODD-DIMENSIONAL GENERALIZED STRUCTURES

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Abstract. In this paper, we consider the integrability of generalized almost contact and contact manifolds after conformal changes. We also study conditions under which the generalized almost contact and normal generalized contact structures, be normal after conformal changes.

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1. Introduction

The notion of a generalized complex structure, introduced by Hitchin [3], is a geometric framework that unifies both complex and symplectic structures. Gualtieri has developed the theory of generalized complex structures and introduced generalized Kähler structures which come with additional conditions [2].

Vaisman introduced the odd-dimensional analog of these structures, generalized almost contact structures, and had defined generalized Sasakian structures from the viewpoint of generalized Kähler structures [7, 8]. He has also defined conformal changes of generalized complex structures and investigated invariant generalized geometry under conformal changes [6]. Poon and Wade have studied integrability conditions of generalized almost contact structures. This framework unifies almost contact, contact and cosymplectic structures [4, 5]. Even more, there is a more general context of generalized contact bundle that is introduced by Vitagliano and Wade [10], in which contact structures do not possess any global contact one-form.

Although Poon-Wade's generalized contact structures are special cases of generalized contact bundles, there are a lot of gaps that can be filled in many special cases, which definitely provide new ideas in more general cases.

In this paper, we consider integrability and normalization of a conformal change of odd-dimensional generalized structures.

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This paper is divided into four sections. In the next Section, we recall the needed background including definitions and theorems about generalized structures. In Section 3, we characterize generalized almost contact and contact manifolds to become integrable after a conformal change and we give an example of a generalized contact manifold which remains invariant under a nonhomothetic conformal change. In Section 4, we carry out a detailed study of geometric properties of normal generalized contact structures. We give geometric conditions expressing the normalization of a generalized almost contact structure. Then we use them to characterize the conformal changes of generalized almost contact and normal generalized contact structures. Also we give an example of a normal generalized contact structure which remains invariant under a nonhomothetic conformal change.

2. Preliminaries

Let M be a smooth manifold and consider the big tangent bundle $\mathbb{T}M = TM \oplus TM^*$. A natural inner product on $\mathbb{T}M = TM \oplus TM^*$ is defined by

$$\langle X+\alpha, Y+\beta\rangle = \frac{1}{2}(\beta(X)+\alpha(Y))$$

and the Courant bracket by

$$[\![X+\alpha, Y+\beta]\!] = [X,Y] + \pounds_X\beta - \pounds_Y\alpha - \frac{1}{2}\mathrm{d}(i_X\beta - i_Y\alpha)$$

where $X, Y \in TM$ and $\alpha, \beta \in TM^*$. A subbundle of $TM \oplus TM^*$ is said to be involutive if its sections are closed under the Courant bracket.

A generalized almost complex structure on M is an endomorphism $\mathcal{J}$ of $TM \oplus TM^*$ such that $\mathcal{J} + \mathcal{J}^* = 0$ and $\mathcal{J}^2 = -\mathrm{Id}$. Since $\mathcal{J}^2 = -\mathrm{Id}$, $\mathcal{J}$ has eigenvalues $\pm$i. Let $E \subset \mathbb{T}M \otimes \mathbb{C}$ be the i eigenbundle of $\mathcal{J}$, E is maximal isotropic with respect to $\langle\,,\rangle$ and it satisfies $E \cap \bar{E} = 0$. Conversely, any such maximal isotropic subbundle E of $\mathbb{T}M \otimes \mathbb{C}$ defines an almost generalized complex structure on M. $\mathcal{J}$ is called a generalized complex structure (or, $\mathcal{J}$ is integrable) if E is involutive [2]. The integrability of $\mathcal{J}$ amounts the nullity of the Nijenhuis tensor of $\mathcal{J}$, i.e., for any $X+\alpha, Y+\beta \in \Gamma(E)$, we have

$$\begin{aligned} N_{\mathcal{J}}(X+\alpha, Y+\beta) =& [\![\mathcal{J}(X+\alpha), \mathcal{J}(Y+\beta)]\!] + \mathcal{J}^2[\![X+\alpha, Y+\beta]\!] \\ &- \mathcal{J}[\![X+\alpha, \mathcal{J}(Y+\beta)]\!] - \mathcal{J}[\![\mathcal{J}(X+\alpha), Y+\beta]\!] = 0. \end{aligned}$$

A generalized Riemannian metric G is an automorphism

$$G : TM \oplus TM^* \longrightarrow TM \oplus TM^*$$

which is self-adjoint (symmetric) operator, i.e., $G^* = G$, and squares to identity, i.e., $G^2 = \mathrm{Id}$ such that G is positive definite metric [2]. It turns out that a generalized Riemannian metric is equivalent with a pair (γ, ψ) where γ is a classical Riemannian metric on M and $\psi \in \Omega^2(M)$. More exactly

$$G = \begin{pmatrix} \mathcal{A} & \gamma^\sharp \\ \sigma^\flat & \mathcal{A}^* \end{pmatrix}$$

where $\mathcal{A} \in \mathrm{End}(\mathbb{T}M)$, $\psi = -\gamma^\flat \mathcal{A}$ and $\sigma^\flat = \gamma^\flat \circ (\mathrm{Id} - \mathcal{A}^2)$. The condition $G^2 = \mathrm{Id}$ implies that $\mathcal{A}$ is skew-symmetric with respect to both metrics γ and σ, i.e., $\sigma(\mathcal{A}X, Y) = -\sigma(X, \mathcal{A}Y)$ and $\gamma(\mathcal{A}^*\alpha, \beta) = -\gamma(\alpha, \mathcal{A}^*\beta)$ [2].
The analog of generalized almost complex structure for odd-dimensional spaces is generalized almost contact structure. We mention here the definition of these geometric structures. But first, it will be worthwhile to recall the formal definition of geometric structures for odd-dimensional spaces on a manifold to use them in generalized cases.

Let M^{2n+1} be a smooth manifold with a one-form η such that $\eta \wedge (\mathrm{d}\eta)^n \neq 0$, then the one-form η is a contact structure or a contact one-form. Given a contact one-form, there is a unique vector field ξ such that $\eta(\xi) = 1$ and $i_\xi \mathrm{d}\eta = 0$. This vector field is known as the Reeb vector field of the contact form η. Other geometric structures for odd-dimensional spaces that is associated with the contact and generalized structures are cosymplectic structures. In terms of tensors, an almost cosymplectic structure (η, θ) is equivalent to the choice of a one-form η and a two-from θ such that $\eta \wedge \theta^n \neq 0$ at every point of the manifold. Subsequently, an almost cosymplectic structure (η, θ) is a cosymplectic structure if it is integrable or equivalently, if both η and θ are closed. It is immediate that contact forms constitute a subclass of almost cosymplectic structures with $\theta = \mathrm{d}\eta$ [5].

An almost contact metric structure on M is given by tensors (φ, ξ, η, g) where φ is a $(1,1)$-tensor field, ξ is a vector field and η is a one-form on M, satisfying the following conditions

$$\varphi^2 = -\mathrm{Id} + \eta \otimes \xi, \qquad \eta(\xi) = 1$$

and where g is a Riemannian metric compatible with almost contact structure, that means

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for vector fields X and Y. We can use the Riemannian metric g and φ to construct the fundamental two-form $\Theta(X, Y) = g(\varphi X, Y)$. Then an almost contact metric structure (φ, ξ, η, g) is called a contact metric structure iff $\Theta = \mathrm{d}\eta$. Furthermore an almost contact metric structure on M is normal if the Nijenhuis tensor of φ

$$N_{\varphi}(X,Y)=\varphi^2[X,Y]-\varphi[X,\varphi Y]-\varphi[\varphi X,Y]+[\varphi X,\varphi Y]$$

satisfies $N_{\varphi}=-2\xi\otimes \mathrm{d}\eta$ [1]. Now, we are ready to return to the definition of generalized structures on odd-dimensional spaces.

Using the definition given in [5], for an odd-dimensional manifold M, a pair $(\Phi, F+\eta)$ is called generalized almost contact structure iff

$$\begin{aligned} \Phi+\Phi^*=0, \qquad \Phi^2=-\mathrm{Id}+F\odot\eta \\ \eta(F)=1, \qquad \Phi(F)=0 \qquad and \qquad \Phi(\eta)=0 \end{aligned} \tag{1}$$

where Φ is an endomorphism of $TM\oplus TM^*$, and $F+\eta$ is a section of $TM\oplus TM^*$ and $F\odot\eta(X+\alpha):=\eta(X)F+\alpha(F)\eta$, for any $X+\alpha\in\Gamma(\mathbb{T}M)$.

Given a generalized almost contact pair $(\Phi, F+\eta)$, define

$$\begin{aligned} E^{(1,0)} &= \{X+\alpha-\mathrm{i}\Phi(X+\alpha);\, X+\alpha\in \ker\eta\oplus\ker F\} \\ E^{(0,1)} &= \{X+\alpha+\mathrm{i}\Phi(X+\alpha);\, X+\alpha\in \ker\eta\oplus\ker F\}. \end{aligned}$$

The endomorphism Φ is linearly extended to the complexified bundle $\mathbb{T}M\otimes\mathbb{C}$. It has three eigenvalues, namely, $\lambda=0$ and $\lambda=\mathrm{i}$ and $\lambda=-\mathrm{i}$. The corresponding eigenbundles are $L_F\oplus L_\eta$, $E^{(1,0)}$ and $E^{(0,1)}$, where L_F and L_η are the complex vector bundles of rank 1 generated by F and η, respectively. Define

$$\begin{aligned} L:=L_F\oplus E^{(1,0)}, \qquad L^*:=L_\eta\oplus E^{(0,1)} \\ \bar{L}:=L_F\oplus E^{(0,1)}, \qquad \bar{L}^*:=L_\eta\oplus E^{(1,0)}. \end{aligned}$$

We say that the generalized almost contact pair $(\Phi, F+\eta)$ is a generalized contact structure or $(\Phi, F+\eta)$ is integrable if L is involutive.

Since Φ has a matrix form as

$$\Phi=\begin{pmatrix} \varphi & \pi^\sharp \\ \theta^\flat & -\varphi^* \end{pmatrix}$$

one sees that a generalized almost contact pair is equivalent to a quintuplet $(\varphi, \pi^\sharp, \theta^\flat, F, \eta)$ where F is a vector field, η a one-form, φ a $(1,1)$-tensor field, π a bivector field, and θ a 2-form that according to (1), they satisfy the following relations

$$i) \quad \varphi^2+\pi^\sharp\theta^\flat=-\mathrm{Id}+F\otimes\eta, \qquad ii) \quad \varphi{*}^2+\theta^\flat\pi^\sharp=-\mathrm{Id}+\eta\otimes F \tag{2}$$

$$i) \quad \theta(\varphi X,Y)=\theta(X,\varphi Y), \qquad ii) \quad \pi(\alpha,\varphi^*\beta)=\pi(\varphi^*\alpha,\beta) \tag{3}$$

$$i) \;\; \eta\circ\varphi=0, \;\; ii)\; \eta\circ\pi^\sharp=0, \;\; iii)\; i_F\varphi=0, \;\; iv)\; i_F\theta=0, \;\; v)\; i_F\eta=1. \tag{4}$$

In this classical form, the integrability conditions of $(\varphi, \pi^\sharp, \theta^\flat, F, \eta)$ are stated in following Theorem.

Theorem 1 ([11]). *A generalized almost contact pair corresponding to the quintuplet* $(\varphi, \pi^\sharp, \theta^\flat, F, \eta)$ *is integrable if and only if the following relations are satisfied*:

$$A_1)\quad \frac{1}{2}[\pi,\pi] = F\wedge(\pi^\sharp\otimes\pi^\sharp)\mathrm{d}\eta, \qquad [F,\pi] = -F\wedge\pi^\sharp\pounds_F\eta$$

$$A_2)\quad \varphi^*\{\alpha,\beta\}_\pi = \pounds_{\pi^\sharp\alpha}\varphi^*\beta - \pounds_{\pi^\sharp\beta}\varphi^*\alpha - \mathrm{d}\pi(\varphi^*\alpha,\beta)$$

$$A_3)\quad N_\varphi(X,Y) + \mathrm{d}\eta(\varphi X,\varphi Y)F = \pi^\sharp(i_{X\wedge Y}\mathrm{d}\theta)$$

$$A_4)\quad \mathrm{d}\theta_\varphi(X,Y,Z) = \mathrm{d}\theta(\varphi X,Y,Z) + \mathrm{d}\theta(X,\varphi Y,Z) + \mathrm{d}\theta(X,Y,\varphi Z)$$

$$A_5)\quad \pounds_F\varphi = 0, \qquad \pounds_F\theta = 0$$

where the bracket is the Schouten-Nijenhuis bracket as explained in [9], $\{\alpha,\beta\}_\pi = \pounds_{\pi^\sharp\alpha}\beta - \pounds_{\pi^\sharp\beta}\alpha - \mathrm{d}\pi(\alpha,\beta)$, $i_{X\wedge Y}\mathrm{d}\theta = \pounds_X\theta^\flat(Y) - \pounds_Y\theta^\flat(X) - \mathrm{d}\theta(X,Y)$, *and* $\theta_\varphi(X,Y) = \theta(\varphi X, Y)$.

In a generalized almost contact structure, if both L and L^* be involutive, the pair $(\Phi, F+\eta)$ is called a strong generalized contact structure. The strong generalized contact structure $(\Phi, F+\eta)$ is called a normal generalized contact structure if $\pounds_F\eta = 0$ [7]. A generalized almost contact metric structure $(\Phi, F+\eta, G)$ is a generalized almost contact structure with a generalized Riemannian metric G that satisfies

$$-\Phi G\Phi = G - F\otimes F - \eta\otimes\eta.$$

3. Conformal Integrable Structures

Consider the automorphism $C_\tau : \mathbb{T}M \longrightarrow \mathbb{T}M$ defined in [6]

$$C_\tau(X,\alpha) := (X, \mathrm{e}^\tau\alpha), \qquad \tau\in C^\infty(M).$$

There Vaisman called it a conformal change of $\mathbb{T}M$ because it produces a conformal change of the natural inner product $\langle\,,\rangle$ such that

$$\langle C_\tau(X+\alpha), C_\tau(Y+\beta)\rangle = \mathrm{e}^\tau\langle X+\alpha, Y+\beta\rangle.$$

Furthermore if τ is locally constant the change will be called a homothety [6]. Applying the conformal change on Φ and G in a generalized almost contact structure $(\Phi, F+\eta, G)$, resultes

$$\Phi\mapsto\tilde\Phi = C_{-\tau}\circ\Phi\circ C_\tau, \qquad G\mapsto\tilde G = C_{-\tau}\circ G\circ C_\tau.$$

Accordingly, one gets

$$\tilde\Phi = \begin{pmatrix} \varphi & \mathrm{e}^\tau\pi^\sharp \\ \mathrm{e}^{-\tau}\theta^\flat & -\varphi^* \end{pmatrix}, \qquad \tilde G = \begin{pmatrix} A & \mathrm{e}^\tau\gamma^\sharp \\ \mathrm{e}^{-\tau}\sigma^\flat & A^* \end{pmatrix}.$$

It follows that if G is related to (γ,ψ), then $\tilde G$ is related to $(\mathrm{e}^{-\tau}\gamma, \mathrm{e}^{-\tau}\psi)$.

Remark 2. *One can see that if $(\Phi, F+\eta, G)$ is a generalized almost contact metric structure, then $(\tilde{\Phi}, \tilde{F}+\tilde{\eta}, \tilde{G})$ is a generalized almost contact metric structure too, where $\tilde{F} = \mathrm{e}^{-\frac{\tau}{2}} F$ and $\tilde{\eta} = \mathrm{e}^{\frac{\tau}{2}} \eta$.*

In [6] Vaisman had considered also the conditions under which conformal changes of generalized almost complex structures and almost Hermitian structures became integrable and Kähler structure, respectively. We give an analog consideration for the integrability of generalized almost contact structure after conformal changes.

Definition 3. *A generalized almost contact structure $(\Phi, F+\eta)$ is called conformal integrable, if there exists a conformal change C_τ such that $(\tilde{\Phi}, \tilde{F}+\tilde{\eta})$ is integrable in which $\tilde{\Phi} = C_{-\tau}\Phi C_\tau$, $\tilde{F} = \mathrm{e}^{-\frac{\tau}{2}} F$ and $\tilde{\eta} = \mathrm{e}^{\frac{\tau}{2}} \eta$.*

Proposition 4. *The generalized almost contact structure $(\varphi, \pi^\sharp, \theta^\flat, F, \eta)$ is conformal integrable, if there exists a function $\tau \in C^\infty(M)$ such that $\varpi = \mathrm{d}\tau$ satisfies the conditions*

$$\begin{aligned}
B_1)\quad & [\pi,\pi] - 2F \wedge (\pi^\sharp \otimes \pi^\sharp)\mathrm{d}\eta = -2\pi^\sharp\varpi \wedge \pi \quad \textit{and} \\
& [F,\pi] + F \wedge \pi^\sharp \pounds_F \eta = -\varpi(F)\pi \\
B_2)\quad & \varphi^*\{\alpha,\beta\}_\pi - (\pounds_{\pi^\sharp\alpha}\varphi^*\beta - \pounds_{\pi^\sharp\beta}\varphi^*\alpha - \mathrm{d}\pi(\varphi^*\alpha,\beta)) \\
& = -\pi(\alpha,\beta)\varphi^*\varpi + \pi(\varphi^*\alpha,\beta)\varpi \\
B_3)\quad & N_\varphi(X,Y) + \mathrm{d}\eta(\varphi X, \varphi Y)F - \pi^\sharp(i_{X\wedge Y}\mathrm{d}\theta) \\
& = \theta(X,Y)\pi^\sharp\varpi - \varpi(X)\pi^\sharp\theta^\flat(Y) + \varpi(Y)\pi^\sharp\theta^\flat(X) \\
B_4)\quad & \mathrm{d}\theta_\varphi(X,Y,Z) - \sum_{\mathrm{cycle}(X,Y,Z)} \mathrm{d}\theta(\varphi X, Y, Z) \\
& = - \sum_{\mathrm{cycle}(X,Y,Z)} (\varpi \wedge \theta_\varphi + (\varpi \circ \varphi) \wedge \theta)(X,Y,Z) \\
B_5)\quad & 2(\pounds_F\varphi) = -\varphi^*(\varpi) \otimes F, \qquad \textit{and} \quad (\pounds_F\theta^\flat) = -\varpi(F)\theta^\flat.
\end{aligned}$$

Proof: Let $(\varphi, \pi^\sharp, \theta^\flat, F, \eta)$ be a generalized almost contact structure and $(\varphi, \tilde{\pi}^\sharp, \tilde{\theta}^\flat, \tilde{F}, \tilde{\eta})$ its conformal change by C_τ, which is integrable. Then the first part of condition $A_1)$ is satisfied and we have $[\tilde{\pi}, \tilde{\pi}] = 2\tilde{F} \wedge (\tilde{\pi}^\sharp \otimes \tilde{\pi}^\sharp)\mathrm{d}\tilde{\eta}$, on the other hand, we get

$$\begin{aligned}
[\tilde{\pi},\tilde{\pi}] &= D(\mathrm{e}^\tau\pi \wedge \mathrm{e}^\tau\pi) - 2D(\mathrm{e}^\tau\pi) \wedge (\mathrm{e}^\tau\pi) \\
&= \mathrm{e}^{2\tau}[\pi,\pi] + 2\mathrm{e}^{2\tau}((\pi\wedge\pi)(\varpi) - \pi^\sharp(\varpi)\wedge\pi) \\
&= \mathrm{e}^{2\tau}([\pi,\pi] + 2(\pi^\sharp(\varpi)\wedge\pi))
\end{aligned}$$

where D is the generalized divergence which generates the Schouthen bracket. Also we have $\tilde{F} \wedge (\tilde{\pi}^\sharp \otimes \tilde{\pi}^\sharp)\mathrm{d}\tilde{\eta} = \mathrm{e}^{2\tau} F \wedge (\pi^\sharp \otimes \pi^\sharp)\mathrm{d}\eta$. Comparing these two relations we get the first part of $B_1)$. Similarly, by the second part of $A_1)$ we have $[\tilde{F}, \tilde{\pi}] = -\tilde{F} \wedge \tilde{\pi}^\sharp \pounds_{\tilde{F}} \tilde{\eta}$, then since

$$\begin{aligned}
[\tilde{F}, \tilde{\pi}] &= D(\mathrm{e}^{\frac{\tau}{2}} F \wedge \pi) - D(\mathrm{e}^{\frac{-\tau}{2}} F) \wedge (\mathrm{e}^{\tau}\pi) - D(\mathrm{e}^{\tau}\pi) \wedge (\mathrm{e}^{\frac{-\tau}{2}} F) \\
&= \mathrm{e}^{\frac{\tau}{2}} [F, \pi] + \frac{\mathrm{e}^{\frac{\tau}{2}}}{2} ((F \wedge \pi)(\varpi) + F(\varpi) \wedge \pi - \pi^\sharp(\varpi) \wedge F) \\
&= \mathrm{e}^{\frac{\tau}{2}} ([F, \pi] + \varpi(F)\pi)
\end{aligned}$$

and $\tilde{F} \wedge \tilde{\pi}^\sharp \pounds_{\tilde{F}} \tilde{\eta} = \mathrm{e}^{\frac{\tau}{2}} F \wedge \pi^\sharp \pounds_F \eta$, we get the second part of $B_1)$. Also by $A_2)$ we have $\varphi^* \{\alpha, \beta\}_{\tilde{\pi}} = (\pounds_{\tilde{\pi}^\sharp \alpha} \varphi^* \beta - \pounds_{\tilde{\pi}^\sharp \beta} \varphi^* \alpha - \mathrm{d}\tilde{\pi}(\varphi^* \alpha, \beta))$, then a straightforward calculation gives $B_2)$. Furthermore by $A_3)$, $N_\varphi(X, Y) = -\mathrm{d}\tilde{\eta}(\varphi X, \varphi Y)\tilde{F} + \tilde{\pi}^\sharp(i_{X \wedge Y} \mathrm{d}\tilde{\theta})$, then from $\mathrm{d}\eta(\varphi X, \varphi Y)F = \mathrm{d}\tilde{\eta}(\varphi X, \varphi Y)\tilde{F}$, we get

$$\begin{aligned}
N_\varphi(X, Y) &= -\mathrm{d}\eta(\varphi X, \varphi Y)F + \tilde{\pi}^\sharp(\pounds_X \tilde{\theta}^\flat(Y) - \pounds_Y \tilde{\theta}^\flat(X) - \mathrm{d}\tilde{\theta}(X, Y)) \\
&= -\mathrm{d}\eta(\varphi X, \varphi Y)F + \pi^\sharp(\pounds_X \theta^\flat(Y) - \pounds_Y \theta^\flat(X) - \mathrm{d}\theta(X, Y)) \\
&\quad - \pi^\sharp(\varpi(X)\theta^\flat(Y) - \varpi(Y)\theta^\flat(X) - \theta(X, Y)\varpi)
\end{aligned}$$

and $B_3)$ is proved. Considering $A_4)$ for $\tilde{\theta}$, a straightforward calculation gives $B_4)$. By the first part of $A_5)$ we have $\pounds_{\tilde{F}} \varphi = 0$, Using (4) we get

$$\begin{aligned}
0 &= (\pounds_{\tilde{F}} \varphi)X = \pounds_{\tilde{F}} \varphi X - \varphi(\pounds_{\tilde{F}} X) = [\mathrm{e}^{-\frac{\tau}{2}} F, \varphi X] - \varphi[\mathrm{e}^{-\frac{\tau}{2}} F, X] \\
&= \mathrm{e}^{-\frac{\tau}{2}} (([F, \varphi X] - \varphi[F, X]) + \frac{1}{2}\varpi(\varphi X)F) = \mathrm{e}^{-\frac{\tau}{2}} (\pounds_F \varphi)X + \frac{1}{2}\varpi(\varphi X)F)
\end{aligned}$$

that gives the first part of $B_5)$. Finally by the second part of $A_5)$ and (4ii), we get

$$\begin{aligned}
0 &= (\pounds_{\tilde{F}} \tilde{\theta})X = \pounds_{\tilde{F}} \tilde{\theta}^\flat X - \tilde{\theta}^\flat(\pounds_{\tilde{F}} X) \\
&= i_{\mathrm{e}^{-\frac{\tau}{2}} F} \circ \mathrm{d}(\mathrm{e}^{-\tau}\theta^\flat)X + \mathrm{d} \circ i_{\mathrm{e}^{-\frac{\tau}{2}} F}(\mathrm{e}^{-\tau}\theta^\flat)X - \mathrm{e}^{-\tau}\theta^\flat([\mathrm{e}^{-\frac{\tau}{2}} F, X]) \\
&= \mathrm{e}^{\frac{-3\tau}{2}} ((i_F \circ \mathrm{d})\theta^\flat(X) + \varpi(F)\theta^\flat(X) + \mathrm{d} \circ i_F \theta^\flat(X) - \theta^\flat(\pounds_F X)) \\
&= \mathrm{e}^{\frac{-3\tau}{2}} ((\pounds_F \theta^\flat)X + \varpi(F)\theta^\flat(X))
\end{aligned}$$

and this completes the proof. ∎

Now, we will investigate a necessary and sufficient condition under which $(\tilde{\Phi}, \tilde{F} + \tilde{\eta})$, the conformal change of generalized contact structure $(\Phi, F + \eta)$, is a generalized contact structure.

Theorem 5. *Let $(M, \Phi, F+\eta)$ be a generalized contact manifold,* $\dim M > 3$ *and $(\tilde{\Phi}, \tilde{F}+\tilde{\eta})$ its conformal change by C_τ. Assume that $(\Phi, F+\eta)$ satisfies one of the following conditions*

(1)- *rank* $\pi > 2$ *and* (2)- *φ_x has no real eigenvalue, for all $x \in M$. Then $(\tilde{\Phi}, \tilde{F}+\tilde{\eta})$ is a generalized contact structure if and only if the conformal change is homothety.*

Proof: By assumption $(\Phi, F+\eta)$ is integrable, thus $(\tilde{\Phi}, \tilde{F}+\tilde{\eta})$ is integrable too if and only if the right hand side of the equalities $B_1)$-$B_5)$ vanishes. Considering (2ii), the vanishing of the second part of $B_1)$ and the first part of $B_5)$, results in

$$\theta^\flat \pi^\sharp \varpi = -\varpi. \tag{5}$$

Furthermore condition $\pi^\sharp \varpi \wedge \pi = 0$ which is obtained from the vanishing of the first part of $B_1)$ holds, if and only if either $rank\ \pi = 2$ or $\pi^\sharp \varpi = 0$, then in case (1), we must have $\pi^\sharp \varpi = 0$. Thus by using (5), we get $\mathrm{d}\tau = \varpi = 0$.

To discuss case (2), assume that $\mathrm{d}_x \tau \neq 0$ on a neighborhood U_x. Since $B_2)$ holds for every one-form β, its vanishing resultes $(\varphi^* \varpi) X \pi^\sharp(\alpha) = \varpi(X)\pi^\sharp(\varphi^*\alpha)$ for a vector field X on U_x. since $\mathrm{d}_x\tau \neq 0$, it yields $f\pi^\sharp(\alpha) = \varphi_x \pi^\sharp(\alpha)$ in which $f = \frac{\varphi^* \varpi_x X}{\varpi_x X} \in C^\infty(\mathbb{T}M)$. Thus, replacing α by a one-form $\theta^\flat(Y)$ for any arbitrary vector field Y and using (2) and (4), we see that $\varphi\,|_{U_x}$ satisfies

$$\varphi^3 - f\varphi^2 + \varphi - f(I + \eta \otimes F) = 0$$

and therefore φ must have a real eigenvalue. Thus the hypothesis of case (2) implies $\mathrm{d}\tau = \varpi = 0$. ■

Let η be a contact structure on M with ξ the corresponding Reeb vector field, then $-\flat(X) := i_X \mathrm{d}\eta - \eta(X)\eta$ is an isomorphism from the tangent bundle TM to the cotangent bundle TM^*. Thus by defining a bivector field [5]

$$\pi(\alpha, \beta) := \mathrm{d}\eta(\flat^{-1}(\alpha), \flat^{-1}(\beta))$$

where $\alpha, \beta \in T^*M$, we have a generalized contact structure $(\Phi, F+\eta)$ in which

$$\Phi = \begin{pmatrix} 0 & \pi^\sharp \\ \mathrm{d}\eta^\flat & 0 \end{pmatrix}, \qquad F = \xi.$$

Thus by Proposition 4, the conformal integrability conditions are reduced to

$$\begin{aligned}
&a_1) \quad 2\pi^\sharp \varpi \wedge \pi = 0, \ \ and \ \ \varpi(F)\pi = 0 \\
&a_2) \quad \mathrm{d}\eta(X,Y)\pi^\sharp\varpi - \varpi(X)\pi^\sharp \mathrm{d}\eta^\flat(Y) + \varpi(Y)\pi^\sharp \mathrm{d}\eta^\flat(X) = 0 \\
&a_3) \quad \varpi(F)\mathrm{d}\eta^\flat = 0.
\end{aligned}$$

Therefore, with the help of the hypotheses of Theorem 5, we have the next result.

Proposition 6. *The generalized contact manifold $(M, \Phi, F + \eta)$, $\dim M > 3$, associated to a classical contact one-form η is conformal integrable if and only if the conformal change is homothety.*

Proof: Since Φ is full rank on $\ker\eta$ and $\Phi = \begin{pmatrix} 0 & \pi^\sharp \\ \mathrm{d}\eta^\flat & 0 \end{pmatrix}$, then $\pi^\sharp$ is also full rank on $\ker\eta$. From $\dim M > 3$, we have rank $\pi > 2$ and by Theorem 5, $(\Phi, F + \eta)$ is conformal integrable if and only if the conformal change is homothety. ■

The following example shows that when $\dim M = 3$, the conformal change need not be homothety.

Example 7. *Let $M = \mathrm{SU}(2)$ on the Lie algebra $\mathfrak{su}(2)$ and choose a basis $\{X_1, X_2, X_3\}$ and a dual basis $\{\sigma^1, \sigma^2, \sigma^3\}$ such that $[X_i, X_j] = -X_k$, thus $\mathrm{d}\sigma^i = \sigma^j \wedge \sigma^k$ for cyclic permutations of $\{i, j, k\}$. We know from* [5] *that one can construct a generalized contact structure associated to a classical contact one-form $\eta = \sigma^3$ by taking*

$$F = X_3, \qquad \theta = \mathrm{d}\sigma^3 = \sigma^1 \wedge \sigma^2, \qquad \pi = X_1 \wedge X_2 \ \ \textit{and} \ \ \Phi = \begin{pmatrix} 0 & \pi^\sharp \\ \theta^\flat & 0 \end{pmatrix}$$

then $L = \mathrm{span}\{X_3, X_1 - \mathrm{i}\sigma^2, X_2 + \mathrm{i}\sigma^1\}$. Now, we consider integrability of conformal change $(\tilde{\Phi}, \tilde{F} + \tilde{\eta})$ for nonconstant function τ such that $\mathrm{d}\tau = \varepsilon\sigma^1$ for a real constant ε. Let $\tilde{L}$ be the conformal changes of L by C_τ, then

$$\tilde{L} = \mathrm{span}\{\mathrm{e}^{-\tau/2} X_3, X_1 - \mathrm{e}^{-\tau}\mathrm{i}\sigma^2, X_2 + \mathrm{e}^{-\tau}\mathrm{i}\sigma^1\}.$$

Then the Courant brackets give

$$\begin{gathered} [\![\mathrm{e}^{-\tau/2} X_3, X_1 - \mathrm{e}^{-\tau}\mathrm{i}\sigma^2]\!] = -\mathrm{e}^{-\tau/2}(X_2 + \mathrm{e}^{-\tau}\mathrm{i}\sigma^1) + \frac{\varepsilon\mathrm{e}^{-\tau/2}}{2} X_3 \\ [\![\mathrm{e}^{-\tau/2} X_3, X_2 + \mathrm{e}^{-\tau}\mathrm{i}\sigma^1]\!] = \mathrm{e}^{-\tau/2}(X_1 - \mathrm{e}^{-\tau}\mathrm{i}\sigma^2) \\ [\![X_1 - \mathrm{e}^{-\tau}\mathrm{i}\sigma^2, X_2 + \mathrm{e}^{-\tau}\mathrm{i}\sigma^1]\!] = -X_3. \end{gathered}$$

Thus $(\tilde{\Phi}, \tilde{F} + \tilde{\eta})$ is a generalized contact structure too.

4. Conformal Normal Structures

Wade [11] has already described the integrability of generalized almost contact structures. Now, we will continue her computational method and describe geometric conditions expressing the normalization of generalized almost contact structure.

Theorem 8. *A generalized almost contact pair* $(\Phi, F+\eta)$ *corresponding to the quintuplet* $(\varphi, \pi^\sharp, \theta^\flat, F, \eta)$ *is normal if and only if it satisfies conditions* $A_1)$*-*$A_5)$*, and* $\pounds_F\eta = 0$*, and the following relations hold*

$$\begin{aligned} &C_1) \quad \pounds_{\pi^\sharp\alpha}\eta = 0 \\ &C_2) \quad \mathrm{d}\eta(\varphi X, Y) - \mathrm{d}\eta(\varphi Y, X) = 0. \end{aligned}$$

Proof: Let $(\Phi, F+\eta)$ be a generalized almost contact pair on M. Then by definition, M is normal if and only if both L and L^* be involutive and $\pounds_F\eta = 0$. It is known that L is involutive if and only if conditions (A_1)-(A_5) are satisfied. Now, we prove that L^* is involutive if and only if (C_1) and (C_2) are satisfied. Given any one-form α on M, we denote $e_\alpha = \alpha + \mathrm{i}\Phi\alpha = \mathrm{i}\pi^\sharp\alpha + (\alpha - \mathrm{i}\varphi^*\alpha)$. Then $[\![\eta, e_\alpha]\!] \in \Gamma(L^*)$ if and only if $\Phi[\![\eta, e_\alpha]\!] = -\mathrm{i}[\![\eta, e_\alpha]\!]$. Since we have

$$[\![\eta, e_\alpha]\!] = [\![\eta, \mathrm{i}\pi^\sharp\alpha + (\alpha - \mathrm{i}\varphi^*\alpha)]\!] = \mathrm{i}\pounds_{\pi^\sharp\alpha}\eta$$

condition $\Phi[\![\eta, e_\alpha]\!] = -\mathrm{i}[\![\eta, e_\alpha]\!]$ can be expressed as

$$\mathrm{i}\pi^\sharp(\pounds_{\pi^\sharp\alpha}\eta) - \mathrm{i}\varphi^*(\pounds_{\pi^\sharp\alpha}\eta) = -(\pounds_{\pi^\sharp\alpha}\eta).$$

Thus $[\![\eta, e_\alpha]\!] \in \Gamma(L^*)$ if and only if $\pounds_{\pi^\sharp\alpha}\eta = 0$. Now, let X be a section of $\ker\eta$, and $e_X := X + \mathrm{i}\Phi X = (X + \mathrm{i}\varphi X) + \mathrm{i}\theta^\flat(X)$. We have

$$[\![\eta, (X + \mathrm{i}\varphi X) + \mathrm{i}\theta^\flat(X)]\!] = -\pounds_X\eta - \mathrm{i}\pounds_{\varphi X}\eta$$

thus $[\![\eta, e_X]\!] \in \Gamma(L^*)$ if and only if $\Phi[\![\eta, e_X]\!] = -\mathrm{i}[\![\eta, e_X]\!]$. This condition can be expressed as

$$\varphi^*(\pounds_X\eta) = -(\pounds_{\varphi X}\eta)$$

or equivalently, for any section Y and by using (4i), we get

$$\begin{aligned} (\pounds_X\eta)\varphi Y = &- (\pounds_{\varphi X}\eta)Y \\ &\Rightarrow X.\eta(\varphi Y) - \eta[X, \varphi Y] = -\varphi X.\eta(Y) + \eta[\varphi X, Y] \\ &\Rightarrow \mathrm{d}\eta(\varphi Y, X) = \mathrm{d}\eta(\varphi X, Y). \end{aligned}$$

Hence $[\![\eta, e_X]\!]$ is a section of L^* if and only if $\mathrm{d}\eta(\varphi X, Y) - \mathrm{d}\eta(\varphi Y, X) = 0$. ■

Definition 9. *A generalized almost contact structure* $(\Phi, F+\eta)$ *is conformal normal, if there exists a conformal change by* C_τ *such that* $(\tilde\Phi, \tilde F + \tilde\eta)$ *is normal.*

Now that all pieces are in place for expressing normal conditions after a conformal change.

Proposition 10. *The generalized almost contact structure* $(\varphi, \pi^\sharp, \theta^\flat, F, \eta)$ *is conformal normal, if there exists a function* $\tau \in C^\infty(M)$ *such that* $\varpi = \mathrm{d}\tau$ *satisfies the conditions* B_1)-B_5) *and*

$$\begin{aligned} &D1) \quad 2(\pounds_F \eta) = -(\varpi \wedge \eta)F \\ &D2) \quad 2(\pounds_{\pi^\sharp\alpha}\eta) = \pi(\varpi, \alpha)\eta \\ &D3) \quad 2\{\mathrm{d}\eta(\varphi X, Y) - \mathrm{d}\eta(\varphi Y, X)\} = -(\varphi^*\varpi \wedge \eta)(X, Y). \end{aligned}$$

Proof: Let $(\varphi, \pi^\sharp, \theta^\flat, F, \eta)$ be some generalized almost contact structure and $(\varphi, \tilde{\pi}^\sharp, \tilde{\theta}^\flat, \tilde{F}, \tilde{\eta})$ its conformal change by C_τ, which is normal. Then $\pounds_{\tilde{F}}\tilde{\eta} = 0$. Thus we get

$$\begin{aligned} 0 = (\pounds_{\tilde{F}}\tilde{\eta}) &= i_{\mathrm{e}^{\frac{-\tau}{2}}F} \circ \mathrm{d}(\mathrm{e}^{\frac{\tau}{2}}\eta) + \mathrm{d} \circ i_{\mathrm{e}^{\frac{-\tau}{2}}F}\mathrm{e}^{\frac{\tau}{2}}\eta \\ &= i_{\mathrm{e}^{\frac{-\tau}{2}}F}(\frac{\mathrm{e}^{\frac{\tau}{2}}}{2}\varpi \wedge \eta + \mathrm{e}^{\frac{\tau}{2}}\mathrm{d}\eta) + \mathrm{d} \circ i_F\eta = \frac{1}{2}(\varpi \wedge \eta)F + \pounds_F\eta. \end{aligned}$$

Also by C_1), we have $\pounds_{\tilde{\pi}^\sharp\alpha}\tilde{\eta} = 0$, then by using (4ii), we have

$$\begin{aligned} 0 = (\pounds_{\tilde{\pi}^\sharp\alpha}\tilde{\eta}) &= i_{\mathrm{e}^\tau\pi^\sharp\alpha} \circ \mathrm{d}(\mathrm{e}^{\frac{\tau}{2}}\eta) + \mathrm{d} \circ i_{\mathrm{e}^\tau\pi^\sharp\alpha}\mathrm{e}^{\frac{\tau}{2}}\eta \\ &= i_{\mathrm{e}^\tau\pi^\sharp\alpha}(\frac{\mathrm{e}^{\frac{\tau}{2}}}{2}\varpi \wedge \eta + \mathrm{e}^{\frac{\tau}{2}}\mathrm{d}\eta) + \mathrm{e}^{\frac{3\tau}{2}}\mathrm{d} \circ i_{\pi^\sharp\alpha}\eta \\ &= \frac{\mathrm{e}^{\frac{3\tau}{2}}}{2}(\varpi \wedge \eta)(\pi^\sharp\alpha) + \mathrm{e}^{\frac{3\tau}{2}}\pounds_{\pi^\sharp\alpha}\eta = \mathrm{e}^{\frac{3\tau}{2}}(-\frac{1}{2}\alpha\pi^\sharp(\varpi)\eta + \pounds_{\pi^\sharp\alpha}\eta). \end{aligned}$$

Finally by C_2), we have $\mathrm{d}\tilde{\eta}(\varphi X, Y) - \mathrm{d}\tilde{\eta}(\varphi Y, X) = 0$, then by using (4i), we get

$$\begin{aligned} 0 &= \mathrm{d}\tilde{\eta}(\varphi X, Y) - \mathrm{d}\tilde{\eta}(\varphi Y, X) = \mathrm{d}(\mathrm{e}^{\frac{\tau}{2}}\eta)(\varphi X, Y) - \mathrm{d}(\mathrm{e}^{\frac{\tau}{2}}\eta)(\varphi Y, X) \\ &= (\frac{\mathrm{e}^{\frac{\tau}{2}}}{2}\varpi \wedge \eta + \mathrm{e}^{\frac{\tau}{2}}\mathrm{d}\eta)(\varphi X, Y) - (\frac{\mathrm{e}^{\frac{\tau}{2}}}{2}\varpi \wedge \eta + \mathrm{e}^{\frac{\tau}{2}}\mathrm{d}\eta)(\varphi Y, X) \\ &= \frac{\mathrm{e}^{\frac{\tau}{2}}}{2}\{\varpi(\varphi X)\eta(Y) - \varpi(\varphi Y)\eta(X)) + \mathrm{d}\eta(\varphi X, Y) - \mathrm{d}\eta(\varphi Y, X)\} \end{aligned}$$

this completes the proof. ■

Theorem 11. *Let* $(M, \Phi, F + \eta)$ *be a normal generalized contact manifold such that* $\dim M > 3$. *If one of the following conditions is satisfied:* 1) - *rank* $\pi > 2$, 2) - φ_x *has no real eigenvalue, for all* $x \in M$, *then the conformal change* C_τ *of* M, $(\tilde{\Phi}, \tilde{F} + \tilde{\eta})$, *is normal if and only if the conformal change is a homothety.*

Proof: By the above Proposition and Theorem 5, one can simply deduce the proof. ■

The following example shows that if none of the conditions (1) and (2) of the above theorem is satisfied, then the conformal change is not necessarily a homothety.

Example 12. *Let $M = \mathbb{R}^5$ and choose a local frame $\{X_1, X_2, X_3, X_4, X_5\}$ and its dual local frame $\{\sigma^1, \sigma^2, \sigma^3, \sigma^4, \sigma^5\}$ such that*

$$[X_5, X_1] = X_4, \qquad [X_5, X_2] = -X_3, \qquad [X_5, X_3] = -X_2$$
$$[X_5, X_4] = X_1, \qquad [X_i, X_j] = 0.$$

Thus we have

$$\mathrm{d}\sigma^1 = \sigma^4 \wedge \sigma^5, \qquad \mathrm{d}\sigma^2 = -\sigma^3 \wedge \sigma^5, \qquad \mathrm{d}\sigma^3 = -\sigma^2 \wedge \sigma^5, \qquad \mathrm{d}\sigma^4 = \sigma^1 \wedge \sigma^5$$

and σ^5 is closed. To construct a normal generalized contact structure, one takes generalized almost contact structure components with $\varphi = X_2 \otimes \sigma^1 - X_1 \otimes \sigma^2 + X_4 \otimes \sigma^3 - X_3 \otimes \sigma^4$, $\Phi = \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi^* \end{pmatrix}$, $F = X_5$ and $\eta = \sigma^5$, where $(\varphi^*\alpha)X = \alpha(\varphi X)$ and $X + \alpha \in \mathbb{T}M$. One computes easily that

$$L = \mathrm{span}\{X_5, X_1 - \mathrm{i}X_2, X_3 - \mathrm{i}X_4, \sigma^1 - \mathrm{i}\sigma^2, \sigma^3 - \mathrm{i}\sigma^4\}$$
$$L^* = \mathrm{span}\{\sigma^5, X_1 + \mathrm{i}X_2, X_3 + \mathrm{i}X_4, \sigma^1 + \mathrm{i}\sigma^2, \sigma^3 + \mathrm{i}\sigma^4\}.$$

For L, the relevant Courant brackets give

$$[\![X_5, X_1 - \mathrm{i}X_2]\!] = \mathrm{i}(X_3 - \mathrm{i}X_4), \qquad [\![X_5, X_3 - \mathrm{i}X_4]\!] = -\mathrm{i}(X_1 - \mathrm{i}X_2)$$
$$[\![X_5, \sigma^1 - \mathrm{i}\sigma^2]\!] = -\mathrm{i}(\sigma^3 - \mathrm{i}\sigma^4), \qquad [\![X_5, \sigma^3 - \mathrm{i}\sigma^4]\!] = \mathrm{i}(\sigma^1 - \mathrm{i}\sigma^2)$$

and the rest of the brackets are equal to zero. Similarly, for L^* we compute the Courant brackets and we see that all of them is equal to zero as well as $\pounds_{X_5}\sigma^5 = \mathrm{d}\sigma^5(X_5) = 0$. Thus $(\Phi, F + \eta)$ is a normal generalized contact structure.

Now, we consider normality of conformal change $(\tilde{\Phi}, \tilde{F} + \tilde{\eta})$ for nonconstant function τ such that $\mathrm{d}\tau = \varepsilon\sigma^5$ for an arbitrary constant function ε . Then for L and L^*, the Courant brackets give

$$[\![\tilde{F}, X_1 - \mathrm{i}X_2]\!] = \mathrm{i}\mathrm{e}^{-\tau}(X_3 - \mathrm{i}X_4), \qquad [\![\tilde{F}, X_3 - \mathrm{i}X_4]\!] = -\mathrm{i}\mathrm{e}^{-\tau}(X_1 - \mathrm{i}X_2)$$
$$[\![\tilde{F}, \sigma^1 - \mathrm{i}\sigma^2]\!] = -\mathrm{i}\mathrm{e}^{-\tau}(\sigma^3 - \mathrm{i}\sigma^4), \qquad [\![\tilde{F}, \sigma^3 - \mathrm{i}\sigma^4]\!] = \mathrm{i}\mathrm{e}^{-\tau}(\sigma^1 - \mathrm{i}\sigma^2)$$

and the others are equal to zero as well as

$$\pounds_{\tilde{F}}\tilde{\eta} = \mathrm{d}\sigma^5(X_5) + \frac{1}{2}(\mathrm{d}\tau \wedge \sigma^5)(X_5) = 0.$$

Thus $(\tilde{\Phi}, \tilde{F} + \tilde{\eta})$ is a normal generalized contact structure.

Let (φ, ξ, η) be a normal almost contact structure on a manifold M^{2n+1}. It is shown in [5] that we have a normal generalized contact structure $(\Phi, F + \eta)$ in which

$$\Phi = \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi^* \end{pmatrix}, \qquad F = \xi.$$

Therefore, by Proposition 10, it is conformal normal if and only if

$$(\varpi \wedge \eta)F = 0$$

that means $\varpi = \varpi(F)\eta$. Thus we have the following result.

Proposition 13. *The normal generalized contact structure associated to a classical normal almost contact structure (φ, ξ, η) is conformal normal if and only if ϖ is a section of L_η.*

Similar to what we recall for contact structure, let (η, θ) be a cosymplectic structure with ξ the corresponding Reeb vector field, then $-\flat(X) := i_X\theta - \eta(X)\eta$ is an isomorphism from the tangent bundle TM to the cotangent bundle TM^*. Thus by defining a bivector field

$$\pi(\alpha, \beta) := \theta(\flat^{-1}(\alpha), \flat^{-1}(\beta))$$

where $\alpha, \beta \in T^*M$, we have a normal generalized contact structure $(\Phi, F + \eta)$ in which

$$\Phi = \begin{pmatrix} 0 & \pi^\sharp \\ \theta^\flat & 0 \end{pmatrix}, \qquad F = \xi.$$

Therefore, by Proposition 10, the conditions for being conformal normal reduce to

$$\begin{aligned}
&b_1) \quad 2\pi^\sharp\varpi \wedge \pi = 0, \quad \mathit{and} \quad \varpi(F)\pi = 0 \\
&b_2) \quad \theta(X,Y)\pi^\sharp\varpi - \varpi(X)\pi^\sharp\theta^\flat(Y) + \varpi(Y)\pi^\sharp\theta^\flat(X) = 0 \\
&b_3) \quad \varpi(F)\theta^\flat = 0, \quad \mathit{and} \quad b_4)\ \alpha\pi^\sharp\varpi \otimes \eta = 0.
\end{aligned}$$

Considering (2ii) and the vanishing of $\pi^\sharp\varpi$ and $\varpi(F)$ results in $\varpi = 0$. Thus we have the following

Proposition 14. *The normal generalized contact manifold $(M, \Phi, F + \eta)$ associated to a classical cosymplectic structure (η, θ) is conformal normal if and only if the conformal change is homothety.*

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