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n-CHARACTERISTIC VECTOR FIELDS OF CONTACT MANIFOLDS

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Abstract. In present paper we define and study n-characteristic vector fields. We present definition of Tanaka-Webster connection, then use it for studying the behavior of n-characteristic vector fields. Also we show some results about of these vector fields by Tanaka-Webster connection.

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1. Introduction

The main goal of this paper is to study a special type of vector fields. These vector fields are defined in contact metric manifolds and called n-characteristic vector fields or briefly n-char vector fields. All of them are commutate with characteristic vector field and the bracket of both n-char vector fields is multiple of characteristic vector field and it is proved that the bracket of n-char vector fields commutate with other components of tangent bundle. It has been shown if tangent space of each contact metric manifold contained a n-char vector field, then characteristic vector is commutate with all vector fields. The *Tanaka-Webster connection* [3] first time defined by Shukichi Tanno for contact manifold. The study of n-char vector fields with *Tanaka-Webster connection* resulted in interesting results.

2. Preliminaries

Let M be an almost contact manifold, i.e., it is a $(2m+1)$-dimensional smooth manifold with an almost contact structure (φ, ξ, η) consisting of an endomorphism ϕ of the tangent bundle, a vector field ξ, its dual one-form η as well as M is equipped with a Riemannian metric g, so that the following relations are valid

$$\varphi\xi = 0, \qquad \phi^2 = -\mathrm{Id} + \eta \otimes \xi, \qquad \eta\xi = 1 \tag{1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

where Id is the identity and $X, Y \in TM$ are arbitrary vector fields. Let Φ denote the two-form in M given by $\Phi(X, Y) = g(X, \varphi Y)$. The two-form Φ is called the fundamental two-form in M and the manifold is said to be a contact metric manifold if $\Phi = \mathrm{d}\eta$. If ξ is a Killing vector field with respect to g, the contact metric structure is called a K-contact structure. It is easy to prove that a contact metric manifold is K-contact if and only if $\nabla_X \xi = -\varphi X$, for any $X \in TM$, where ∇ denotes the Levi-Civita connection of M. It is defined Nijenhuis torsion of ϕ

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]. \tag{3}$$

Now we define four tensor $N^{(1)}$, $N^{(2)}$, $N^{(3)}$, $N^{(4)}$ and consider them separately

$$N^{(1)}(X, Y) = [\phi, \phi](X, Y) + 2\mathrm{d}\eta(X, Y)\xi \tag{4}$$

$$N^{(2)}(X, Y) = (\mathcal{L}_{\phi X}\eta)(Y) - (\mathcal{L}_{\phi Y}\eta)(X) \tag{5}$$

$$N^{(3)} = (\mathcal{L}_\xi \phi)X \tag{6}$$

$$N^{(4)} = (\mathcal{L}_\xi \eta)X. \tag{7}$$

An almost contact structure (φ, ξ, η) is normal if and only if these four tensors vanish [2, 4].

3. n-Characteristic Vector Fields

In this section we will study a Riemannian manifold M with a contact structure (φ, η, ξ). First we give main definition.

Definition 1. *Let $X \in TM$ be an arbitrary element, if $g(X, Y) = n$, for any $Y \in TM$, such that $[nX, Y] = \xi$, then X is a n-characteristic vector field.*

In the first step we show the two most prominent properties of n-char vector fields.

Lemma 2. *If X be n-characteristic vector field, then $\eta(X) = 0$,*

Proof: Let $\eta(X) = m$, then from definition we have $[\xi, mX] = \xi$. Using only these identities and combining a few permutations of variables obtain the formula

$$\begin{aligned} g(\nabla_X Y, Z) = \frac{1}{2}(&Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &- g([X, Y], Z) + g([Z, X], Y) + g([Z, Y], X)). \end{aligned} \tag{8}$$

Using equation (8) we get

$$\begin{aligned} 2g(\nabla_\xi mX, \xi) &= \xi\eta(mX) + mXg(\xi,\xi) - \xi\eta(mX) \\ &= mXg(\xi,\xi) = 2g(\nabla_{mX}\xi, \xi) \end{aligned}$$

and additionally we have $\nabla_\xi mX = \nabla_{mX}\xi$, thus $[\xi, mX] = 0$, now the proof is trivial. ■

Lemma 3. *Let M be a contact metric manifold. If $X \in TM$ is a n-char vector field, then there is no $Y \in TM$ so that $g(X,Y) = 0$.*

Proof: Let $g(X,Y) = n$, then $[nX, Y] = \xi$. From definition of Levi-Civita connection we have

$$n[X,Y] = n(\nabla_X Y - \nabla_Y X) = \nabla_{nX} Y - \nabla_Y nX = [nX, Y] = \xi.$$

We conclude that $[X,Y] = \dfrac{1}{n}\xi$. Thus $n \neq 0$ and proof is completed. ■

In the following Theorem we show that the right hand sides of equations (4), (5), (6) and (7) are zero when vector fields are n-char.

Theorem 4. *Let M be a contact metric manifold, then*

$$N^{(1)}(X,Y) = N^{(2)}(X,Y) = N^{(3)} = N^{(4)} = 0$$

when $X, Y \in TM$ are n-char vector fields.

Proof: From Lemma 3 we have $[X,Y] = \dfrac{1}{g(X,Y)}\xi$. Also, we know that

$$2g(X, \varphi Y) = 2\mathrm{d}\eta(X,Y) = -\eta([X,Y]) = -\frac{1}{g(X,Y)}. \tag{9}$$

Thus

$$[X,Y] = -2g(X,\varphi Y)\xi. \tag{10}$$

Using equation (10) and by direct calculations we get

$$[\varphi X, Y] = -2g(X,Y)\xi \tag{11}$$

$$[\varphi X, \varphi Y] = 2g(\varphi X, Y)\xi \tag{12}$$

$$[X, \varphi Y] = 2g(X,Y)\xi. \tag{13}$$

Hence $[\varphi X, Y] = -[X, \varphi Y]$ and $[X, Y] = [\varphi X, \varphi Y]$. From (3), (4), (11), (12) and (13) we arrive at $N^{(1)}(X, Y) = N^{(2)}(X, Y) = 0$. Furthermore, using Lemma 3 it is trivial to establish that $N^{(4)} = 0$. Using $N^{(1)} = 0$, we set $Y = \xi$, then $\eta([\xi, \varphi X]) = 0$ and we conclude $N^{(3)} = 0$ and proof is completed. ■

Immediately we get some facts in the next corollary.

Corollary 5. *Let M be a contact metric manifold such that TM is contained some n-char vector fields, then*

$$[Z, \xi] = 0$$

for any $Z \in TM$.

Proof: From Lemma 2 we have

$$[Z, [X, Y]] = [[Z, X], Y] + [X, [Z, Y]] = -[\xi, Y] - [X, \xi] = 0$$

for all $X, Y \in TM$, such that X,Y are n-char vector fields and $g(X, Z) = n$, $g(Y, Z) = m$, therefore $[X, Z] = \frac{1}{n}\xi$ and $[Y, Z] = \frac{1}{m}\xi$. Using (10) we have

$$[Z, [X, Y]] = -2[Z, g(X, \varphi Y)\xi] = 0$$

and the rest of the proof is trivial. ■

The next Lemma will have an interesting result.

Lemma 6. **([1])** *On a contact metric manifold h is a symmetric operator, i.e.,*

$$\nabla_X \xi = -\varphi X - \varphi h X$$

which anticommutes with φ and $\mathrm{trh} = 0$.

Corollary 7. *Let M be a contact metric manifold, then*

$$\nabla_X \xi = -\varphi X$$

where X is a n-char vector field.

Proof: From Theorem 4 the proof is trivial. ■

Therefore *n-char vector fields* have same property with vector fields of tangent bundle of K-contact manifolds.

Define the generalized Tanaka-Webster [3] connection for contact metric manifold by

$$\breve{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\varphi Y. \tag{14}$$

We consider the *n-char vector fields* with the Tanaka-Webster connection.

Theorem 8. *Let M be a contact metric manifold and $X, Y \in TM$ are n-char vector fields, then*

$$\breve{\nabla}_X Y \neq 0$$

where $\breve{\nabla}$ is Tanaka-Webster connection.

Proof: Using Lemma 2, Corollary 5 and equation (8) we have

$$\breve{\nabla}_X Y = \nabla_X Y - \eta([X,Y])\xi.$$

If $\breve{\nabla}_X Y = 0$, then $\nabla_X Y = \frac{1}{n}\xi$, on the other hand $[X,Y] = \frac{1}{n}\xi$, therefore $[X,Y] = \nabla_X Y$ and $\nabla_Y X = 0$. From Corollary 7 and equation (8) we have

$$2g(\nabla_X Y, \xi) = -\xi g(X,Y) + g([X,Y],\xi) = -\xi g(X,Y) + \frac{1}{n}\xi. \tag{15}$$

Then

$$-\frac{1}{n}\xi = \xi g(X,Y). \tag{16}$$

Using Theorem 8 we obtain

$$2g(\varphi X, Y) = \xi g(X,Y). \tag{17}$$

Using Corollary 7 we have $\nabla_X \xi = \nabla_\xi X$, therefore

$$\xi g(X,Y) = g(\nabla_\xi X, Y) + g(X, \nabla_\xi Y) = -g(\varphi X, Y) - g(X, \varphi Y) = 0.$$

By equality (16) we conclude $g(\varphi X, Y) = 0$, and by (9) we realize that it is impossible and the proof is completed. ■

From definition of Tanaka-Webster connection and straightforward calculations we arrive at the following facts

1) $\breve{\nabla}_X Y = \nabla_X Y - \frac{1}{n}\xi$

2) $\breve{\nabla}_X \varphi Y = \nabla_X \varphi Y + 2g(X,Y)\xi$

3) $\breve{\nabla}_{\varphi X} Y = \nabla_{\varphi X} Y + 2g(X,Y)\xi$

4) $\breve{\nabla}_{\varphi X} \varphi Y = \nabla_{\varphi X} \varphi Y - \frac{1}{n}\xi$

5) $(\breve{\nabla}_X \varphi)\varphi Y = (\nabla_X \varphi)\varphi Y + \frac{1}{n}\xi$

6) $\breve{\nabla}_X \xi = 0$

7) $\breve{\nabla}_\xi X = -2\varphi X.$

Also, relying on equations (2) and (3) we get

$$\breve{\nabla}_X \varphi Y - \breve{\nabla}_{\varphi X} Y = \nabla_X \varphi Y - \nabla_{\varphi X} Y$$

where X and Y are n-*char vector fields*.

Lemma 9. **([1])** *On a contact metric manifold*

$$(\nabla_X \varphi)Y + (\nabla_{\varphi X} \varphi)\varphi Y = 2g(X,Y)\xi - \eta(Y)(X + hX + \eta(X)\xi)$$

for all $X, Y \in TM$.

Lemma 10. *Let* M *be a contact metric manifold and* $X, Y \in TM$ *are arbitrary* n-char vector fields. *Then*

$$(\nabla_{\varphi X} \varphi)Y + \varphi \nabla_X \varphi Y + \nabla_X Y = 4g(\varphi X, Y).$$

Proof: Using Lemma 9 we will have

$$(\nabla_X \varphi)Y + (\nabla_{\varphi X} \varphi)\varphi Y = 2g(X,Y)\xi. \tag{18}$$

Then

$$\nabla_X \varphi Y - \varphi \nabla_X Y - \nabla_{\varphi X} Y - \varphi \nabla_{\varphi X} \varphi Y = 2g(X,Y)\xi. \tag{19}$$

Applying φ on (19) we have

$$\varphi \nabla_X \varphi Y + \nabla_X Y - \eta(\nabla_X Y)\xi - \varphi \nabla_{\varphi X} Y + \nabla_{\varphi X} \varphi Y - \eta(\nabla_{\varphi X} \varphi Y)\xi = 0. \tag{20}$$

Using equation (9) and Theorem 8 we get

$$\eta(\nabla_X Y) = -\xi g(X,Y) + \frac{1}{n} \tag{21}$$

and

$$\eta(\nabla_{\varphi X} \varphi Y) = -\xi g(X,Y) - \frac{1}{n}. \tag{22}$$

From Corollary 5 and Corollary 7 we conclude

$$\xi g(X,Y) = g(\nabla_\xi X, Y) + g(\nabla_\xi Y, X) = -g(\varphi X, Y) - g(\varphi Y, X) = 0.$$

Then

$$\eta(\nabla_X Y) + \eta(\nabla_{\varphi X}\varphi Y) = \frac{2}{n}\cdot$$

Therefore

$$\varphi\nabla_X\varphi Y + \nabla_X Y - \varphi\nabla_{\varphi X} Y + \nabla_{\varphi X}\varphi Y = \frac{2}{n}$$

and the proof is trivial. ■

Corollary 11. **([1])** *For a contact metric structure the formula of* Lemma 6.1 *becomes*

$$2g((\nabla_X\varphi)Y, Z) = g(N^{(1)}(Y,Z), \varphi X) + 2\mathrm{d}\eta(\varphi Y, X)\eta(Z) - 2\mathrm{d}\eta(\varphi Z, X)\eta(Y).$$

Using previous Lemma we consider curvature tensor with n*-char vector fields*.

Lemma 12. *On a contact metric manifold*

$$R(X,Y)\xi = 0$$

where X and Y are arbitrary n-char vector fields.

Proof:

$$\begin{aligned} R(X,Y)\xi &= \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi \\ &= -\nabla_X\varphi Y + \nabla_Y\varphi X \\ &= -\nabla_X\varphi Y + \varphi\nabla_X Y - \varphi\nabla_X Y + \nabla_Y\varphi X - \varphi\nabla_Y X + \varphi\nabla_Y X \\ &= -(\nabla_X\varphi)Y + (\nabla_Y\varphi)X + \varphi[Y,X] \\ &= (\nabla_Y\varphi)X - (\nabla_X\varphi)Y. \end{aligned}$$

Using Corollary 11 we get

$$2g((\nabla_X\varphi)Y, Z) = 2\mathrm{d}\eta(\varphi Y, X)\eta(Z) = 2g(X,Y)\eta(Z)$$

for any $Z \in TM$(Z is not a n*-char vector field*). Thus

$$(\nabla_X\varphi)Y = 2g(X,Y)\xi \tag{23}$$

and therefore $(\nabla_X\varphi)Y = (\nabla_Y\varphi)X$ and the proof is trivial. ■

Corollary 13. *Let* M *be a contact metric manifold, then*

$$(\nabla_X\varphi)Y \neq 0$$

where $X, Y \in TM$ *are* n-char vector fields.

Proof: Taking into account equations (13) and (23) we have

$$(\nabla_X \varphi)Y = [X, \varphi Y] = 2g(X, Y)\xi$$

and the rest of the proof is trivial. ■

Corollary 14. *Let M be a contact metric manifold, then*

$$(\breve{\nabla}_X \varphi)Y = 0$$

where $X, Y \in TM$ are n-char vector fields.

Proof: From (14) we have

$$(\breve{\nabla}_X \varphi)Y = (\nabla_X \varphi)Y - 2g(X, Y)\xi. \tag{24}$$

Then using equation (23) we can complete the proof. ■

Corollary 15. *Let M be a contact metric manifold, then*

$$\varphi \nabla_X \varphi Y + \nabla_X Y = \frac{1}{n}\xi$$

where $X, Y \in TM$ are n-char vector fields.

Proof: From (23) we have

$$\nabla_X \varphi Y - \varphi \nabla_X Y = 2g(X, Y)\xi. \tag{25}$$

Applying φ we obtain

$$\varphi \nabla_X \varphi Y + \nabla_X Y - \eta(\nabla_X Y)\xi = 0. \tag{26}$$

Using equation (8) and Corollary 5 the proof is trivial. ■

Corollary 16. *Let M be a contact metric manifold, then*

$$(\nabla_{\varphi X} \varphi)\varphi Y = 0 \tag{27}$$

where $X, Y \in TM$ are n-char vector fields.

Proof: From Theorem 8 and (23) the proof is trivial. ■

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