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ON THE GEOMETRY OF BIHARMONIC SUBMANIFOLDS IN SASAKIAN SPACE FORMS

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Abstract. We classify all proper-biharmonic Legendre curves in a Sasakian space form and point out some of their geometric properties. Then we provide a method for constructing anti-invariant proper-biharmonic submanifolds in the Sasakian space forms. Finally, using the Boothby-Wang fibration, we determine all proper-biharmonic Hopf cylinders over homogeneous real hypersurfaces in complex projective spaces.

1. Introduction

As defined by Eells and Sampson in [14], *harmonic maps* $f : (M, g) \to (N, h)$ are the critical points of the *energy functional*

$$E(f) = \frac{1}{2} \int_M \|\mathrm{d}f\|^2 \, v_g$$

and they are solutions of the associated Euler-Lagrange equation

$$\tau(f) = \mathrm{tr}_g \nabla \mathrm{d}f = 0$$

where $\tau(f)$ is called the *tension field* of f. When f is an isometric immersion with mean curvature vector field H, then $\tau(f) = mH$ and f is harmonic if and only if it is minimal.

The *bienergy functional* (proposed also by Eells and Sampson in 1964, [14]) is defined by

$$E_2(f) = \frac{1}{2} \int_M \|\tau(f)\|^2 \, v_g.$$

The critical points of E_2 are called *biharmonic maps* and they are solutions of the Euler-Lagrange equation (derived by Jiang in 1986, [20]):

$$\tau_2(f) = -\Delta^f \tau(f) - \mathrm{tr}_g R^N(\mathrm{d}f, \tau(f))\mathrm{d}f = 0$$

where Δ^f is the Laplacian on sections of $f^{-1}TN$ and $R^N(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ is the curvature operator on N; $\tau_2(f)$ is called the *bitension field* of f. Since all harmonic maps are biharmonic, we are interested in studying those which are biharmonic but non-harmonic, called *proper-biharmonic* maps.

Now, if $f : M \to N_c$ is an isometric immersion into a space form of constant sectional curvature c, then

$$\tau(f) = mH \quad \text{and} \quad \tau_2(f) = -m\Delta^f H + cm^2 H.$$

Thus f is biharmonic if and only if

$$\Delta^f H = mcH.$$

In a different way, Chen defined the biharmonic submanifolds in an Euclidean space as those with harmonic mean curvature vector field ([10]). Replacing $c = 0$ in the above equation we just reobtain Chen's definition. Moreover, let $f : M \to \mathbb{R}^n$ be an isometric immersion. Set $f = (f^1, \ldots, f^n)$ and $H = (H^1, \ldots, H^n)$. Then $\Delta^f H = (\Delta H^1, \ldots, \Delta H^n)$, where Δ is the Beltrami-Laplace operator on M, and f is biharmonic if and only if

$$\Delta^f H = \Delta(\frac{-\Delta f}{m}) = -\frac{1}{m}\Delta^2 f = 0.$$

There are several classification results for the proper-biharmonic submanifolds in Euclidean spheres and non-existence results for such submanifolds in the space forms manifolds N_c, $c \leq 0$ ([4, 5, 7–10, 13]), while in spaces of non-constant sectional curvature only a few results were obtained ([1, 12, 18, 19, 25, 29]).

We recall that the proper-biharmonic curves of the unit Euclidean two-dimensional sphere $\mathbb{S}^2$ are the circles of radius $\frac{1}{\sqrt{2}}$, and the proper-biharmonic curves of $\mathbb{S}^3$ are the geodesics of the minimal Clifford torus $\mathbb{S}^1\left(\frac{1}{\sqrt{2}}\right) \times \mathbb{S}^1\left(\frac{1}{\sqrt{2}}\right)$ with the slope different from ± 1. The proper-biharmonic curves of $\mathbb{S}^3$ are helices. Further, the proper-biharmonic curves of $\mathbb{S}^n$, $n > 3$, are those of $\mathbb{S}^3$ (up to a totally geodesic embedding). Concerning the hypersurfaces of $\mathbb{S}^n$, it was conjectured in [4] that the only proper-biharmonic hypersurfaces are the open parts of $\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{2}}\right)$ or $\mathbb{S}^{m_1}\left(\frac{1}{\sqrt{2}}\right) \times \mathbb{S}^{m_2}\left(\frac{1}{\sqrt{2}}\right)$ with $m_1 + m_2 = n - 1$ and $m_1 \neq m_2$.

Since odd dimensional unit Euclidean spheres $\mathbb{S}^{2n+1}$ are Sasakian space forms with constant φ-sectional curvature one, the next step is to study the biharmonic submanifolds of Sasakian space forms. In this paper we mainly gather the results obtained in [15–17].

We note that the proper-biharmonic submanifolds in pseudo-Riemannian manifolds are also intensively-studied (for example, see [2, 3, 11]).

For a general account of biharmonic maps see [22] and *The Bibliography of Biharmonic Maps* [28].

Conventions. We work in the C^∞ category, that means manifolds, metrics, connections and maps are smooth. The Lie algebra of the vector fields on N is denoted by $C(TN)$.

2. Sasakian Space Forms

In this section we briefly recall some basic facts from the theory of Sasakian manifolds. For more details see [6].

A *contact metric structure* on a manifold N^{2n+1} is given by (φ, ξ, η, g), where φ is a tensor field of type $(1, 1)$ on N, ξ is a vector field on N, η is an one-form on N and g is a Riemannian metric, such that

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad g(X, \varphi Y) = \mathrm{d}\eta(X, Y)$$

for any $X, Y \in C(TN)$.

A contact metric structure (φ, ξ, η, g) is *Sasakian* if it is *normal*, i.e.,

$$N_\varphi + 2\mathrm{d}\eta \otimes \xi = 0$$

where for all $X, Y \in C(TN)$

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y]$$

is the Nijenhuis tensor field of φ.

The *contact distribution* of a Sasakian manifold $(N, \varphi, \xi, \eta, g)$ is defined by $\{X \in TN;\ \eta(X) = 0\}$, and any integral curve of the contact distribution is called *Legendrian curve*.

A submanifold M of N which is tangent to ξ is said to be *anti-invariant* if φ maps any vector tangent to M and normal to ξ to a vector normal to M.

Let $(N, \varphi, \xi, \eta, g)$ be a Sasakian manifold. The sectional curvature of a two-plane generated by X and φX, where X is an unit vector orthogonal to ξ, is called

φ-sectional curvature determined by X. A Sasakian manifold with constant φ-sectional curvature c is called a *Sasakian space form* and it is denoted by $N(c)$.

A contact metric manifold $(N, \varphi, \xi, \eta, g)$ is called *regular* if for any point $p \in N$ there exists a cubic neighborhood of p such that any integral curve of ξ passes through the neighborhood at most once, and *strictly regular* if all integral curves are homeomorphic to each other.

Let $(N, \varphi, \xi, \eta, g)$ be a regular contact metric manifold. Then the orbit space $\bar{N} = N/\xi$ has a natural manifold structure and, moreover, if N is compact then N is a principal circle bundle over $\bar{N}$ (the Boothby-Wang Theorem). In this case the fibration $\pi : N \to \bar{N}$ is called *Boothby-Wang fibration*. The Hopf fibration $\pi : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ is a well-known example of a Boothby-Wang fibration.

Theorem 1 ([24]) *Let $(N, \varphi, \xi, \eta, g)$ be a strictly regular Sasakian manifold. Then on $\bar{N}$ can be given the structure of a Kähler manifold. Moreover, if $(N, \varphi, \xi, \eta, g)$ is a Sasakian space form $N(c)$, then $\bar{N}$ has constant sectional holomorphic curvature $c + 3$.*

Even if N is non-compact, we still call the fibration $\pi : N \to \bar{N}$ of a strictly regular Sasakian manifold, the Boothby-Wang fibration.

3. Biharmonic Legendre Curves in Sasakian Space Forms

Let (N^n, g) be a Riemannian manifold and $\gamma : I \to N$ a curve parametrized by arc length. Then γ is called a *Frenet curve of osculating order* r, $1 \leq r \leq n$, if there exists orthonormal vector fields $E_1, E_2, \ldots, E_r$ along γ such that $E_1 = \gamma' = T$, $\nabla_T E_1 = \kappa_1 E_2$, $\nabla_T E_2 = -\kappa_1 E_1 + \kappa_2 E_3$, $\ldots$, $\nabla_T E_r = -\kappa_{r-1} E_{r-1}$, where $\kappa_1, \ldots, \kappa_{r-1}$ are positive functions on I.

A geodesic is a Frenet curve of osculating order one, a *circle* is a Frenet curve of osculating order two with $\kappa_1 = \text{const}$, a *helix of order* r, $r \geq 3$, is a Frenet curve of osculating order r with $\kappa_1, \ldots, \kappa_{r-1}$ constants and a helix of order three is called, simply, helix.

In [16] we studied the biharmonicity of Legendre Frenet curves and we obtained the following results.

Let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a Sasakian space form with constant φ-sectional curvature c and $\gamma : I \to N$ a Legendre Frenet curve of osculating order r. Then γ is biharmonic if and only if

$$
\begin{aligned}
\tau_2(\gamma) &= \nabla_T^3 T - R(T, \nabla_T T)T \\
&= (-3\kappa_1\kappa_1')E_1 + \Big(\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2 + \frac{(c+3)\kappa_1}{4}\Big)E_2 \\
&\quad + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')E_3 + \kappa_1\kappa_2\kappa_3 E_4 + \frac{3(c-1)\kappa_1}{4} g(E_2, \varphi T)\varphi T \\
&= 0.
\end{aligned}
$$

The expression of the bitension field $\tau_2(\gamma)$ imposed a case-by-case analysis as follows.

Case I ($c = 1$)

Theorem 2 ([16]) *If $c = 1$ then γ is proper-biharmonic if and only if $n \geq 2$ and either γ is a circle with $\kappa_1 = 1$ or γ is a helix with $\kappa_1^2 + \kappa_2^2 = 1$.*

Case II ($c \neq 1$ and $E_2 \perp \varphi T$)

Theorem 3 ([16]) *Assume that $c \neq 1$ and $E_2 \perp \varphi T$. We have*

1) *if $c \leq -3$ then γ is biharmonic if and only if it is a geodesic*;

2) *if $c > -3$ then γ is proper-biharmonic if and only if either*

   a) $n \geq 2$ *and γ is a circle with* $\kappa_1^2 = \dfrac{c+3}{4}$, *or*

   b) $n \geq 3$ *and γ is a helix with* $\kappa_1^2 + \kappa_2^2 = \dfrac{c+3}{4}\cdot$

Case III ($c \neq 1$ and $E_2 \parallel \varphi T$)

Theorem 4 ([16]) *If $c \neq 1$ and $E_2 \parallel \varphi T$, then $\{T, \varphi T, \xi\}$ is the Frenet frame field of γ and we have*

1) *if $c < 1$ then γ is biharmonic if and only if it is a geodesic*

2) *if $c > 1$ then γ is proper-biharmonic if and only if it is a helix with $\kappa_1^2 = c - 1$ and $\kappa_2 = 1$.*

Remark 5. *In dimension three the result was obtained by Inoguchi in* [19] *and explicit examples are given in* [15].

Case IV ($c \neq 1$ and $g(E_2, \varphi T)$ is not constant 0, 1 or -1)

Theorem 6 ([16]) *Let $c \neq 1$ and γ a Legendre Frenet curve of osculating order r such that $g(E_2, \varphi T)$ is not constant 0, 1 or -1. We have*

1) *if $c \leq -3$ then γ is biharmonic if and only if it is a geodesic*;

2) *if $c > -3$ then γ is proper-biharmonic if and only if $r \geq 4$, $\varphi T = \cos\alpha_0 E_2 + \sin\alpha_0 E_4$ and*

$$\kappa_1, \kappa_2, \kappa_3 = \text{const} > 0$$
$$\kappa_1^2 + \kappa_2^2 = \frac{c+3}{4} + \frac{3(c-1)}{4}\cos^2\alpha_0$$
$$\kappa_2\kappa_3 = -\frac{3(c-1)}{8}\sin(2\alpha_0)$$

where $\alpha_0 \in (0, 2\pi) \setminus \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ is a constant such that

$$c + 3 + 3(c-1)\cos^2\alpha_0 > 0, \qquad 3(c-1)\sin(2\alpha_0) < 0.$$

In order to obtain explicit examples of proper-biharmonic Legendre curves given by Theorem 2 we used the unit Euclidean sphere $\mathbb{S}^{2n+1}$ as a model of a Sasakian space form with $c = 1$ and we proved the following

Theorem 7 ([16]) *Let $\gamma : I \to \mathbb{S}^{2n+1}(1)$, $n \geq 2$, be a proper-biharmonic Legendre curve parametrized by arc length. Then the parametric equation of γ in the Euclidean space $\mathbb{E}^{2n+2} = (\mathbb{R}^{2n+2}, \langle\,,\rangle)$ is either*

$$\gamma(s) = \frac{1}{\sqrt{2}}\cos\left(\sqrt{2}s\right)e_1 + \frac{1}{\sqrt{2}}\sin\left(\sqrt{2}s\right)e_2 + \frac{1}{\sqrt{2}}e_3$$

where $\{e_i, \mathcal{I}e_j\}$ are constant unit vectors orthogonal to each other, or

$$\gamma(s) = \frac{1}{\sqrt{2}}\cos(As)e_1 + \frac{1}{\sqrt{2}}\sin(As)e_2 + \frac{1}{\sqrt{2}}\cos(Bs)e_3 + \frac{1}{\sqrt{2}}\sin(Bs)e_4$$

where $A = \sqrt{1+\kappa_1}$, $B = \sqrt{1-\kappa_1}$, $\kappa_1 \in (0,1)$, $\{e_i\}$ are constant unit vectors orthogonal to each other such that

$$\langle e_1, \mathcal{I}e_3\rangle = \langle e_1, \mathcal{I}e_4\rangle = \langle e_2, \mathcal{I}e_3\rangle = \langle e_2, \mathcal{I}e_4\rangle = 0$$

$$A\langle e_1, \mathcal{I}e_2\rangle + B\langle e_3, \mathcal{I}e_4\rangle = 0$$

and $\mathcal{I}$ is the usual complex structure on $\mathbb{R}^{2n+2}$.

Remark 8. *For the Cases II and III we also obtained the explicit equations of proper-biharmonic Legendre curves in odd dimensional spheres endowed with the deformed Sasakian structure introduced in* [27].

In [21] are introduced the complex torsions for a Frenet curve in a complex manifold. In the same way, for $\gamma : I \to N$ a Legendre Frenet curve of osculating order r in a Sasakian manifold $(N^{2n+1}, \varphi, \xi, \eta, g)$, we define the φ-torsions $\tau_{ij} = g(E_i, \varphi E_j) = -g(\varphi E_i, E_j), i, j = 1, \ldots, r, i < j$.
It is easy to see that we can formulate

Proposition 9. *Let $\gamma : I \to N(c)$ be a proper-biharmonic Legendre Frenet curve in a Sasakian space form $N(c)$, $c \neq 1$. Then $c > -3$ and τ_{12} is a constant.*

Moreover

Proposition 10. *If γ is a proper-biharmonic Legendre Frenet curve in a Sasakian space form $N(c)$, $c > -3$, $c \neq 1$, of osculating order $r < 4$, then it is a circle or a helix with constant φ-torsions.*

Proof: From Theorems 3, 4 and 6 we see that if γ is a proper-biharmonic Legendre Frenet curve of osculating order $r < 4$, then $\tau_{12} = 0$ or $\tau_{12} = \pm 1$ and, obviously, we only have to prove that when γ is a helix then τ_{13} and τ_{23} are constants.
Indeed, by using the Frenet equations of γ, we have

$$\begin{aligned}\tau_{13} = g(E_1, \varphi E_3) &= -\frac{1}{\kappa_2} g(\varphi E_1, \nabla_{E_1} E_2 + \kappa_1 E_1) = -\frac{1}{\kappa_2} g(\varphi E_1, \nabla_{E_1} E_2)\\ &= \frac{1}{\kappa_2} g(E_2, \nabla_{E_1} \varphi E_1) = \frac{1}{\kappa_2} g(E_2, \varphi \nabla_{E_1} E_1 + \xi) = 0\end{aligned}$$

since

$$g(E_2, \xi) = \frac{1}{\kappa_1} g(\nabla_{E_1} E_1, \xi) = -\frac{1}{\kappa_1} g(E_1, \nabla_{E_1} \xi) = \frac{1}{\kappa_1} g(E_1, \varphi E_1) = 0.$$

On the other hand, it is easy to see that for any Frenet curve of osculating order three we have $\tau_{23} = \dfrac{1}{\kappa_1}(\tau_{13}' + \kappa_2 \tau_{12} + \eta(E_3))$ and

$$\begin{aligned}\eta(E_3) = g(E_3, \xi) &= \frac{1}{\kappa_2}(g(\nabla_{E_1} E_2, \xi) + \kappa_1 g(E_1, \xi)) = -\frac{1}{\kappa_2} g(E_2, \nabla_{E_1} \xi)\\ &= -\frac{1}{\kappa_2} \tau_{12}.\end{aligned}$$

In conclusion $\tau_{23} = \dfrac{1}{\kappa_1}(\tau'_{13} + \kappa_2\tau_{12} - \dfrac{1}{\kappa_2}\tau_{12}) = \text{const}.$ ■

Proposition 11. *If γ is a proper-biharmonic Legendre Frenet curve in a Sasakian space form $N(c)$ of osculating order $r = 4$, then $c \in (\frac{7}{3}, 5)$ and the curvatures of γ are*

$$\kappa_1 = \frac{\sqrt{c+3}}{2}, \qquad \kappa_2 = \frac{1}{2}\sqrt{\frac{6(c-1)(5-c)}{c+3}}, \qquad \kappa_3 = \frac{1}{2}\sqrt{\frac{3(c-1)(3c-7)}{c+3}}.$$

Moreover, the φ-torsions of γ are given by

$$\tau_{12} = \mp\sqrt{\frac{2(5-c)}{c+3}}, \qquad \tau_{13} = 0, \qquad \tau_{14} = \pm\sqrt{\frac{3c-7}{c+3}}$$

$$\tau_{23} = \mp\frac{3c-7}{\sqrt{3(c-1)(c+3)}}, \qquad \tau_{24} = 0, \qquad \tau_{34} = \pm\sqrt{\frac{2(5-c)(3c-7)}{3(c-1)(c+3)}}.$$

Proof: Let γ be a proper-biharmonic Legendre Frenet curve in $N(c)$ of osculating order $r = 4$. Then $c \neq 1$ and τ_{12} is different from 0, 1 or -1. From Theorem 6 we have $\varphi E_1 = \cos\alpha_0 E_2 + \sin\alpha_0 E_4$. It results that

$$\tau_{12} = -\cos\alpha_0, \qquad \tau_{13} = 0, \qquad \tau_{14} = -\sin\alpha_0 \quad \text{and} \quad \tau_{24} = 0.$$

In order to prove that τ_{23} is constant we differentiate the expression of φE_1 along γ and using the Frenet equations we obtain

$$\begin{aligned}\nabla_{E_1}\varphi E_1 &= \cos\alpha_0\nabla_{E_1}E_2 + \sin\alpha_0\nabla_{E_1}E_4\\ &= -\kappa_1\cos\alpha_0 E_1 + (\kappa_2\cos\alpha_0 - \kappa_3\sin\alpha_0)E_3.\end{aligned}$$

On the other hand, $\nabla_{E_1}\varphi E_1 = \kappa_1\varphi E_2 + \xi$ and therefore we have

$$\kappa_1\varphi E_2 + \xi = -\kappa_1\cos\alpha_0 E_1 + (\kappa_2\cos\alpha_0 - \kappa_3\sin\alpha_0)E_3. \tag{1}$$

We take the scalar product in (1) with ξ and obtain

$$(\kappa_2\cos\alpha_0 - \kappa_3\sin\alpha_0)\eta(E_3) = 1. \tag{2}$$

In the same way as in the proof of Proposition 10 we get

$$\begin{aligned}\eta(E_3) = g(E_3,\xi) &= \frac{1}{\kappa_2}(g(\nabla_{E_1}E_2,\xi) + \kappa_1 g(E_1,\xi))\\ &= -\frac{1}{\kappa_2}g(E_2,\nabla_{E_1}\xi)\\ &= -\frac{1}{\kappa_2}\tau_{12} = \frac{\cos\alpha_0}{\kappa_2}\end{aligned}$$

and then, from (2), $\kappa_2 \sin\alpha_0 = -\kappa_3 \cos\alpha_0$. Therefore $\alpha_0 \in (\frac{\pi}{2}, \pi) \cup (\frac{3\pi}{2}, 2\pi)$.
Next, from Theorem 6, we have

$$\kappa_1^2 = \frac{c+3}{4}, \qquad \kappa_2^2 = \frac{3(c-1)}{4}\cos^2\alpha_0, \qquad \kappa_3^2 = \frac{3(c-1)}{4}\sin^2\alpha_0$$

and so c must be greater than one.
Now, we take the scalar product in (1) with E_3, φE_2 and φE_4, respectively, and we get

$$\kappa_1\tau_{23} = -(\kappa_2\cos\alpha_0 - \kappa_3\sin\alpha_0) + \eta(E_3) = -\frac{\kappa_2}{\cos\alpha_0} + \frac{\cos\alpha_0}{\kappa_2} \tag{3}$$

$$\kappa_1\sin^2\alpha_0 = -(\kappa_2\cos\alpha_0 - \kappa_3\sin\alpha_0)\tau_{23} = -\frac{\kappa_2}{\cos\alpha_0}\tau_{23} \tag{4}$$

$$\begin{aligned} 0 &= \kappa_1\cos\alpha_0\sin\alpha_0 + (\kappa_2\cos\alpha_0 - \kappa_3\sin\alpha_0)\tau_{34} \\ &= \kappa_1\cos\alpha_0\sin\alpha_0 + \frac{\kappa_2}{\cos\alpha_0}\tau_{34} \end{aligned} \tag{5}$$

and then, equations (3) and (4) lead to $\kappa_1^2\sin^2\alpha_0 = \dfrac{\kappa_2^2}{\cos^2\alpha_0} - 1$. We come to the conclusion that $\sin^2\alpha_0 = \dfrac{3c-7}{c+3}$, so $c \in \left(\dfrac{7}{3}, 5\right)$, and then we obtain the expressions of the curvatures and the φ-torsions. ■

Remark 12. *The proper-biharmonic Legendre curves given by* Theorem 7 (*for the case* $c = 1$) *have also constant* φ*-torsions.*

4. A Method to Obtain Biharmonic Submanifolds in a Sasakian Space Form

In [16] we gave a method to obtain proper-biharmonic anti-invariant submanifolds in a Sasakian space form from proper-biharmonic integral submanifolds.

Theorem 13 ([16]) *Let* $(N^{2n+1}, \varphi, \xi, \eta, g)$ *be a strictly regular Sasakian space form with constant* φ*-sectional curvature c and let* $\boldsymbol{i} : M \to N$ *be an* r*-dimensional integral submanifold of* N, $1 \leq r \leq n$. *Consider*

$$F : \widetilde{M} = I \times M \to N, \qquad F(t,p) = \phi_t(p) = \phi_p(t)$$

where $I = \mathbb{S}^1$ *or* $I = \mathbb{R}$ *and* $\{\phi_t\}_{t\in I}$ *is the flow of the vector field* ξ. *Then* $F : (\widetilde{M}, \widetilde{g} = \mathrm{d}t^2 + \boldsymbol{i}^*g) \to N$ *is a Riemannian immersion and it is proper-biharmonic if and only if* M *is a proper-biharmonic submanifold of* N.

The previous Theorem provides a classification result for proper-biharmonic surfaces in a Sasakian space form, which are invariant under the flow-action of ξ.

Theorem 14 ([16]) *Let M^2 be a surface of $N^{2n+1}(c)$ invariant under the flow-action of the characteristic vector field ξ. Then M is proper-biharmonic if and only if, locally, it is given by $x(t,s) = \phi_t(\gamma(s))$, where γ is a proper-biharmonic Legendre curve.*

Also, using the standard Sasakian 3-structure on $\mathbb{S}^7$, by iteration, Theorem 13 leads to examples of three-dimensional proper-biharmonic submanifolds of $\mathbb{S}^7$.

5. Biharmonic Hopf Cylinders in a Sasakian Space Form

Let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a strictly regular Sasakian manifold and $\bar{\mathbf{i}} : \bar{M} \to \bar{N}$ a submanifold of $\bar{N}$. Then $M = \pi^{-1}(\bar{M})$ is the Hopf cylinder over $\bar{M}$, where $\pi : N \to \bar{N} = N/\xi$ is the Boothby-Wang fibration.

In [19] the biharmonic Hopf cylinders in a three-dimensional Sasakian space form are classified.

Theorem 15 ([19]) *Let $S_{\bar{\gamma}}$ be a Hopf cylinder, where $\bar{\gamma}$ is a curve in the orbit space of $N^3(c)$, parametrized by arc length. We have*

1) *if $c \leqslant 1$, then $S_{\bar{\gamma}}$ is biharmonic if and only if it is minimal*;
2) *if $c > 1$, then $S_{\bar{\gamma}}$ is proper-biharmonic if and only if the curvature $\bar{\kappa}$ of $\bar{\gamma}$ is constant $\bar{\kappa}^2 = c - 1$.*

In [17] we obtained a geometric characterization of biharmonic Hopf cylinders of any codimension in an arbitrary Sasakian space form. A special case of our result is the case when $\bar{M}$ is a hypersurface.

Proposition 16 ([17]) *If $\bar{M}$ is a hypersurface of $\bar{N}$, then $M = \pi^{-1}(\bar{M})$ is biharmonic if and only if*

$$\Delta^{\perp} H = \left(-\|B\|^2 + \frac{c(n+1) + 3n - 1}{2} \right) H$$

$$2\operatorname{tr} A_{\nabla^{\perp}_{\cdot} H}(\cdot) + n \operatorname{grad}(\|H\|^2) = 0$$

where B, A and H are the second fundamental form of M in N, the shape operator and the mean curvature vector field, respectively, and $\nabla^{\perp}$ and $\Delta^{\perp}$ are the normal connection and Laplacian on the normal bundle of M in N.

Proposition 17 ([17]) *If $\bar{M}$ is a hypersurface and $\|\bar{H}\| = \mathrm{const} \neq 0$, then $M = \pi^{-1}(\bar{M})$ is proper-biharmonic if and only if*

$$\|\bar{B}\|^2 = \frac{c(n+1) + 3n - 5}{2}\cdot$$

Remark 18. *From the last result we see that there exist no proper-biharmonic hypersurfaces of constant mean curvature $M = \pi^{-1}(\bar{M})$ in $N(c)$ if $c \leq \dfrac{5-3n}{n+1}$, which implies that such hypersurfaces do not exist if $c \leq -3$, whatever the dimension of N is.*

In [26] Takagi classified all homogeneous real hypersurfaces in the complex projective space $\mathbb{CP}^n$, $n > 1$, and found five types of such hypersurfaces (see also [23]). The first type (with subtypes $A1$ and $A2$) are described in the following. We shall consider $u \in (0, \frac{\pi}{2})$ and r a positive constant given by $\dfrac{1}{r^2} = \dfrac{c+3}{4}\cdot$

Theorem 19 ([26]) *The geodesic spheres (Type $A1$) in complex projective space $\mathbb{CP}^n(c+3)$ have two distinct principal curvatures: $\lambda_2 = \frac{1}{r}\cot u$ of multiplicity $2n-2$ and $a = \frac{2}{r}\cot 2u$ of multiplicity one.*

Theorem 20 ([26]) *The hypersurfaces of Type $A2$ in complex projective space $\mathbb{CP}^n(c+3)$ have three distinct principal curvatures: $\lambda_1 = -\frac{1}{r}\tan u$ of multiplicity $2p$, $\lambda_2 = \frac{1}{r}\cot u$ of multiplicity $2q$, and $a = \frac{2}{r}\cot 2u$ of multiplicity one, where $p > 0$, $q > 0$, and $p + q = n - 1$.*

We note that if $c = 1$ and $\bar{M}$ is of type $A1$ or $A2$ then $\pi^{-1}(\bar{M}) = \mathbb{S}^1(\cos u) \times \mathbb{S}^{2n-1}(\sin u) \subset \mathbb{S}^{2n+1}$ or $\pi^{-1}(\bar{M}) = \mathbb{S}^{2p+1}(\cos u) \times \mathbb{S}^{2q+1}(\sin u)$, respectively.
By using Takagi's result we classified in [17] the biharmonic Hopf cylinders $M = \pi^{-1}(\bar{M})$ in a Sasakian space form N^{2n+1} over homogeneous real hypersurfaces in $\mathbb{CP}^n$, $n > 1$.

Theorem 21 ([17]) *Let $M = \pi^{-1}(\bar{M})$ be the Hopf cylinder over $\bar{M}$.*

1) *If $\bar{M}$ is of Type $A1$, then M is proper-biharmonic if and only if either*

   a) $c = 1$ and $\tan^2 u = 1$, or

b) $c \in \Big[\dfrac{-3n^2+2n+1+8\sqrt{2n-1}}{n^2+2n+5}, +\infty\Big) \setminus \{1\}$ *and*

$$\tan^2 u = n + \tfrac{2c-2}{c+3} \pm \tfrac{\sqrt{c^2(n^2+2n+5)+2c(3n^2-2n-1)+9n^2-30n+13}}{c+3}.$$

2) *If $\bar{M}$ is of Type $A2$, then M is proper-biharmonic if and only if either*

a) $c = 1$, $\tan^2 u = 1$ *and* $p \neq q$, *or*

b) $c \in \Big[\dfrac{-3(p-q)^2-4n+4+8\sqrt{(2p+1)(2q+1)}}{(p-q)^2+4n+4}, +\infty\Big) \setminus \{1\}$ *and*

$$\tan^2 u = \tfrac{n}{2p+1} + \tfrac{2c-2}{(c+3)(2p+1)} \pm \tfrac{\sqrt{c^2((p-q)^2+4n+4)+2c(3(p-q)^2+4n-4)+9(p-q)^2-12n+4}}{(c+3)(2p+1)}.$$

Theorem 22 ([17]) *There are no proper-biharmonic hypersurfaces $M = \pi^{-1}(\bar{M})$ when $\bar{M}$ is a hypersurface of Type B, C, D or E in the complex projective space $\mathbb{CP}^n(c+3)$.*

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