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DECOMPOSITION THEOREMS FOR HILBERT MODULAR FORMS

Benjamin Linowitz

Abstract: Let $\mathscr{S}_k^+(\mathcal{N},\Phi)$ denote the space generated by Hilbert modular newforms (over a fixed totally real field K) of weight k, level $\mathcal{N}$ and Hecke character Φ. In this paper we examine the behavior of $\mathscr{S}_k^+(\mathcal{N},\Phi)$ under twists (by a Hecke character). We show how this space may be decomposed into a direct sum of twists of other spaces of newforms. This sheds light on the behavior of a newform under a character twist: the exact level of the twist of a newform, when such a twist is itself a newform, and when a newform may be realized as the twist of a primitive newform. In certain cases it is shown that the entire space $\mathscr{S}_k^+(\mathcal{N},\Phi)$ can be represented as a direct sum of twists of primitive nebenspaces. This adds perspective to the Jacquet-Langlands correspondence, which characterizes those elements of $\mathscr{S}_k^+(\mathcal{N},\Phi)$ not representable as theta series arising from a quaternion algebra as being precisely those forms which are twists of primitive nebenforms. It follows that in these cases no newforms arise from a quaternion algebra. These results were proven for elliptic modular forms by Hijikata, Pizer and Shemanske by employing the Eichler-Selberg trace formula.

Keywords: Hilbert modular form, newform, character twist.

1. Introduction

Let K be a totally real number field. In this paper we study the structure of spaces of Hilbert modular cusp forms over K. Shimura [11] showed that these spaces possess a newform theory, so it suffices to restrict our attention to spaces of Hilbert modular newforms. In order to understand these spaces, one often studies the behavior of newforms under character twists. Motivation for studying newforms via their character twists ranges from Weil's converse theorem [13] to the Jacquet-Langlands correspondence [7, Theorem 16.1], which characterizes those newforms not representable as a theta series arising from a quaternion algebra as being precisely those which are twists of primitive nebenforms.

More specifically, we are interested in extending to the Hibert modular setting a number of results about elliptic modular newforms which concern the behavior under character twists of both individual newforms and spaces generated by new-

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forms. For the elliptic case there were two perspectives. The first was that of Atkin and Li [1], who proved a number of important theorems about twists of newforms. In particular they determined the exact level of the twist of a newform and shed light on when the twist of a newform is itself a newform. The second perspective was that of Hijikata, Pizer and Shemanske [6], who used the Eichler-Selberg trace formula for Hecke operators in order to prove structural results about the space generated by newforms of a fixed weight, level and character and in particular to decompose such a space into sums of twists of other spaces generated by newforms. The latter perspective, in which one studies the behavior of the space generated by newforms under character twists, is an important strengthening of the former perspective. Indeed, one can deduce many properties about twists of individual newforms in a straightforward manner from the aforementioned decomposition theorems.

It is not difficult to show that when one twists a (Hilbert modular) newform by a character whose conductor is coprime to the level, the resulting modular form is always a newform (of suitable character and level). The complementary situation, in which the twist is by a character whose conductor is a power of a prime dividing the level, is considerably more subtle and is the focus of this paper. That the latter situation is indeed more subtle can be seen in the following theorem (our Theorem 3.11), which makes clear the fragile dependence of the twist of a newform being a newform upon both the conductor of the twisting character and the level of the newform being twisted (see Section 2 for notation and terminology).

Theorem 1.1. *Let $\mathcal{N}$ be an integral ideal which we decompose as $\mathcal{N} = \mathcal{P}\mathcal{N}_0$ for $\mathcal{P}$ a power of a prime ideal $\mathfrak{p}$ coprime to $\mathcal{N}_0$. Set $\nu = ord_{\mathfrak{p}}(\mathcal{P})$. Let ϕ be a numerical character modulo $\mathcal{N}$ and Φ a Hecke character extending $\phi\phi_\infty$. Write $\Phi = \Phi_{\mathcal{P}}\Phi_{\mathcal{N}_0}$ and denote the finite part of the conductor of $\Phi_{\mathcal{P}}$ by $\mathfrak{p}^{e(\Phi_{\mathcal{P}})}$. Let Ψ be a Hecke character whose conductor has finite part a power of $\mathfrak{p}$.*

If $0 < e(\Psi) < \frac{\nu}{2}$ and $e(\Phi_{\mathcal{P}}) + e(\Psi) < \nu$ then

$$\mathscr{S}_k^+(\mathcal{N}, \Phi)^\Psi = \mathscr{S}_k^+(\mathcal{N}, \Psi^2\Phi).$$

In the vein of studying twists of individual newforms we have Theorem 3.14, which gives necessary and sufficient conditions for a newform to be represented as the twist of a newform in terms of the vanishing of certain Fourier coefficients, the level of the newform and the conductor of the twisting character.

Many of our theorems deal with the structure of the entire space generated by newforms and its twists by various characters. Our motivation for studying structural properties of twists of the space generated by newforms is the Jacquet-Langlands correspondence, which implies that the complement of the subspace generated by theta series arising from a quaternion algebra is a sum of twists of primitive nebenspaces. A concrete realization of this can be found in [6], where explicit structure theorems were developed in order to facilitate a solution to Eichler's basis problem [5].

As the object of our study is the space generated by newforms, it is natural to make use of Hecke theory. The general approach of Hijikata, Pizer and Shemanske [6] was to establish isomorphisms between spaces of newforms and twists of spaces of newforms as modules for the Hecke algebra. They did this by establishing identities between traces of Hecke operators acting on these spaces. This involves using the Eichler-Selberg trace formula and in particular Hijikata's explicit formula [4] for the trace of a Hecke operator acting on a space of cusp forms.

Of course using trace formulae in the Hilbert modular setting might be seen as a natural way to extend some of the aforementioned theorems. However, formidable complications quickly arise, even when one tries to compute an explicit formula for the trace of the Hecke operator $T(1)$. In our generalization we are able in a number of cases to avoid the use of trace formulae and use only properties of newforms that were proven in the elliptic case by Li [8], and Atkin-Li [1], and later extended to the Hilbert modular case by Shemanske-Walling [9]. Our results are therefore new for Hilbert modular forms over totally real number fields other than $\mathbb{Q}$ and provide simplified proofs when specialized to the elliptic case.

The following theorem (our Theorem 3.12; see Section 2 for notation and terminology) adds perspective to the Jacquet-Langlands correspondence by showing that in certain cases the entire space generated by newforms can be represented as a sum of twists of primitive nebenspaces. In these cases it follows that no newforms arise from theta series coming from a quaternion algebra.

Theorem 1.2. *Let $\mathcal{N}$ be an integral ideal which we decompose as $\mathcal{N} = \mathcal{P}\mathcal{N}_0$ for $\mathcal{P}$ a power of a prime ideal $\mathfrak{p}$ coprime to $\mathcal{N}_0$. Set $\nu = ord_{\mathfrak{p}}\,\mathcal{P}$. Let ϕ be a numerical character modulo $\mathcal{N}$ and Φ be a Hecke character extending $\phi\phi_\infty$ which satisfies $\frac{\nu}{2} < e(\Phi_{\mathcal{P}}) = ord_{\mathfrak{p}}(\mathfrak{f}_{\Phi_{\mathcal{P}}}) < \nu$. Then*

$$\mathscr{S}_k^+(\mathcal{N},\Phi) = \bigoplus_{e(\Psi)=\nu-e(\Phi_{\mathcal{P}})} \mathscr{S}_k^+(\mathfrak{p}^{e(\Phi_{\mathcal{P}})}\mathcal{N}_0, \Psi^2\Phi)^{\overline{\Psi}},$$

where the sum $\bigoplus_{e(\Psi)=\nu-e(\Phi_{\mathcal{P}})}$ is taken over all $\mathfrak{p}$-primary Hecke characters Ψ with conductor $\mathfrak{p}^{\nu-e(\Phi_{\mathcal{P}})}$ and infinite part $\Psi_\infty(a) = sgn(a)^l$ for $l \in \mathbb{Z}^n$ and $a \in K_\infty^\times$.

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2. Notation and Preliminaries

For the most part we follow the notation of [9, 10, 11]. However, to make this paper somewhat self-contained, we shall briefly review the basic definitions of the functions and operators which we shall study.

Let K be a totally real number field of degree n over $\mathbb{Q}$ with ring of integers $\mathcal{O}$, group of units $\mathcal{O}^\times$ and totally positive units $\mathcal{O}_+^\times$. Let $\mathfrak{d}$ be the different of K. If $\mathfrak{q}$ is a finite prime of K, we denote by $K_{\mathfrak{q}}$ the completion of K at $\mathfrak{q}$, $\mathcal{O}_{\mathfrak{q}}$ the valuation ring of $K_{\mathfrak{q}}$, and $\pi_{\mathfrak{q}}$ a local uniformizer.

We denote by K_A the ring of K-adeles and by $K_A^\times$ the group of K-ideles. As usual we view K as a subgroup of K_A via the diagonal embedding. If $\tilde{\alpha} \in K_A^\times$, we let $\tilde{\alpha}_\infty$ denote the archimedean part of $\tilde{\alpha}$ and $\tilde{\alpha}_0$ the finite part of $\tilde{\alpha}$. If $\mathcal{J}$ is an integral ideal we let $\tilde{\alpha}_\mathcal{J}$ denote the $\mathcal{J}$-part of $\tilde{\alpha}$.

For an integral ideal $\mathcal{N}$ we define a numerical character ϕ modulo $\mathcal{N}$ to be a character $\phi : (\mathcal{O}/\mathcal{N})^\times \to \mathbb{C}^\times$, and a Hecke character to be a continuous character on the idele class group: $\Phi : K_A^\times / K^\times \to \mathbb{C}^\times$. We denote the induced character on $K_A^\times$ by Φ as well. Every Hecke character is of the form $\Phi(\tilde{\alpha}) = \prod_\nu \Phi_\nu(\alpha_\nu)$ where each Φ_ν is a character $\Phi_\nu : K_\nu^\times \longrightarrow \mathbb{C}^\times$. The conductor, cond$(\Phi)$, of Φ is defined to be the modulus whose finite part is $\mathfrak{f}_\Phi$ (see [3]) and whose infinite part is the formal product of those archimedean primes ν for which Φ_ν is nontrivial. In the case that $\mathfrak{f}_\Phi$ is a power of a single prime $\mathfrak{q}$, we define the exponential conductor $e(\Phi)$ to be the integer such that $\mathfrak{f}_\Phi = \mathfrak{q}^{e(\Phi)}$. We adopt the convention that ϕ and ψ will always denote numerical characters and Φ and Ψ will denote Hecke characters.

For a fractional ideal $\mathcal{I}$ and integral ideal $\mathcal{N}$, define

$$\Gamma_0(\mathcal{N}, \mathcal{I}) = \{A \in \begin{pmatrix} \mathcal{O} & \mathcal{I}^{-1}\mathfrak{d}^{-1} \\ \mathcal{N}\mathcal{I}\mathfrak{d} & \mathcal{O} \end{pmatrix} : \det A \in \mathcal{O}_+^\times\}.$$

Let $\theta : \mathcal{O}_+^\times \to \mathbb{C}^\times$ be a character of finite order and note that there exists an element $m \in \mathbb{R}^n$ such that $\theta(a) = a^{im}$ for all totally positive a. While such an m is not unique, we shall fix one such m for the remainder of this paper.

Let $k = (k_1, ..., k_n) \in \mathbb{Z}_+^n$ and ϕ be a numerical character modulo $\mathcal{N}$.

Following Shimura [10, 11], we define $M_k(\Gamma_0(\mathcal{N}, \mathcal{I}), \phi, \theta)$ to be the complex vector space of classical Hilbert modular forms on $\Gamma_0(\mathcal{N}, \mathcal{I})$.

It is well known that the space classical Hilbert modular forms of a fixed weight, character and congruence subgroup is not invariant under the entire Hecke algebra (see the introduction to [12] for a related discussion). We therefore consider the larger space of adelic Hilbert modular forms, which *is* invariant under the Hecke algebra. Our construction closely follows that of Shimura [11].

Fix a set of strict ideal class representatives $\mathcal{I}_1, ..., \mathcal{I}_h$ of K, set $\Gamma_\lambda = \Gamma_0(\mathcal{N}, \mathcal{I}_\lambda)$, and put

$$\mathfrak{M}_k(\mathcal{N}, \phi, \theta) = \prod_{\lambda=1}^{h} M_k(\Gamma_\lambda, \phi, \theta).$$

We are interested in studying h-tuples $(f_1, ..., f_h) \in \mathfrak{M}_k(\mathcal{N}, \phi, \theta)$.

Let $G_A = GL_2(K_A)$ and view $G_K = GL_2(K)$ as a subgroup of G_A via the diagonal embedding. Denote by $G_\infty = GL_2(\mathbb{R})^n$ the archimedean part of G_A. For an integral ideal $\mathcal{N}$ of $\mathcal{O}$ and a prime $\mathfrak{p}$, let

$$Y_\mathfrak{p}(\mathcal{N}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathcal{O}_\mathfrak{p} & \mathfrak{d}^{-1}\mathcal{O}_\mathfrak{p} \\ \mathcal{N}\mathfrak{d}\mathcal{O}_\mathfrak{p} & \mathcal{O}_\mathfrak{p} \end{pmatrix} : \det A \in K_\mathfrak{p}^\times, (a\mathcal{O}_\mathfrak{p}, \mathcal{N}\mathcal{O}_\mathfrak{p}) = 1 \right\},$$

$$W_\mathfrak{p}(\mathcal{N}) = \left\{ x \in Y_\mathfrak{p}(\mathcal{N}) : \det x \in \mathcal{O}_\mathfrak{p}^\times \right\}$$

and put

$$Y = Y(\mathcal{N}) = G_A \cap \left(G_{\infty+} \times \prod_{\mathfrak{p}} Y_{\mathfrak{p}}(\mathcal{N}) \right), \qquad W = W(\mathcal{N}) = G_{\infty+} \times \prod_{\mathfrak{p}} W_{\mathfrak{p}}(\mathcal{N}).$$

Given a numerical character ϕ modulo $\mathcal{N}$ define a homomorphism $\phi_Y : Y \to \mathbb{C}^\times$ by setting $\phi_Y(\begin{pmatrix} \tilde{a} & * \\ * & * \end{pmatrix}) = \phi(\tilde{a}_{\mathcal{N}} \bmod \mathcal{N})$.

Given a fractional ideal $\mathcal{I}$ of K define $\tilde{\mathcal{I}} = (\mathcal{I}_\nu)_\nu$ to be a fixed idele such that $\mathcal{I}_\infty = 1$ and $\tilde{\mathcal{I}}\mathcal{O} = \mathcal{I}$. For $\lambda = 1, ..., h$, set $x_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{I}_\lambda \end{pmatrix} \in G_A$. By the Strong Approximation theorem

$$G_A = \bigcup_{\lambda=1}^{h} G_K x_\lambda W = \bigcup_{\lambda=1}^{h} G_K x_\lambda^{-\iota} W,$$

where ι denotes the canonical involution on two-by-two matrices.

For an h-tuple $(f_1, ..., f_h) \in \mathfrak{M}_k(\mathcal{N}, \phi, \theta)$ we define a function $\mathbf{f} : G_A \to \mathbb{C}$ by

$$\mathbf{f}(\alpha x_\lambda^{-\iota} w) = \phi_Y(w^\iota) \det(w_\infty)^{im} (f_\lambda \mid w_\infty)(\mathbf{i})$$

for $\alpha \in G_K$, $w \in W(\mathcal{N})$ and $\mathbf{i} = (i, ..., i)$ (with $i = \sqrt{-1}$). Here

$$f_\lambda \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau) = (ad - bc)^{\frac{k}{2}} (c\tau + d)^{-k} f_\lambda \left(\frac{a\tau + b}{c\tau + d} \right).$$

We identify $\mathfrak{M}_k(\mathcal{N}, \phi, \theta)$ with the set of functions $\mathbf{f} : G_A \to \mathbb{C}$ satisfying

1. $\mathbf{f}(\alpha x w) = \phi_Y(w^\iota)\, \mathbf{f}(x)$ for all $\alpha \in G_K, x \in G_A, w \in W(\mathcal{N}), w_\infty = 1$
2. For each λ there exists an element $f_\lambda \in M_k$ such that

$$\mathbf{f}(x_\lambda^{-\iota} y) = \det(y)^{im} (f_\lambda \mid y)(\mathbf{i})$$

   for all $y \in G_{\infty+}$.

We denote by $\mathfrak{S}_k(\mathcal{N}, \phi, \theta)$ the subspace of cusp forms of $\mathfrak{M}_k(\mathcal{N}, \phi, \theta)$.

Let $\phi_\infty : K_A^\times \to \mathbb{C}^\times$ be defined by $\phi_\infty(\tilde{a}) = \mathrm{sgn}(\tilde{a}_\infty)^k |\tilde{a}_\infty|^{2im}$, where m was defined in the definition of θ. We say that a Hecke character Φ extends $\phi\phi_\infty$ if $\Phi(\tilde{a}) = \phi(\tilde{a}_{\mathcal{N}} \bmod \mathcal{N})\phi_\infty(\tilde{a})$ for all $\tilde{a} \in K_\infty^\times \times \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^\times$. If $\mathfrak{P}_\infty$ denotes the K-modulus consisting of the product of all the infinite primes of K, then any Hecke character Φ extending $\phi\phi_\infty$ has conductor dividing $\mathcal{N}\mathfrak{P}_\infty$. Henceforth we will use the word conductor to refer to the finite part of the conductor.

If ϕ is a numerical character modulo $\mathcal{P}\mathcal{N}_0$ where $\mathcal{P} = \mathfrak{p}^a$ is a power of a prime $\mathfrak{p}$ and $(\mathfrak{p}, \mathcal{N}_0) = 1$, then by the Chinese Remainder Theorem we have a decomposition $\phi = \phi_{\mathcal{P}}\phi_{\mathcal{N}_0}$ where $\phi_{\mathcal{P}}$ is a numerical character modulo $\mathcal{P}$ and $\phi_{\mathcal{N}_0}$ is a numerical character modulo $\mathcal{N}_0$. If $\Phi_{\mathcal{P}}$ is a Hecke character extending $\phi_{\mathcal{P}}$ (i.e. trivial infinite part) and $\Phi_{\mathcal{N}_0}$ is a Hecke character extending $\phi_{\mathcal{N}_0}\phi_\infty$ then it is clear that $\Phi = \Phi_{\mathcal{P}}\Phi_{\mathcal{N}_0}$. Throughout this paper we shall adopt this convention and decompose a Hecke character Φ extending a numerical character modulo $\mathcal{P}\mathcal{N}_0$ as $\Phi = \Phi_{\mathcal{P}}\Phi_{\mathcal{N}_0}$ where $\Phi_{\mathcal{P}}$ has trivial infinite part.

Given a Hecke character Φ extending $\phi\phi_\infty$ we define an ideal character Φ^* modulo $\mathcal{N}\mathfrak{P}_\infty$ by

$$\begin{cases} \Phi^*(\mathfrak{p}) = \Phi(\tilde{\pi}_{\mathfrak{p}}) & \text{for } \mathfrak{p} \nmid \mathcal{N} \text{ and } \tilde{\pi}\mathcal{O} = \mathfrak{p}, \\ \Phi^*(\mathfrak{a}) = 0 & \text{if } (\mathfrak{a}, \mathcal{N}) \neq 1 \end{cases}$$

For $\tilde{s} \in K_A^\times$, define $\mathbf{f}^{\tilde{s}}(x) = \mathbf{f}(\tilde{s}x)$. The map $\tilde{s} \longrightarrow \left(\mathbf{f} \mapsto \mathbf{f}^{\tilde{s}}\right)$ defines a unitary representation of $K_A^\times$ in $\mathfrak{M}_k(\mathcal{N}, \phi, \theta)$. By Schur's Lemma the irreducible subrepresentations are all one-dimensional (since $K_A^\times$ is abelian). For a character Φ on $K_A^\times$, let $\mathscr{M}_k(\mathcal{N}, \Phi)$ denote the subspace of $\mathfrak{M}_k(\mathcal{N}, \phi, \theta)$ consisting of all $\mathbf{f}$ for which $\mathbf{f}^{\tilde{s}} = \Phi(\tilde{s})\,\mathbf{f}$ and let $\mathscr{S}_k(\mathcal{N}, \Phi) \subset \mathscr{M}_k(\mathcal{N}, \Phi)$ denote the subspace of cusp forms. If $s \in K^\times$ then $\mathbf{f}^s = \mathbf{f}$. It follows that $\mathscr{M}_k(\mathcal{N}, \Phi)$ is nonempty only when Φ is a Hecke character.

If $\mathbf{f} = (f_1, ..., f_h) \in \mathfrak{M}_k(\mathcal{N}, \phi, \theta)$, then each f_λ has a Fourier expansion

$$f_\lambda(\tau) = a_\lambda(0) + \sum_{0 \ll \xi \in \mathcal{I}_\lambda} a_\lambda(\xi) e^{2\pi i \operatorname{tr}(\xi\tau)}.$$

If $\mathfrak{m}$ is an integral ideal then we define the $\mathfrak{m}$-th 'Fourier' coefficient of $\mathbf{f}$ by

$$C(\mathfrak{m}, \mathbf{f}) = \begin{cases} N(\mathfrak{m})^{\frac{k_0}{2}} a_\lambda(\xi)\xi^{-\frac{k}{2} - im} & \text{if } \mathfrak{m} = \xi\mathcal{I}_\lambda^{-1} \subset \mathcal{O} \\ 0 & \text{otherwise} \end{cases}$$

where $k_0 = \max\{k_1, ..., k_n\}$.

Given $\mathbf{f} \in \mathfrak{M}_k(\mathcal{N}, \phi, \theta)$ and $y \in G_A$ define a slash operator by setting $(\mathbf{f} \mid y)(x) = \mathbf{f}(xy^\iota)$.

For an integral ideal $\mathfrak{r}$ define the shift operator $B_\mathfrak{r}$ by

$$\mathbf{f} \mid B_\mathfrak{r} = N(\mathfrak{r})^{-\frac{k_0}{2}}\, \mathbf{f} \mid \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\mathfrak{r}}^{-1} \end{pmatrix}.$$

The shift operator maps $\mathscr{M}_k(\mathcal{N}, \Phi)$ to $\mathscr{M}_k(\mathfrak{r}\mathcal{N}, \Phi)$ and takes cusp forms to cusp forms. Further, $C(\mathfrak{m}, \mathbf{f} \mid B_\mathfrak{r}) = C(\mathfrak{m}\mathfrak{r}^{-1}, \mathbf{f})$. It is clear that $\mathbf{f} \mid B_{\mathfrak{r}_1} \mid B_{\mathfrak{r}_2} = \mathbf{f} \mid B_{\mathfrak{r}_1\mathfrak{r}_2}$.

For an integral ideal $\mathfrak{r}$ the Hecke operator $T_\mathfrak{r} = T_\mathfrak{r}^{\mathcal{N}}$ maps $\mathscr{M}_k(\mathcal{N}, \Phi)$ to itself regardless of whether or not $(\mathfrak{r}, \mathcal{N}) = 1$. This action is given on Fourier coefficients by

$$C(\mathfrak{m}, \mathbf{f} \mid T_\mathfrak{r}) = \sum_{\mathfrak{m}+\mathfrak{r} \subset \mathfrak{a}} \Phi^*(\mathfrak{a}) N(\mathfrak{a})^{k_0 - 1} C(\mathfrak{a}^{-2}\mathfrak{m}\mathfrak{r}, \mathbf{f}).$$

Like the shift operator, $T_\mathfrak{r}$ takes cusp forms to cusp forms. Also note that if $(\mathfrak{a}, \mathfrak{r}) = 1$ then $B_\mathfrak{a} T_\mathfrak{r} = T_\mathfrak{r} B_\mathfrak{a}$. Given $\mathbf{f} \in \mathscr{S}_k(\mathcal{N}, \Phi)$ we define the annihilator operator $A_\mathfrak{p}$ by

$$\mathbf{f} \mid A_\mathfrak{p} = \mathbf{f} - \mathbf{f} \mid T_\mathfrak{p} \mid B_\mathfrak{p}.$$

Let $\mathscr{S}_k^-(\mathcal{N},\Phi)$ be the subspace of $\mathscr{S}_k(\mathcal{N},\Phi)$ generated by all $\mathbf{g} \mid B_{\mathcal{Q}}$ where $\mathbf{g} \in \mathscr{S}_k(\mathcal{N}',\Phi)$ for some proper divisor $\mathcal{N}'$ of $\mathcal{N}$ with $\mathcal{Q}\mathcal{N}' \mid \mathcal{N}$. This space is invariant under the action of the Hecke operators $T_\mathfrak{r}$ with $(\mathfrak{r},\mathcal{N})=1$.

Shimura defines ((2.28) of [11]) a Petersson inner product $\langle \mathbf{f},\mathbf{g}\rangle$ for $\mathbf{f},\mathbf{g} \in \mathscr{S}_k(\mathcal{N},\Phi)$. With respect to this inner product the Hecke operators satisfy

$$\Phi^*(\mathfrak{m})\langle \mathbf{f} \mid T_\mathfrak{m},\mathbf{g}\rangle = \langle \mathbf{f},\mathbf{g} \mid T_\mathfrak{m}\rangle$$

for integral ideals $\mathfrak{m}$ coprime to $\mathcal{N}$. Let $\mathscr{S}_k^+(\mathcal{N},\Phi)$ denote the orthogonal complement of $\mathscr{S}_k^-(\mathcal{N},\Phi)$ in $\mathscr{S}_k(\mathcal{N},\Phi)$. It follows from our discussion above that $\mathscr{S}_k^+(\mathcal{N},\Phi)$ is invariant under the Hecke operators $T_\mathfrak{r}$ with $(\mathfrak{r},\mathcal{N})=1$.

Definition 2.1. *A newform $\boldsymbol{f}$ in $\mathscr{S}_k(\mathcal{N},\Phi)$ is a form in $\mathscr{S}_k^+(\mathcal{N},\Phi)$ which is a simultaneous eigenform for all Hecke operators $T_\mathfrak{q}$ with $\mathfrak{q}$ a prime not dividing $\mathcal{N}$. We say that $\boldsymbol{f}$ is normalized if $C(\mathcal{O},\boldsymbol{f})=1$.*

As in the classical case, if $\mathbf{f} \in \mathscr{S}_k(\mathcal{N},\Phi)$ is a newform with Hecke eigenvalues $\{\lambda_\mathfrak{p} : \mathfrak{p} \text{ is prime}\}$, then $C(\mathfrak{p},\mathbf{f}) = \lambda_\mathfrak{p} C(\mathcal{O},\mathbf{f})$ for all primes $\mathfrak{p} \nmid \mathcal{N}$.

Since $\{T_\mathfrak{q} : \mathfrak{q} \nmid \mathcal{N}\}$ is commuting family of hermitian operators, $\mathscr{S}_k^+(\mathcal{N},\Phi)$ has an orthogonal basis consisting of newforms. If $\mathbf{g} \in \mathscr{S}_k^-(\mathcal{N},\Phi)$ is a simultaneous eigenform for all $T_\mathfrak{q}$ with $\mathfrak{q} \nmid \mathcal{N}$ then there exists a newform $\mathbf{h} \in \mathscr{S}_k^+(\mathcal{N}',\Phi)$ with $\mathcal{N}' \mid \mathcal{N}$ having the same eigenvalues as $\mathbf{g}$ for all such $T_\mathfrak{q}$.

Finally, if $\mathbf{f},\mathbf{g} \in \mathscr{S}_k(\mathcal{N},\Phi)$ are both simultaneous eigenforms for all Hecke operators $T_\mathfrak{q}$ with $\mathfrak{q}$ a prime not dividing $\mathcal{N}$ having the same Hecke eigenvalues, then we say that $\mathbf{f}$ is equivalent to $\mathbf{g}$ and write $\mathbf{f} \sim \mathbf{g}$. If $\mathbf{f}$ is a newform and $\mathbf{f} \sim \mathbf{g}$, then there exists $c \in \mathbb{C}^\times$ such that $\mathbf{f} = c\,\mathbf{g}$. This follows from Theorem 3.5 of [9].

3. Twists of Newforms

Throughout this section $\mathfrak{p}$ will denote a fixed prime ideal of $\mathcal{O}$.

Fix an integral ideal $\mathcal{N}$ and write $\mathcal{N} = \mathcal{P}\mathcal{N}_0$ where $\mathcal{P}$ is the $\mathfrak{p}$-primary part of $\mathcal{N}$ and $(\mathcal{P},\mathcal{N}_0)=1$.

Fix a space $\mathscr{S}_k(\mathcal{N},\Phi) \subset \mathfrak{S}_k(\mathcal{N},\phi,\theta)$, where Φ is a Hecke character extending $\phi\phi_\infty$.

Definition 3.1. *If $\boldsymbol{f} \in \mathscr{S}_k(\mathcal{N},\Phi)$ and Ψ is a Hecke character then we define the twist of $\boldsymbol{f}$ by Ψ, denoted $\boldsymbol{f}_\Psi$, by*

$$\boldsymbol{f}_\Psi(x) = \tau(\overline{\Psi})^{-1}\Psi(\det x) \sum_{r \in \mathfrak{f}_\Psi^{-1}\mathfrak{d}^{-1}/\mathfrak{d}^{-1}} \overline{\Psi}_\infty(r)\overline{\Psi}^*(r\mathfrak{f}_\Psi\mathfrak{d})\,\boldsymbol{f} \,|\, \left(\begin{smallmatrix}1 & r\\ 0 & 1\end{smallmatrix}\right)_0(x),$$

where $\tau(\overline{\Psi})$ is the Gauss sum associated to $\overline{\Psi}$ defined in (9.31) of [10] and the subscript 0 denotes the projection onto the nonarchimedean part. Additionally, set $\mathscr{S}_k^+(\mathcal{N},\Phi)^\Psi = \{\boldsymbol{f}_\Psi : \boldsymbol{f} \in \mathscr{S}_k^+(\mathcal{N},\Phi)\}$.

Proposition 3.2. *Let notation be as above and set* $\mathcal{L} = lcm\{\mathcal{N}, \mathfrak{f}_\Phi \mathfrak{f}_\Psi, \mathfrak{f}_\Psi^2\}$. *If* $\boldsymbol{f} \in \mathscr{S}_k(\mathcal{N}, \Phi)$ *is a normalized newform then* $\boldsymbol{f}_\Psi \in \mathscr{S}_k(\mathcal{L}, \Psi^2\Phi)$ *and* $C(\mathfrak{m}, \boldsymbol{f}_\Psi) = \Psi^*(\mathfrak{m})C(\mathfrak{m}, \boldsymbol{f})$ *for all integral ideals* $\mathfrak{m}$.

Proof. This follows from Propositions 4.4 and 4.5 of [11]. ■

The following proposition is trivial to verify using the action of the Hecke operators on Fourier coefficients.

Proposition 3.3. *Let notation be as above and* $\mathfrak{q}$ *be a prime with* $\mathfrak{q} \nmid \mathfrak{f}_\Psi$. *For* $\boldsymbol{f} \in \mathscr{S}_k(\mathcal{N}, \Phi)$ *we have* $\boldsymbol{f}_\Psi \mid T_\mathfrak{q} = \Psi^*(\mathfrak{q})(\boldsymbol{f} \mid T_\mathfrak{q})_\Psi$.

Although Proposition 3.2 gives an upper bound for the exact level of $\mathbf{f}_\Psi$, one can obtain better bounds in certain special cases. Of particular interest to us is the case in which $\Psi = \overline{\Phi}_\mathcal{P}$. The following proposition gives an improved bound on the level of $\mathbf{f}_\Psi$ in this special case and generalizes Proposition 3.6 of [1].

Proposition 3.4. *Let* $\mathfrak{f}$ *be the conductor of* $\Phi_\mathcal{P}$. *Set*

$$\mathcal{L} = \begin{cases} \mathcal{N} & \text{if } ord_\mathfrak{p}(\mathfrak{f}) < ord_\mathfrak{p}(\mathcal{P}) \\ \mathfrak{p}\mathcal{N} & \text{if } ord_\mathfrak{p}(\mathfrak{f}) = ord_\mathfrak{p}(\mathcal{P}) \end{cases}$$

If $\boldsymbol{f} \in \mathscr{S}_k(\mathcal{N}, \Phi)$ *then* $\boldsymbol{f}_{\overline{\Phi}_\mathcal{P}} \in \mathscr{S}_k(\mathcal{L}, \overline{\Phi}_\mathcal{P}\Phi_{\mathcal{N}_0})$.

Proof. The proof of Proposition 3.4 is analogous to the proof of Proposition 3.6 in [1], though somewhat more tedious as we work adelically. We therefore provide a rough sketch of the proof and leave the details to the interested reader.

Let $\alpha \in G_K, x \in G_A$ and $w \in W(\mathcal{L})$ with $w_\infty = 1$. We must show that

$$\mathbf{f}_{\overline{\Phi}_\mathcal{P}}(\alpha x w) = (\phi_{\mathcal{N}_0}\overline{\phi}_\mathcal{P})_Y(w^\iota)\mathbf{f}_{\overline{\Phi}_\mathcal{P}}(x).$$

The first step of the proof is to show that given $r \in \mathfrak{f}^{-1}\mathfrak{d}^{-1}/\mathfrak{d}^{-1}$ there is a unique $r' \in \mathfrak{f}^{-1}\mathfrak{d}^{-1}/\mathfrak{d}^{-1}$ such that

$$w \begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix}_0 = \begin{pmatrix} 1 & -r' \\ 0 & 1 \end{pmatrix}_0 w', \tag{3.1}$$

for some $w' \in W(\mathcal{N})$.

By definition,

$$\begin{aligned} \mathbf{f}_{\overline{\Phi}_\mathcal{P}}(\alpha x w) &= \tau(\Phi_\mathcal{P})^{-1}\overline{\Phi}_\mathcal{P}(\det(\alpha x w)) \sum_{r \in \mathfrak{f}^{-1}\mathfrak{d}^{-1}/\mathfrak{d}^{-1}} \Phi^*_\mathcal{P}(r\,\mathfrak{f}\,\mathfrak{d})\,\mathbf{f} \mid \left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right)_0 (\alpha x w) \\ &= \tau(\Phi_\mathcal{P})^{-1}\overline{\Phi}_\mathcal{P}(\det(x))\overline{\Phi}_\mathcal{P}(\det(w)) \\ &\quad \times \sum_{r \in \mathfrak{f}^{-1}\mathfrak{d}^{-1}/\mathfrak{d}^{-1}} \Phi^*_\mathcal{P}(r\,\mathfrak{f}\,\mathfrak{d})\,\mathbf{f}(\alpha x w \left(\begin{smallmatrix} 1 & -r \\ 0 & 1 \end{smallmatrix}\right)_0) \end{aligned} \tag{3.2}$$

The proof proceeds by substituting equation (3.1) into equation (3.2). Straightforward manipulations now suffice to finish the proof. ■

Throughout this paper we will examine the behavior of normalized newforms under character twists. The case in which the conductor of the twisting character is coprime to the level of the newform is straightforward in light of the following theorem.

Theorem 3.5. *Let $\boldsymbol{f} \in \mathscr{S}_k^+(\mathcal{N}, \Phi)$ be a normalized newform and Ψ be a Hecke character whose conductor is coprime to $\mathcal{N}$. Then $\boldsymbol{f}_\Psi$ is a normalized newform of $\mathscr{S}_k^+(\mathfrak{f}_\Psi^2 \mathcal{N}, \Psi^2\Phi)$.*

Proof. This is Theorem 5.5 of [9]. ∎

The situation when the conductor of Ψ and the level of $\mathbf{f}$ are not coprime is much more subtle and will be studied throughout the remainder of this paper. Clearly it suffices to consider characters whose conductor is a power of a single prime dividing the level $\mathcal{N}$. We therefore suppose that Ψ is a $\mathfrak{p}$-primary Hecke character.

Henceforth we assume that Ψ is a Hecke character with conductor dividing $\mathcal{P}$. The infinite part of Ψ has the form $\Psi_\infty(a) = \operatorname{sgn}(a)^l |a|^{ir}$ for $l \in \mathbb{Z}^n$, $r \in \mathbb{R}^n$ and $a \in K_\infty^\times$. In what follows we shall always choose Ψ so that $r = 0$.

We will see that the vanishing of $C(\mathfrak{p}, \mathbf{f})$ lies at the heart of the question of whether or not $\mathbf{f}_\Psi$ is a newform of $\mathscr{S}_k(\mathcal{N}, \Psi^2\Phi)$. We present a slightly strengthened version of Theorem 3.3 of [9], which will allow us to determine when $C(\mathfrak{p}, \mathbf{f}) \neq 0$.

Theorem 3.6. *Let $\boldsymbol{f}$ be a normalized newform lying in $\mathscr{S}_k(\mathcal{N}, \Phi)$.*

1. *The Dirichlet series attached to $\boldsymbol{f}$, $D(s, f) = \sum_{\mathfrak{m} \subset \mathcal{O}} C(\mathfrak{m}, \boldsymbol{f}) N(\mathfrak{m})^{-s}$ has an Euler product*

$$D(s, \boldsymbol{f}) = \prod_{\mathfrak{q}_0 | \mathcal{N}} (1 - C(\mathfrak{q}_0, \boldsymbol{f}) N(\mathfrak{q}_0)^{-s})^{-1} \times \prod_{\mathfrak{q}_1 \nmid \mathcal{N}} (1 - C(\mathfrak{q}_1, \boldsymbol{f}) N(\mathfrak{q}_1)^{-s} + \Phi^*(\mathfrak{q}_1) N(\mathfrak{q}_1)^{k_0 - 1 - 2s})^{-1}$$

2. *If ϕ is not defined modulo $\mathcal{N}\mathfrak{p}^{-1}$, then $|C(\mathfrak{p}, \boldsymbol{f})| = N(\mathfrak{p})^{\frac{(k_0-1)}{2}}$.*
3. *If ϕ is a character modulo $\mathcal{N}\mathfrak{p}^{-1}$, then $C(\mathfrak{p}, \boldsymbol{f}) = 0$ if $\mathfrak{p}^2 \mid \mathcal{N}$ and $|C(\mathfrak{p}, \boldsymbol{f})|^2 = N(\mathfrak{p})^{k_0 - 2}$ if $\mathfrak{p}^2 \nmid \mathcal{N}$.*

Proof. The statement of this theorem differs from Theorem 3.3 of [9] only in that part 2 of the latter showed that either $C(\mathfrak{p}, \mathbf{f}) = 0$ or $|C(\mathfrak{p}, \mathbf{f})| = N(\mathfrak{p})^{\frac{(k_0-1)}{2}}$ and that $C(\mathfrak{p}, \mathbf{f})$ was non-zero for a set of primes having density 1. Kevin Buzzard has recently shown that in fact, $C(\mathfrak{p}, \mathbf{f})$ is never zero (see [2]), allowing us to state the above theorem in its strengthened form. ∎

Henceforth we use the letter ν to denote $ord_{\mathfrak{p}}(\mathcal{P}) = ord_{\mathfrak{p}}(\mathcal{N})$.

Lemma 3.7. *Assume that $\nu \geqslant 2$ and that $e(\Phi_\mathcal{P}) < \nu$. If $\boldsymbol{f} \in \mathscr{S}_k^+(\mathcal{N}, \Phi)$ is a normalized newform then $\boldsymbol{f}_{\overline{\Psi}\Psi} = \boldsymbol{f}$. In particular,*

$$\mathscr{S}_k^+(\mathcal{N}, \Phi)^{\overline{\Psi}\Psi} = \mathscr{S}_k^+(\mathcal{N}, \Phi).$$

Proof. It follows immediately from Theorem 3.6.(3) that $C(\mathfrak{p}, \mathbf{f}) = 0$. Because $\mathbf{f}$ is an eigenform of $T_\mathfrak{p}$ with eigenvalue $C(\mathfrak{p}, \mathbf{f})$, $C(\mathscr{I}\mathfrak{p}, \mathbf{f}) = C(\mathscr{I}, \mathbf{f})C(\mathfrak{p}, \mathbf{f}) = 0$ for all integral ideals $\mathscr{I}$. Thus the annihilator operator $A_\mathfrak{p}$ acts as the identity operator on the newforms of level $\mathcal{N}$ and character Φ. The first part therefore follows from the observation that $\mathbf{f}_{\overline{\Psi}\Psi} = \mathbf{f} \mid A_\mathfrak{p}$. As newforms generate the space $\mathscr{S}_k^+(\mathcal{N}, \Phi)$, we have the second part as well. ■

Proposition 3.8. *Assume that* $\nu \geqslant 2$ *and that* $0 < e(\Phi_\mathcal{P}) < \nu$. *If* $\boldsymbol{f} \in \mathscr{S}_k^+(\mathcal{N}, \Phi)$ *is a normalized newform then* $\boldsymbol{f}_{\overline{\Phi}_\mathcal{P}} \in \mathscr{S}_k^+(\mathcal{N}, \overline{\Phi}_\mathcal{P}\Phi_{\mathcal{N}_0})$ *is a newform as well.*

Proof. Let $\mathbf{f} \in \mathscr{S}_k^+(\mathcal{N}, \Phi)$ be a normalized newform. By Propositions 3.3 and 3.4, $\mathbf{f}_{\overline{\Phi}_\mathcal{P}} \in \mathscr{S}_k(\mathcal{N}, \overline{\Phi}_\mathcal{P}\Phi_{\mathcal{N}_0})$ and is an eigenfunction of all the Hecke operators $T_\mathfrak{q}$ with $\mathfrak{q} \nmid \mathcal{N}$. Thus there exists an ideal $\mathcal{N}_0' \mid \mathcal{N}_0$, an integer μ satisfying $1 \leqslant e(\Phi_\mathcal{P}) \leqslant \mu \leqslant \nu$ and a newform $\mathbf{g} \in \mathscr{S}_k^+(\mathfrak{p}^\mu \mathcal{N}_0', \overline{\Phi}_\mathcal{P}\Phi_{\mathcal{N}_0})$ such that $\mathbf{f}_{\overline{\Phi}_\mathcal{P}} \sim \mathbf{g}$. We claim that $\mathcal{N}_0' = \mathcal{N}_0$. Note that $\mathbf{f} = \mathbf{f}_{\overline{\Phi}_\mathcal{P}\Phi_\mathcal{P}} \sim \mathbf{g}_{\Phi_\mathcal{P}}$ by Lemma 3.7, where $\mathbf{g}_{\Phi_\mathcal{P}}$ has level $\mathfrak{p}^\lambda \mathcal{N}_0'$ for some non-negative integer λ. Thus $\mathcal{N}_0 \mid \mathcal{N}_0'$, hence $\mathcal{N}_0 = \mathcal{N}_0'$.

If $\mu = \nu$ then $\mathbf{f}_{\overline{\Phi}_\mathcal{P}}$ and $\mathbf{g}$ are of the same level and are both normalized, hence $\mathbf{f}_{\overline{\Phi}_\mathcal{P}} = \mathbf{g}$ is a newform, finishing the proof. We may therefore suppose that $\mu < \nu$. We claim that $e(\overline{\Phi}_\mathcal{P}) < \mu$. To show this, we will assume that $e(\overline{\Phi}_\mathcal{P}) = e(\Phi_\mathcal{P}) = \mu$ and derive a contradiction.

Because $\mathbf{f}_{\overline{\Phi}_\mathcal{P}} \sim \mathbf{g}$, we have $\mathbf{f}_{\overline{\Phi}_\mathcal{P}\Phi_\mathcal{P}} \sim \mathbf{g}_{\Phi_\mathcal{P}}$ as well. By Lemma 3.7, $\mathbf{f}_{\overline{\Phi}_\mathcal{P}\Phi_\mathcal{P}} = \mathbf{f}$, hence $\mathbf{f} \sim \mathbf{g}_{\Phi_\mathcal{P}}$. By Proposition 3.4, $\mathbf{g}_{\Phi_\mathcal{P}} \in \mathscr{S}_k(\mathfrak{p}^{\mu+1}\mathcal{N}_0, \Phi)$. Therefore $\nu \leqslant \mu + 1$, meaning that

$$\mu + 1 \geqslant \nu > \mu.$$

It is thus clear than $\nu = \mu + 1$. This means that $\mathbf{f}$ is a newform of level $\mathfrak{p}^{\mu+1}\mathcal{N}_0$ and character Φ and $\mathbf{g}_{\Phi_\mathcal{P}}$ is a normalized cuspform in the same space which is equivalent to it. Therefore $\mathbf{f} = \mathbf{g}_{\Phi_\mathcal{P}}$. As $C(\mathfrak{p}, \mathbf{g}) \neq 0$ by Theorem 3.6(2), this contradicts Corollary 6.4 of [9], which implies that $\mathbf{g}_{\Phi_\mathcal{P}}$ is not a newform of any level.

We conclude that $e(\overline{\Phi}_\mathcal{P}) < \mu$. If $\mu \geqslant 2$ then Theorem 3.6(3) implies that the $\mathfrak{p}$-th coefficient $C(\mathfrak{p}, \mathbf{g})$ of $\mathbf{g}$ is zero. Since $C(\mathfrak{p}, \mathbf{g}) = 0$ we have $\mathbf{g} = \mathbf{g} \mid A_\mathfrak{p}$. But

$$\mathbf{f}_{\overline{\Phi}_\mathcal{P}} = c_\mathcal{O}\, \mathbf{g} + c_\mathfrak{p}\, \mathbf{g} \mid B_\mathfrak{p}$$

and one easily checks by comparing Fourier coefficients that $c_\mathcal{O} = 1$ and $c_\mathfrak{p} = -C(\mathfrak{p}, \mathbf{g})$. Then $\mathbf{f}_{\overline{\Phi}_\mathcal{P}} = \mathbf{g} - C(\mathfrak{p}, \mathbf{g})\, \mathbf{g} \mid B_\mathfrak{p} = \mathbf{g} \mid A_\mathfrak{p} = \mathbf{g}$. Therefore $\mathbf{f}_{\overline{\Phi}_\mathcal{P}}$ is a newform and we're done.

Now suppose that $\mu = 1$. Then $e(\overline{\Phi}_\mathcal{P}) < \mu$ implies that $\Phi_\mathcal{P}$ is trivial. This contradicts our hypothesis that $\Phi_\mathcal{P}$ is nontrivial. ■

Proposition 3.9. *Assume that* $0 < e(\Psi) < \frac{\nu}{2}$ *and* $e(\Phi_\mathcal{P}) + e(\Psi) < \nu$.

If $\boldsymbol{f} \in \mathscr{S}_k^+(\mathcal{N}, \Phi)$ *is a normalized newform then* $\boldsymbol{f}_\Psi \in \mathscr{S}_k^+(\mathcal{N}, \Psi^2\Phi)$ *is a newform as well.*

Proof. We begin by noting that our hypotheses imply that $\nu \geqslant 3$. By Proposition 3.2, $\mathbf{f}_\Psi \in \mathscr{S}_k(\mathfrak{p}^\nu \mathcal{N}_0, \Psi^2\Phi)$. Thus there exists an ideal $\mathcal{N}'_0 \mid \mathcal{N}_0$, an integer μ satisfying $0 \leqslant e(\Phi_\mathcal{P}\Psi^2) \leqslant \mu \leqslant \nu$ and a newform $\mathbf{g} \in \mathscr{S}_k^+(\mathfrak{p}^\mu \mathcal{N}'_0, \Psi^2\Phi)$ such that $\mathbf{f}_\Psi \sim \mathbf{g}$. An argument identical to the one used in Proposition 3.8 shows that $\mathcal{N}'_0 = \mathcal{N}_0$.

We will show that $e(\Phi_\mathcal{P}\Psi^2) < \mu$ by assuming that $e(\Phi_\mathcal{P}\Psi^2) = \mu$ and deriving a contradiction. Let $L = \max\{\mu, e(\Phi_\mathcal{P}\Psi^2) + e(\Psi), 2e(\Psi)\}$. As $\mathbf{f}_\Psi \sim \mathbf{g}$, we have, by Lemma 3.7, $\mathbf{f} = \mathbf{f}_{\Psi\overline{\Psi}} \sim \mathbf{g}_{\overline{\Psi}}$ where $\mathbf{g}_{\overline{\Psi}} \in \mathscr{S}_k(\mathfrak{p}^L \mathcal{N}_0, \Phi)$ by Proposition 3.2. Therefore $L \geqslant \nu$. We have three cases to consider.

Case 1: $L = 2e(\Psi)$. In this case $2e(\Psi) \geqslant \nu$ implies that $e(\Psi) \geqslant \frac{\nu}{2}$, contradicting our hypothesis that $e(\Psi) < \frac{\nu}{2}$.

Case 2: $L = e(\Phi_\mathcal{P}\Psi^2) + e(\Psi)$. We have three subcases to consider. First suppose that $e(\Phi_\mathcal{P}) > e(\Psi)$. Then $e(\Phi_\mathcal{P}\Psi^2) = e(\Phi_\mathcal{P})$, hence $L \geqslant \nu$ implies that $e(\Phi_\mathcal{P}) + e(\Psi) \geqslant \nu$, contradicting our hypothesis that $e(\Phi_\mathcal{P}) + e(\Psi) < \nu$. If $e(\Psi) > e(\Phi_\mathcal{P})$, then $e(\Psi) \geqslant e(\Phi_\mathcal{P}\Psi^2)$, hence $L \geqslant \nu$ implies that $2e(\Psi) \geqslant \nu$, which we have already seen results in a contradiction. Finally, suppose that $e(\Phi_\mathcal{P}) = e(\Psi)$. Then $e(\Psi) < \frac{\nu}{2}$ implies that $e(\Phi_\mathcal{P}) < \frac{\nu}{2}$ and consequently that $e(\Phi_\mathcal{P}\Psi^2) < \frac{\nu}{2}$. But this means that $L = e(\Phi_\mathcal{P}\Psi^2) + e(\Psi) < \nu$, contradicting the fact that $L \geqslant \nu$.

Case 3: $L = \mu$. This case cannot occur as we have assumed that $e(\Phi_\mathcal{P}\Psi^2) = \mu$, meaning that $e(\Phi_\mathcal{P}\Psi^2) + e(\Psi) > \mu$ by the non-triviality of Ψ.

We conclude that $e(\Phi_\mathcal{P}\Psi^2) < \mu$. Suppose first that $\mu > 1$. Then Theorem 3.6(3) implies that $c(\mathfrak{p}, \mathbf{g}) = 0$. As in the proof of Proposition 3.8 we may easily show that $\mathbf{f}_\Psi = \mathbf{g} \mid A_\mathfrak{p}$. But we've just shown that $\mathbf{g} \mid A_\mathfrak{p} = \mathbf{g}$. Therefore $\mathbf{f}_\Psi$ is a newform and we're done.

We show that the case $\mu = 1$ cannot occur. Indeed, suppose that $\mu = 1$ (and hence $e(\Phi_\mathcal{P}\Psi^2) = 0$). Then $\mathbf{g}$ is a newform of $\mathscr{S}_k(\mathfrak{p}\mathcal{N}_0, \Phi)$. As $\mathbf{f}_\Psi \sim \mathbf{g}$, we also have $\mathbf{f}_{\Psi\overline{\Psi}} \sim \mathbf{g}_{\overline{\Psi}}$. Our hypotheses imply that $\nu \geqslant 3$, so Lemma 3.7 implies that $\mathbf{f} = \mathbf{f}_{\Psi\overline{\Psi}}$; hence $\mathbf{f} \sim \mathbf{g}_{\overline{\Psi}}$. Theorem 6.1 of [9] implies that $\mathbf{g}_{\overline{\Psi}}$ is a newform of $\mathscr{S}_k(\mathfrak{p}^{2e(\Psi)} \mathcal{N}_0, \Phi)$, hence Theorem 3.5 of [9] implies that in fact we have $\mathbf{f} = \mathbf{g}_{\overline{\Psi}}$. By comparing the levels of $\mathbf{f}$ and $\mathbf{g}_{\overline{\Psi}}$, we see that this means that $2e(\Psi) = \nu$; i.e. $e(\Psi) = \frac{\nu}{2}$. We assumed that $e(\Psi) < \frac{\nu}{2}$ however, so we obtain a contradiction, finishing our proof. ■

Theorem 3.10. *If* $e(\Phi_\mathcal{P}) < \nu$ *then* $\mathscr{S}_k^+(\mathcal{N}, \Phi) = \mathscr{S}_k^+(\mathcal{N}, \overline{\Phi}_\mathcal{P}\Phi_{\mathcal{N}_0})^{\Phi_\mathcal{P}}$.
If $e(\Phi_\mathcal{P}) = \nu$ *and* $\boldsymbol{f}$ *is a normalized newform in* $\mathscr{S}_k^+(\mathcal{N}, \overline{\Phi}_\mathcal{P}\Phi_{\mathcal{N}_0})$*, then*

$$\boldsymbol{f}_{\Phi_\mathcal{P}} = \boldsymbol{g} - C(\mathfrak{p}, \boldsymbol{g}) \cdot \boldsymbol{g} \mid B_\mathfrak{p}$$

for some normalized newform $\boldsymbol{g}$ *in* $\mathscr{S}_k^+(\mathcal{N}, \Phi)$.

Proof. When $K = \mathbb{Q}$ this is Corollary 3.4 of [6].

Note first that the theorem is vacuously true when $e(\Phi_\mathcal{P}) = 0$. We therefore assume that $e(\Phi_\mathcal{P}) \geqslant 1$. As a consequence, $\nu \geqslant 2$.

Let $\mathbf{f} \in \mathscr{S}_k^+(\mathcal{N}, \overline{\Phi}_\mathcal{P}\Phi_{\mathcal{N}_0})$ be a newform. Applying Proposition 3.8 shows that $\mathbf{f}_{\Phi_\mathcal{P}} \in \mathscr{S}_k^+(\mathcal{N}, \Phi)$ is a newform. As $\mathscr{S}_k^+(\mathcal{N}, \overline{\Phi}_\mathcal{P}\Phi_{\mathcal{N}_0})$ is generated by newforms, we have the inclusion

$$\mathscr{S}_k^+(\mathcal{N}, \overline{\Phi}_\mathcal{P}\Phi_{\mathcal{N}_0})^{\Phi_\mathcal{P}} \subset \mathscr{S}_k^+(\mathcal{N}, \Phi). \tag{3.3}$$

Now let $\mathbf{f} \in \mathscr{S}_k^+(\mathcal{N}, \Phi)$. Then as above $\mathbf{f}_{\overline{\Phi}_{\mathcal{P}}} \in \mathscr{S}_k^+(\mathcal{N}, \overline{\Phi}_{\mathcal{P}}\Phi_{\mathcal{N}_0})$ (by interchanging $\Phi_{\mathcal{P}}$ and $\overline{\Phi}_{\mathcal{P}}$ in equation 3.3), hence $\mathbf{f}_{\overline{\Phi}_{\mathcal{P}}\Phi_{\mathcal{P}}} \in \mathscr{S}_k^+(\mathcal{N}, \overline{\Phi}_{\mathcal{P}}\Phi_{\mathcal{N}_0})^{\Phi_{\mathcal{P}}}$. This gives us the chain of inclusions

$$\mathscr{S}_k^+(\mathcal{N}, \Phi)^{\overline{\Phi}_{\mathcal{P}}\Phi_{\mathcal{P}}} \subset \mathscr{S}_k^+(\mathcal{N}, \overline{\Phi}_{\mathcal{P}}\Phi_{\mathcal{N}_0})^{\Phi_{\mathcal{P}}} \subset \mathscr{S}_k^+(\mathcal{N}, \Phi).$$

Lemma 3.7 shows that $\mathscr{S}_k^+(\mathcal{N}, \Phi)^{\overline{\Phi}_{\mathcal{P}}\Phi_{\mathcal{P}}} = \mathscr{S}_k^+(\mathcal{N}, \Phi)$, and it follows that

$$\mathscr{S}_k^+(\mathcal{N}, \Phi) = \mathscr{S}_k^+(\mathcal{N}, \overline{\Phi}_{\mathcal{P}}\Phi_{\mathcal{N}_0})^{\Phi_{\mathcal{P}}}.$$

We now prove the second assertion. Suppose that $e(\Phi_{\mathcal{P}}) = \nu$. First note that by Proposition 3.4, $\mathbf{f}_{\Phi_{\mathcal{P}}} \in \mathscr{S}_k(\mathfrak{p}^{\nu+1}\mathcal{N}_0, \Phi)$. By Proposition 3.3, $\mathbf{f}_{\Phi_{\mathcal{P}}}$ is a Hecke eigenform for all $T_{\mathfrak{q}}$ with $\mathfrak{q}$ a prime not dividing $\mathcal{N}$. Thus there exists an integer μ with $e(\Phi_{\mathcal{P}}) = \nu \leqslant \mu \leqslant \nu+1$ and a normalized newform $\mathbf{g} \in \mathscr{S}_k^+(\mathfrak{p}^{\mu}\mathcal{N}_0, \Phi)$ such that $\mathbf{f}_{\Phi_{\mathcal{P}}} \sim \mathbf{g}$. We claim that the case $\mu = \nu+1$ cannot occur. Indeed, if $\mu = \nu+1$ then $\mathbf{g}$ and $\mathbf{f}_{\Phi_{\mathcal{P}}}$ would both lie in $\mathscr{S}_k^+(\mathfrak{p}^{\nu+1}\mathcal{N}_0, \Phi)$ and our remarks at the end of Section 2 would imply that $\mathbf{f}_{\Phi_{\mathcal{P}}} = \mathbf{g}$ is a newform. But Theorem 3.6 shows that $C(\mathfrak{p}, \mathbf{f}) \neq 0$, so that Corollary 6.4 of [9] implies that $\mathbf{f}_{\Phi_{\mathcal{P}}}$ is not a newform of any level. This contradiction allows us to conclude that $\mu = \nu$. It then follows from Proposition 3.4 that $\mathbf{g}_{\overline{\Phi}_{\mathcal{P}}} \in \mathscr{S}_k(\mathfrak{p}^{\nu+1}\mathcal{N}_0, \overline{\Phi}_{\mathcal{P}}\Phi_{\mathcal{N}_0})$. Using the fact that $\mathbf{g}$ is an eigenform of $T_{\mathfrak{p}}$ (as follows from Theorem 3.5 of [9]), we see that

$$\mathbf{g} - C(\mathfrak{p}, \mathbf{g}) \cdot \mathbf{g} \mid B_{\mathfrak{p}} = \mathbf{g} - \mathbf{g} \mid T_{\mathfrak{p}} \mid B_{\mathfrak{p}} = (\mathbf{g}_{\overline{\Phi}_{\mathcal{P}}})_{\Phi_{\mathcal{P}}} = (c_1\,\mathbf{f} + c_2\,\mathbf{f} \mid B_{\mathfrak{p}})_{\Phi_{\mathcal{P}}} = c_1 \mathbf{f}_{\Phi_{\mathcal{P}}}$$

Comparing Fourier coefficients yields $c_1 = 1$. ∎

Theorem 3.11. *If* $0 < e(\Psi) < \frac{\nu}{2}$ *and* $e(\Phi_{\mathcal{P}}) + e(\Psi) < \nu$ *then*

$$\mathscr{S}_k^+(\mathcal{N}, \Phi)^{\Psi} = \mathscr{S}_k^+(\mathcal{N}, \Psi^2\Phi).$$

Proof. When $K = \mathbb{Q}$ this is Theorem 3.12 of [6]. We begin by noting that our hypotheses imply that $\nu \geqslant 3$. Let $\mathbf{f} \in \mathscr{S}_k^+(\mathcal{N}, \Phi)$ be a newform. By Proposition 3.9, $\mathbf{f}_{\Psi} \in \mathscr{S}_k^+(\mathcal{N}, \Psi^2\Phi)$ is a newform. As $\mathscr{S}_k^+(\mathcal{N}, \Phi)$ is generated by newforms, we have the inclusion

$$\mathscr{S}_k^+(\mathcal{N}, \Phi)^{\Psi} \subset \mathscr{S}_k^+(\mathcal{N}, \Psi^2\Phi). \tag{3.4}$$

Twisting by $\overline{\Psi}$ yields:

$$\mathscr{S}_k^+(\mathcal{N}, \Phi)^{\Psi\overline{\Psi}} \subset \mathscr{S}_k^+(\mathcal{N}, \Psi^2\Phi)^{\overline{\Psi}}. \tag{3.5}$$

We claim that $e(\Psi^2\Phi_{\mathcal{P}}) + e(\Psi) < \nu$. We have two cases to consider.

Case 1: $e(\Phi_{\mathcal{P}}) < \frac{\nu}{2}$ - By hypothesis $e(\Psi) < \frac{\nu}{2}$. Therefore $e(\Psi^2\Phi_{\mathcal{P}}) < \frac{\nu}{2}$, hence $e(\Psi^2\Phi_{\mathcal{P}}) + e(\Psi) < \nu$.

Case 2: $e(\Phi_{\mathcal{P}}) \geqslant \frac{\nu}{2}$ - We have two subcases to consider. Suppose first that $e(\Phi_{\mathcal{P}}) > e(\Psi^2)$. Then $e(\Psi^2\Phi_{\mathcal{P}}) = e(\Phi_{\mathcal{P}}) < \nu - e(\Psi)$. Now suppose that $e(\Phi_{\mathcal{P}}) \leqslant e(\Psi^2)$. Then $e(\Phi_{\mathcal{P}}) \leqslant e(\Psi^2) \leqslant e(\Psi) < \frac{\nu}{2}$. But Case 2 assumes that $e(\Phi_{\mathcal{P}}) \geqslant \frac{\nu}{2}$, so this subcase cannot occur and we have shown our claim.

Having shown that $e(\Psi^2\Phi_{\mathcal{P}}) + e(\Psi) < \nu$, we apply Theorem 5.7 of [9] and Proposition 3.9 to show that

$$\mathscr{S}_k^+(\mathcal{N}, \Psi^2\Phi)^{\overline{\Psi}} \subset \mathscr{S}_k^+(\mathcal{N}, \Phi). \tag{3.6}$$

Combining equations (3.5) and (3.6) gives us the chain of inclusions:

$$\mathscr{S}_k^+(\mathcal{N}, \Phi)^{\Psi\overline{\Psi}} \subset \mathscr{S}_k^+(\mathcal{N}, \Psi^2\Phi)^{\overline{\Psi}} \subset \mathscr{S}_k^+(\mathcal{N}, \Phi).$$

Lemma 3.7 implies that $\mathscr{S}_k^+(\mathcal{N}, \Phi) = \mathscr{S}_k^+(\mathcal{N}, \Psi^2\Phi)^{\overline{\Psi}}$.

Twisting by Ψ then yields:

$$\mathscr{S}_k^+(\mathcal{N}, \Phi)^{\Psi} = \mathscr{S}_k^+(\mathcal{N}, \Psi^2\Phi)^{\overline{\Psi}\Psi}.$$

As $e(\Psi^2\Phi_{\mathcal{P}}) < \nu$, Lemma 3.7 shows that $\mathscr{S}_k^+(\mathcal{N}, \Psi^2\Phi)^{\overline{\Psi}\Psi} = \mathscr{S}_k^+(\mathcal{N}, \Psi^2\Phi)$, finishing the proof. ■

Theorem 3.12. *If $\frac{\nu}{2} < e(\Phi_{\mathcal{P}}) < \nu$ then*

$$\mathscr{S}_k^+(\mathcal{N}, \Phi) = \bigoplus_{e(\Psi)=\nu-e(\Phi_{\mathcal{P}})} \mathscr{S}_k^+(\mathfrak{p}^{e(\Phi_{\mathcal{P}})}\mathcal{N}_0, \Psi^2\Phi)^{\overline{\Psi}},$$

where the sum $\bigoplus_{e(\Psi)=\nu-e(\Phi_{\mathcal{P}})}$ is taken over all Hecke characters Ψ with conductor $\mathfrak{p}^{\nu-e(\Phi_{\mathcal{P}})}$ and infinite part $\Psi_\infty(a) = sgn(a)^l$ for $l \in \mathbb{Z}^n$ and $a \in K_\infty^\times$.

Proof. When $K = \mathbb{Q}$ this is Theorem 3.9 of [6].

We begin by noting that our hypothesis $\frac{\nu}{2} < e(\Phi_{\mathcal{P}}) < \nu$ implies that $\nu \geqslant 2$. By Theorem 3.6(3) above and Theorem 6.8 of [9] we have the inclusion

$$\mathscr{S}_k^+(\mathcal{N}, \Phi) \subset \sum_{e(\Psi)=\nu-e(\Phi_{\mathcal{P}})} \mathscr{S}_k^+(\mathfrak{p}^{e(\Phi_{\mathcal{P}})}\mathcal{N}_0, \Psi^2\Phi)^{\overline{\Psi}}.$$

Our strategy to complete the proof will be to prove the reverse inclusion and then show that the sum is direct.

Let Ψ be a Hecke character with conductor $\mathfrak{p}^{\nu-e(\Phi_{\mathcal{P}})}$ and infinite part $\Psi_\infty(a) = \mathrm{sgn}(a)^l$, and let $\mathbf{f} \in \mathscr{S}_k^+(\mathfrak{p}^{e(\Phi_{\mathcal{P}})}\mathcal{N}_0, \Psi^2\Phi)$ be a newform. By Theorem 5.7 of [9] we have $\mathbf{f}_{\overline{\Psi}} \in \mathscr{S}_k(\mathcal{N}, \Phi)$ where $\mathcal{N}$ is the exact level of $\mathbf{f}_{\overline{\Psi}}$. By Theorem 3.6(2), $C(\mathfrak{p}, \mathbf{f}) \neq 0$, so by Theorem 6.3 of [9], $\mathbf{f}_{\overline{\Psi}}$ is a newform. Therefore for all $\mathfrak{p}$-primary Hecke characters Ψ with $e(\Psi) = \nu - e(\Phi_{\mathcal{P}})$ we have the inclusion

$$\mathscr{S}_k^+(\mathfrak{p}^{e(\Phi_{\mathcal{P}})}\mathcal{N}_0, \Psi^2\Phi)^{\overline{\Psi}} \subset \mathscr{S}_k^+(\mathcal{N}, \Phi).$$

We have therefore shown that

$$\mathscr{S}_k^+(\mathcal{N}, \Phi) = \sum_{e(\Psi)=\nu-e(\Phi_{\mathcal{P}})} \mathscr{S}_k^+(\mathfrak{p}^{e(\Phi_{\mathcal{P}})}\mathcal{N}_0, \Psi^2\Phi)^{\overline{\Psi}}. \tag{3.7}$$

It therefore remains only to show that the sum on the right hand side of equation 3.7 is direct. We do this by showing that

$$\dim(\mathscr{S}_k^+(\mathcal{N},\Phi)) = \sum_{e(\Psi)=\nu-e(\Phi_{\mathcal{P}})} \dim(\mathscr{S}_k^+(\mathfrak{p}^{e(\Phi_{\mathcal{P}})}\mathcal{N}_0,\Psi^2\Phi)^{\overline{\Psi}}).$$

Given a Hecke character Ψ with $e(\Psi)=\nu-e(\Phi_{\mathcal{P}})$ and infinite part $\Psi_\infty(a)=\mathrm{sgn}(a)^l$, fix a basis S_Ψ of $\mathscr{S}_k^+(\mathfrak{p}^{e(\Phi_{\mathcal{P}})}\mathcal{N}_0,\Psi^2\Phi)$ consisting of normalized newforms $\mathbf{f}_1,\dots,\mathbf{f}_n$.

Define

$$S=\bigcup_{\Psi}\{\mathbf{f}_{\overline{\Psi}}:\mathbf{f}\in S_\Psi\}.$$

We have already shown that the elements of S are all newforms of $\mathscr{S}_k^+(\mathcal{N},\Phi)$ and in fact span the space. It therefore suffices to show

1. The (distinct) elements of S are linearly independent
2. $\#S=\sum_{e(\Psi)=\nu-e(\Phi_{\mathcal{P}})}\#S_\Psi=\sum_{e(\Psi)=\nu-e(\Phi_{\mathcal{P}})}\dim(\mathscr{S}_k^+(\mathfrak{p}^{e(\Phi_{\mathcal{P}})}\mathcal{N}_0,\Psi^2\Phi)^{\overline{\Psi}})$.

Note that (2) is equivalent to the statement that all the elements $\mathbf{f}_{\overline{\Psi}}$ of S are distinct.

We show that the elements of S are linearly independent by assuming the contrary and obtaining a contradiction. Suppose that there is a nontrivial relation

$$\sum_{i=1}^{m} c_i\mathbf{h}_i=0 \tag{3.8}$$

where $\mathbf{h}_i\in S$ (for all i), the $\mathbf{h}_i$ are all distinct, and each c_i is a non-zero scalar. Also assume that $m\geqslant 2$ is minimal in the sense that the elements of any subset of S having fewer than m elements are linearly independent.

For a prime $\mathfrak{q}$ which does not divide $\mathcal{N}$, we can apply the linear operator $T_{\mathfrak{q}}-C(\mathfrak{q},\mathbf{h}_1)\mathrm{Id}$ to equation 3.8 to get

$$\sum_{i=1}^{m} c_i(C(\mathfrak{q},\mathbf{h}_i)-C(\mathfrak{q},\mathbf{h}_1))\mathbf{h}_i.$$

Note that the coefficient of $\mathbf{h}_1$ is zero in the above sum. This means that the sum has fewer than m summands and hence must be trivial by the minimality of m. As each c_i is non-zero, we conclude that $C(\mathfrak{q},\mathbf{h}_i)=C(\mathfrak{q},\mathbf{h}_j)$ for all $1\leqslant i,j\leqslant m$ and $\mathfrak{q}\nmid\mathcal{N}$. As only finitely many primes divide $\mathcal{N}$, Theorem 3.5 of [9] shows that $\mathbf{h}_1=\mathbf{h}_2=\cdots=\mathbf{h}_m$. This contradicts our assumption that the $\mathbf{h}_i$ are distinct, proving that the elements of S are linearly independent.

To prove that

$$\#S=\sum_{e(\Psi)=\nu-e(\Phi_{\mathcal{P}})}\#S_\Psi=\sum_{e(\Psi)=\nu-e(\Phi_{\mathcal{P}})}\dim(\mathscr{S}_k^+(\mathfrak{p}^{e(\Phi_{\mathcal{P}})}\mathcal{N}_0,\Psi^2\Phi)^{\overline{\Psi}}),$$

it suffices to show if $\mathbf{f} \in \mathscr{S}_k^+(\mathfrak{p}^{e(\Phi_\mathcal{P})}\mathcal{N}_0, \Psi_0^2\Phi)$ and $\mathbf{g} \in \mathscr{S}_k^+(\mathfrak{p}^{e(\Phi_\mathcal{P})}\mathcal{N}_0, \Psi_1^2\Phi)$ are normalized newforms (with Ψ_0, Ψ_1 Hecke characters satisfying $e(\Psi_0) = e(\Psi_1) = \nu - e(\Phi_\mathcal{P})$) such that $\mathbf{f}_{\overline{\Psi}_0} = \mathbf{g}_{\overline{\Psi}_1}$ then $\Psi_0 = \Psi_1$ and $\mathbf{f} = \mathbf{g}$.

Suppose that $\mathbf{f}, \mathbf{g}$ are as in the previous paragraph and $\mathbf{f}_{\overline{\Psi}_0} = \mathbf{g}_{\overline{\Psi}_1}$. If $\Psi_0 = \Psi_1$ then Theorem 3.5 of [9] shows that $\mathbf{f} = \mathbf{g}$. Consequently, we may assume that $\Psi_0 \neq \Psi_1$. Then

$$\mathbf{f} \mid A_\mathfrak{p} = \mathbf{f}_{\overline{\Psi}_0\Psi_0} = \mathbf{g}_{\overline{\Psi}_1\Psi_0}.$$

Observe that $e(\Phi_\mathcal{P}\Psi_1^2) = e(\Phi_\mathcal{P})$ (as $e(\Phi_\mathcal{P}) > e(\Psi_1)$) and

$$0 < e(\overline{\Psi}_1\Psi_0) \leqslant \max\{e(\Psi_1), e(\Psi_0)\} < \frac{\nu}{2} < e(\Phi_\mathcal{P})$$

by hypothesis. By Corollary 6.4 of [9], $\mathbf{g}_{\overline{\Psi}_1\Psi_0} \in \mathscr{S}_k^+(\mathfrak{p}^{e(\Phi_\mathcal{P})+e(\overline{\Psi}_1\Psi_0)}\mathcal{N}_0, \Psi_0^2\Phi)$ is a normalized newform. As $\mathbf{f} \sim \mathbf{f} \mid A_\mathfrak{p}$ and $\mathbf{f} \mid A_\mathfrak{p} = \mathbf{g}_{\overline{\Psi}_1\Psi_0}$ we must have $\mathbf{f} = \mathbf{g}_{\overline{\Psi}_1\Psi_0}$ (by Theorem 3.5 of [9]). This means that $\mathbf{f} = \mathbf{f} \mid A_\mathfrak{p}$. In particular, the $\mathfrak{p}$-th coefficient of $\mathbf{f}$ is zero, contradicting Theorem 3.6(2) and finishing the proof. ∎

We conclude by presenting an application of the preceding theorems. This application makes clear the centrality of determining the vanishing of the $\mathfrak{p}$-th 'Fourier' coefficient of a Hilbert modular form in the study of character twists. This is a Hilbert modular analogue of Theorem 3.16 of [6].

Before stating the theorem however, we need a definition.

Definition 3.13. *A newform $\boldsymbol{g} \in \mathscr{S}_k(\mathcal{N}, \Phi)$ is said to be $\mathfrak{p}$-primitive if $\boldsymbol{g}$ is not the twist of any newform of level $\mathcal{N}'$ where $\mathcal{N}'$ is a proper divisor of $\mathcal{N}$ by a Hecke character by a Hecke character whose conductor is a power of $\mathfrak{p}$.*

Theorem 3.14. *Let $\boldsymbol{f} \in \mathscr{S}_k^+(\mathcal{N}, \Phi)$ be a normalized newform. The following are equivalent:*

1. $C(\mathfrak{p}, \boldsymbol{f}) = 0$
2. $\mathfrak{p}^2 \mid \mathcal{N}$ *and* $e(\Phi_\mathcal{P}) < \nu$
3. $\boldsymbol{f} = \boldsymbol{g}_\Psi$ *for some newform $\boldsymbol{g}$ in $\mathscr{S}_k^+(\mathcal{N}', \Phi\overline{\Psi}^2)$ for some ideal $\mathcal{N}'$ dividing $\mathcal{N}$ and some $\mathfrak{p}$-primary Hecke character Ψ.*

Further, assuming (1), if $e(\Phi_\mathcal{P}) > \frac{\nu}{2}$ then in (3) $\boldsymbol{g}$ may be chosen so that $ord_\mathfrak{p}(\mathcal{N}') < ord_\mathfrak{p}(\mathcal{N})$ and $\boldsymbol{g}$ is $\mathfrak{p}$-primitive.

Proof. (1) implies (2) follows immediately from Theorem 3.6. Now assume (2) holds. We have two cases to consider. If $\Phi_\mathcal{P}$ is trivial then let Ψ be a $\mathfrak{p}$-primary Hecke character with $0 < e(\Psi) < \frac{\nu}{2}$. Theorem 3.11 shows that $\mathscr{S}_k^+(\mathcal{N}, \overline{\Psi}^2\Phi_{\mathcal{N}_0})^\Psi = \mathscr{S}_k^+(\mathcal{N}, \Phi_{\mathcal{N}_0})$ and that there exists a newform $\mathbf{g} \in \mathscr{S}_k^+(\mathcal{N}, \overline{\Psi}^2\Phi_{\mathcal{N}_0})$ such that $\mathbf{f} = \mathbf{g}_\Psi$. Now suppose that $\Phi_\mathcal{P}$ is nontrivial. Then Theorem 3.10 shows that there exists a newform $\mathbf{g} \in \mathscr{S}_k^+(\mathcal{N}, \overline{\Phi}_\mathcal{P}\Phi_{\mathcal{N}_0})$ such that $\mathbf{f} = \mathbf{g}_{\Phi_\mathcal{P}}$. We therefore take $\mathcal{N}' = \mathcal{N}$ and $\Psi = \Phi_\mathcal{P}$. Finally, assume (3) holds. Then $C(\mathfrak{p}, \mathbf{f}) = C(\mathfrak{p}, \mathbf{g}_\Psi) = \Psi^*(\mathfrak{p})C(\mathfrak{p}, \mathbf{g}) = 0$ by Proposition 3.2.

For the final assertion, note that $\frac{\nu}{2} < e(\Phi_{\mathcal{P}}) < \nu$ implies, by Theorem 3.12, that there exists a newform $\mathbf{g} \in \mathscr{S}_k^+(\mathfrak{p}^{e(\Phi_{\mathcal{P}})}\mathcal{N}_0, \Psi^2\Phi)$ such that $\mathbf{f} = \mathbf{g}_{\overline{\Psi}}$, where Ψ is a $\mathfrak{p}$-primary Hecke character with $e(\Psi) = \nu - e(\Phi_{\mathcal{P}})$. We show that such a $\mathbf{g}$ is $\mathfrak{p}$-primitive. It clearly suffices to show that $C(\mathfrak{p}, \mathbf{g}) \neq 0$, which follows from Theorem 3.6 as $e(\Psi^2\Phi_{\mathcal{P}}) = e(\Phi_{\mathcal{P}}) = ord_{\mathfrak{p}}(\mathfrak{p}^{e(\Phi_{\mathcal{P}})}\mathcal{N}_0)$. ■

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Address: Benjamin Linowitz: Department of Mathematics, 6188 Kemeny Hall, Dartmouth College, Hanover, NH 03755, USA.

E-mail: benjamin.linowitz@gmail.com

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