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OSCILLATIONS OF FOURIER COEFFICIENTS OF $GL(m)$ HECKE-MAASS FORMS AND NONLINEAR EXPONENTIAL FUNCTIONS AT PRIMES

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Abstract: Let $F(z)$ be a Hecke-Maass form for $SL(m,\mathbb{Z})$ and $A_F(n,1,\dots,1)$ be the coefficients of L-function attached to F. We study the cancellation of $A_F(n,1,\dots,1)$ for twisted with a nonlinear exponential function at primes, namely the sum

$$\sum_{n\leqslant N}\Lambda(n)A_F(n,1,\dots,1)e(\alpha n^{\theta}),$$

where $0<\theta<2/m$. We also strengthen the corresponding previous results for holomorphic cusp forms for $SL(2,\mathbb{Z})$, and improve the estimates of Ren-Ye on the resonance of exponential sums involving Fourier coefficients of a Maass form for $SL(m,\mathbb{Z})$.

Keywords: exponential sums, Fourier coefficients, Hecke-Maass forms.

1. Introduction

Problems concerning the distribution of arithmetic function twisted some exponential function over primes are very classical in analytic number theory. This means that we have to establish estimates for the sum

$$S_\theta(\mathcal{A},N)=\sum_{n\leqslant N}\Lambda(n)a(n)e(\alpha n^{\theta}), \tag{1.1}$$

where $e(x):=\exp(2\pi ix)$ is an additive character, $\Lambda(n)$, as usual, denotes the von Mangoldt function, $a(n)$ is an arithmetic function and $\alpha,\theta>0$.

When $a(n)$ is the constant function, I. M. Vinogradov [15, 16] showed that for $\theta=1$,

$$S_1(1,N)\ll q^{\frac12}N^{\frac12}+q^{-\frac12}N+N\exp\left(-\frac12\sqrt{\log N}\right),$$

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where $|\alpha - a/q| \leqslant 1/q^2$ with $1 \leqslant a \leqslant q$ and $(a,q)=1$, which played a great role in solving the Goldbach problem with three primes; and for $\theta = 1/2$,

$$S_{1/2}(1,N) \ll N^{\frac{7}{8}+\varepsilon},$$

where the implied constant depends on α. It was asserted by Iwaniec and Kowalski [4, Eq. (13.55)] that they could prove, based on Vaughan's identity,

$$S_{1/2}(1,N) \ll N^{\frac{5}{6}} \log^4 N.$$

Recently, Ren [10] applied Heath-Brown's identity and got a more general and stronger result, namely

$$S_\theta(1,N) \ll \left(N^{\frac{1+\theta}{2}} + N^{\frac{4}{5}} + N^{\frac{2-\theta}{2}}\right) \log^A N$$

holds for $0 < \theta < 1$, where A is an absolute positive constant.

For the Fourier coefficients $\lambda_f(n)$ of a holomorphic cusp form f, by the so-called "principle of square-rooting", then one may be led to believe that $S_\theta(f,N) = O\left(N^{\frac{1}{2}+\epsilon}\right)$. Surprisingly, Iwaniec, Luo and Sarnak [5] gave the asymptotic formula, under the assumption of some extremely strong hypotheses,

$$S_{1/2}(f,N) = Z_f N^{\frac{3}{4}} + O(N^{\frac{5}{8}+\epsilon}), \tag{1.2}$$

where Z_f is a non-zero constant that depends on the cusp form f. Zhao [17] first obtained an unconditional bound and showed that

$$S_{1/2}(f,N) \ll N^{\frac{5}{6}} \log^{21} N, \tag{1.3}$$

where the implied constant depends effectively on α in (1.1) and the cusp form f. Later, Pi and Sun [9] used the zeros density-estimates method and proved a more general result, which stated that,for any $\varepsilon > 0$,

$$S_\theta(f,N) \ll \left(N^{\frac{1+\theta}{2}} + N^{\frac{2\theta+9}{12}} + N^{\frac{2-\theta}{2}}\right) N^{\varepsilon}. \tag{1.4}$$

Here $0 < \theta \leqslant 9/32$ or $1/2 \leqslant \theta \leqslant 9/16$. Note that the first term in the right-hand side is ineffective in the range above. Recently, Hou [3] followed Zhao's method and obtained that for $0 < \theta \leqslant 1/2$,

$$S_\theta(f,N) \ll \left(N^{\frac{5}{6}} + N^{\frac{2-\theta}{2}}\right) \log^c N, \tag{1.5}$$

which gave a complement for the range $9/32 < \theta < 1/2$. Here $c > 0$ is an absolute constant.

For the higher rank group $SL(m,\mathbb{Z})$ with $m \geqslant 2$, let $F(z)$ be a Hecke-Maass form of type $\nu = (\nu_1, \nu_2, \dots, \nu_{m-1})$ for $SL(m,\mathbb{Z})$. Then it has the Fourier expansion

$$F(z) = \sum_{\gamma \in U(m-1,\mathbb{Z})\backslash SL(m-1,\mathbb{Z})} \sum_{n_1 \geqslant 1} \cdots \sum_{n_{m-2} \geqslant 1} \sum_{n_{m-1} \neq 0} \frac{A_F(n_1, \dots, n_{m-1})}{\prod_{k=1}^{m} |n_k|^{\frac{k(m-k)}{2}}} \times W_J\left(\begin{pmatrix} n_1 \dots |n_{m-1}| & & & \\ & \ddots & & \\ & & n_1 & \\ & & & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, \nu, \psi_{1,\dots,1,\frac{n_{m-1}}{|n_{m-1}|}} \right), \tag{1.6}$$

where $U_{m-1}(\mathbb{Z})$ denotes the group of $(m-1) \times (m-1)$ upper triangular matrices with 1s on the diagonal and an integer entry above the diagonal, $A_F(n_1, \dots, n_{m-1}) \in \mathbb{C}$, $A_F(1, \dots, 1) = 1$, and W_J denotes the Jacquet Whittaker function. The generalized Ramanujan conjecture asserts that

$$|A_F(n, 1, \dots, 1)| \leqslant d_m(n). \tag{1.7}$$

Here $d_m(n)$ denotes the number of representations of n as the product of m natural numbers. The current best estimate is due to Kim and Sarnak [7] ($2 \leqslant m \leqslant 4$) and Luo, Rudnick and Sarnak ($m \geqslant 5$)

$$|A_F(n)| \leqslant n^{\frac{7}{64}} d(n), \qquad |A_F(n,1)| \leqslant n^{\frac{5}{14}} d_3(n), \qquad |A_F(n,1,1)| \leqslant n^{\frac{9}{22}} d_4(n),$$
$$|A_F(n, 1, \dots, 1)| \leqslant n^{\frac{1}{2} - \frac{1}{m^2+1}} d_m(n) \qquad (m \geqslant 5). \tag{1.8}$$

In this paper, we are interested in estimating an exponential sum over primes with square root amplitude twisted with Fourier coefficients $A_F(n, 1, \dots, 1)$. More precisely, we want to have an estimate for the following sum

$$S_\theta(F, N) = \sum_{n \leqslant N} \Lambda(n) A_F(n, 1, \dots, 1) e(\alpha n^\theta). \tag{1.9}$$

In order to state the result, we first need a technical hypothesis, i.e. the well known Hypothesis H of Rudnick and Sarnak [13]. Then Hypothesis H states the following.

Hypothesis H. *For any fixed* $\nu \geqslant 2$,

$$\sum_p \frac{|a_F(p^\nu)|^2 (\log p)^2}{p^\nu} < \infty, \tag{1.10}$$

where the arithmetic function $a_F(n)$ *is defined as in* (2.6).

Remark 1. For $m = 2, 3$, Hypothesis H follows from the Rankin-Selberg theory [13]. The $GL_4(\mathcal{A}_\mathbb{Q})$ case and the symmetric fourth power $\mathrm{sym}^4 F$ of a cuspidal representation F of $GL_2(\mathcal{A}_\mathbb{Q})$ were proved by Kim [8] based on his proof of the (weak) functoriality of the exterior square $\wedge^2 F$ from a cuspidal representation F of $GL_4(\mathcal{A}_\mathbb{Q})$.

Theorem 1. *Let $F(z)$ be a Hecke-Maass form for $SL(m,\mathbb{Z})$ and $0<\theta<1$. Then we have*

$$S_\theta(F,N) \ll \left(N^{\frac{5}{6}} + N^{\frac{4+m\theta}{6}} + N^{\frac{2-\theta}{2}}\right)\log^{\frac{9}{2}} N + N^{1-\frac{2(m+1)}{3(m^2+1)}+\varepsilon} \tag{1.11}$$

unconditionally for $2 \leqslant m \leqslant 4$ and under Hypothesis H for $m \geqslant 5$, where the implied constant depends effectively on α, θ in (1.9) and F.

Remark 2. One of the ingredients in the proof of Theorem 1 is those estimates in Lemma 2.8. This lemma improves the previous results of Ren-Ye [12, 11] on exponential sums involving Hecke-Maass forms at integers. It is clear that Theorem 1 gives the nontrivial bound for $\theta < 2/m$. The range of θ is restricted by the estimates of exponential sums in Lemma 2.8. Moreover, the condition Hypothesis H is due to our requirements for the estimates of some arithmetic functions in Lemma 2.6.

Remark 3. Let f be the holomorphic form of even integral weight k for the full modular group $SL(2,\mathbb{Z})$, and let $\lambda_f(n)$ be the Fourier coefficients of f. Our method implies the following result

$$S_\theta(f,N) \ll \left(N^{\frac{5}{6}} + N^{\frac{2+\theta}{3}} + N^{\frac{2-\theta}{2}}\right)\log^{\frac{11}{2}} N. \tag{1.12}$$

In particular, for $\theta = 1/2$, we have

$$S_{1/2}(f,N) \ll N^{\frac{5}{6}} \log^{\frac{11}{2}} N. \tag{1.13}$$

Obviously, it strengthens the upper bound in (1.3) on the aspect of the power of logarithm. On the other hand, (1.12) gives an nontrivial bound for $0<\theta<1$. Recall that the previous results only holds for $0<\theta<9/16$, if we combine (1.4) with (1.5). Thus, we enlarge the effective range of θ.

Furthermore, if we employ the estimate of Jutila [6, Theorem 4.6] when $\theta > 3/4$, which gives

$$\sum_{n \leqslant N} \lambda_f(n) e(\alpha n^\theta) \ll \alpha^{\frac{1}{3}} N^{\frac{1}{2}+\frac{\theta}{3}+\varepsilon}$$

for $N^{3/4-\theta} \ll \alpha \ll N^{3/2-\theta}$. Then our method implies that, if $\theta > 3/4$,

$$S_\theta(f,N) \ll N^{\frac{1+\theta}{2}+\varepsilon}, \tag{1.14}$$

which further improves (1.12).

One expects that the idea in the proof of Theorem 1.1 can be also used to investigate sums of the form

$$T_\theta(F,N) = \sum_{n \leqslant N} \mu(n) A_F(n,1,\ldots,1) e(\alpha n^\theta). \tag{1.15}$$

In fact, using the identity

$$\frac{1}{L(s,F)} = 2G(s) - L(s,F)M^2(s) + \Big(\frac{1}{L(s,F)} - M(s)\Big)\Big(1 - L(s,F)M(s)\Big)$$

with $M(s)$ being as in (2.8), one can obtain an analogue of Vaughan's identity for $\mu_F(n)$, which states that any $n > Y$

$$\begin{aligned}\mu_F(n) = &-\sum\sum_{\substack{bc|n\\ b\leqslant X, c\leqslant Y}} \mu_F(b)\mu_F(c)A_F\left(\frac{n}{bc}, 1, \ldots, 1\right)\\ &+\sum\sum_{\substack{bc|n\\ b> X, c> Y}} \mu_F(b)\mu_F(c)A_F\left(\frac{n}{bc}, 1, \ldots, 1\right).\end{aligned}$$

Here $X \geqslant 1$ and $Y \geqslant 1$. Further we are able to prove the same estimate as in Theorem 1.1 for the exponential sum

$$\sum_{n\leqslant N} \mu_F(n)e(n^\theta\alpha). \tag{1.16}$$

Since the details are completely analogous we omit the proof. Nevertheless, unlike the relation (3.1), there does not exist an explicit relation between the sum (1.15) and the sum (1.16). So this method, in this sense, cannot work. If we use the classical Vaughan identity, the variables m and n in Fourier coefficient $A_F\,(mn, 1, \ldots, 1)$ require to be separated. The multiplicative properties of $A_F\,(n, 1, \ldots, 1)$ give us satisfaction only for the case $m = 2$. Thus, we have

Theorem 2. *Let $F(z)$ be a Hecke-Maass form for $SL(2,\mathbb{Z})$ and $0 < \theta < 1$. Then we have*

$$T_\theta(F,N) \ll \left(N^{\frac{5}{6}} + N^{\frac{2+\theta}{3}} + N^{\frac{2-\theta}{2}}\right)\log^4 N, \tag{1.17}$$

where the implied constant depends effectively on α, θ and F.

Note that Hou [3] established

$$T_\theta(F,N) \ll \left(N^{\frac{5}{6}} + N^{\frac{2-\theta}{2}}\right)\log^c N$$

for some effective $c > 0$ and $0 < \theta \leqslant 1/2$.

Remark 4. Let f be the holomorphic form of even integral weight k for the full modular group $SL(2,\mathbb{Z})$, and let $\lambda_f(n)$ be the Fourier coefficients of f. As in Remark 3, we have

$$\sum_{n\leqslant N} \mu(n)\lambda_f(n)e(\alpha n^\theta) \ll \left(N^{\frac{5}{6}} + N^{\frac{2+\theta}{3}} + N^{\frac{2-\theta}{2}}\right)\log^4 N, \tag{1.18}$$

and

$$\sum_{n\leqslant N} \mu(n)\lambda_f(n)e(\alpha n^\theta) \ll N^{\frac{1+\theta}{2}+\varepsilon}, \tag{1.19}$$

if $\theta > 3/4$.

2. Preliminary lemmas

In this section, we quote and prove some results needed later.

2.1. Partition of the generalized von Mangoldt function

The Godement-Jacquet L-function $L(s,F)$ attached to F can be defined for $\Re s>1$ by

$$L(s,F)=\sum_{n=1}^{\infty}\frac{A_F(n,1,\ldots,1)}{n^s}=\prod_p\prod_{1\leqslant j\leqslant m}\left(1-\frac{\alpha_F(p,j)}{p^s}\right)^{-1},$$

where the $\{\alpha_F(p,j)\}, 1\leqslant j\leqslant m$ are the complex roots of the monic polynomial

$$X^m+\sum_{\ell=1}^{m-1}(-1)^{\ell}A_F(\overbrace{1,\ldots,1}^{\ell-1\text{ terms}},p,1,\ldots,1)X^{m-\ell}+(-1)^m\in\mathbb{C}[X]. \tag{2.1}$$

From (2.1), we find that

$$A_F(\overbrace{1,\ldots,1}^{\ell-1},p,1,\ldots,1)=\sum_{1\leqslant j_1<\cdots<j_\ell\leqslant m}\alpha_F(j_1,p)\ldots\alpha_F(p,j_\ell) \tag{2.2}$$

for $1\leqslant\ell\leqslant m-1$.

The equivalent assertion of (1.7) says

$$|\alpha_F(p,j)|=1 \tag{2.3}$$

for all primes p and $j=1,\ldots,m$. And (1.8) is equivalent to

$$|\alpha_F(p,j)|\leqslant p^{\theta_m} \tag{2.4}$$

for all primes p and $1\leqslant j\leqslant m$, where

$$\theta_2:=\frac{7}{64},\qquad\theta_3:=\frac{5}{14},\qquad\theta_4:=\frac{9}{22},\qquad\theta_m:=\frac{1}{2}-\frac{1}{m^2+1}\qquad(m\geqslant5). \tag{2.5}$$

Taking the logarithmic derivatives for $L(s,F)$, we have

$$-\frac{L'}{L}(s,F)=\sum_{n=1}^{\infty}\frac{\Lambda_F(n)}{n^s}=\sum_{n=1}^{\infty}\frac{\Lambda(n)a_F(n)}{n^s}, \tag{2.6}$$

where the arithmetic function $a_F(n)$ are multiplicative, and

$$a_F(p^k)=\sum_{i=1}^{m}\alpha_F(p,j)^k$$

for any $k \geqslant 1$. We define the function $L^{-1}(s,F)$ as

$$L^{-1}(s,F) = \sum_{n=1}^{\infty} \frac{\mu_F(n)}{n^s},$$

where

$$\mu_F(n) = \begin{cases} 0, & p^{m+1}|n \text{ for some prime } p, \\ \prod\limits_{p^\ell \| n} (-1)^\ell & \\ \qquad \times \sum\limits_{1\leqslant j_1<\cdots<j_\ell\leqslant m} \alpha_F(p,j_1)\cdots\alpha_F(p,j_\ell), & \text{for all } \ell \leqslant m. \end{cases}$$

Clearly, $\mu_F(n)$ are multiplicative.

With notations above, the version of Vaughan's identity for the Godement-Jacquet L-function $L(s,F)$ is the following.

Lemma 2.1. *Let $X \geqslant 1$ and $Y \geqslant 1$. Then for any $n > Y$ we have*

$$\begin{aligned} \Lambda_F(n) = & \sum_{\substack{b|n \\ b\leqslant X}} \mu_F(b) A_F\left(\frac{n}{b},1,\ldots,1\right)\log\left(\frac{n}{b}\right) \\ & - \sum\sum_{\substack{bc|n \\ b\leqslant X, c\leqslant Y}} \mu_F(b)\Lambda_F(c)A_F\left(\frac{n}{bc},1,\ldots,1\right) \\ & + \sum\sum_{\substack{bc|n \\ b>X, c>Y}} \mu_F(b)\Lambda_F(c)A_F\left(\frac{n}{bc},1,\ldots,1\right). \end{aligned} \tag{2.7}$$

Proof. Define

$$M(s) = \sum_{n\leqslant X} \mu_F(n)n^{-s}, \qquad N(s) = \sum_{n\leqslant Y} \Lambda_F(n)n^{-s}. \tag{2.8}$$

We then have

$$\begin{aligned} \frac{L'}{L}(s,F) &= L'(s,F)M(s) + L(s,F)M(s)N(s) \\ &\quad + \left(\frac{L'}{L}(s,F) + N(s)\right)(1 - L(s,F)M(s)) - N(s). \end{aligned}$$

On picking out the coefficient of n^{-s} on each side, one obtains the following analogue of Vaughan's identity. ■

The proof of our theorems require a variant of Vaughan's identity for the Godement-Jacquet L-function $L(s,F)$.

Lemma 2.2. *For $N < n \leqslant 2N$, $n \geqslant N^\eta$ with $\eta \leqslant 1/3$, we have*

$$\Lambda_F(n) = \Lambda_{1,F}(n) + \Lambda_{2,F}(n) + \Lambda_{3,F}(n) + \Lambda_{4,F}(n),$$

where

$$\Lambda_{1,F}(n)=\sum_{\substack{ab=n\\ b\leqslant N^{\eta}}}\mu_F(b)A_F\left(a,1,\ldots,1\right)\log a,\ \Lambda_{2,F}(n)$$

$$=-\sum_{\substack{abc=n\\ bc\leqslant N^{\eta}}}\mu_F(b)\Lambda_F(c)A_F\left(a,1,\ldots,1\right),$$

$$\Lambda_{3,F}(n)=-\sum_{\substack{abc=n\\ b,c\leqslant N^{\eta}\\ N^{1-2\eta}<a<2N^{1-\eta}}}\mu_F(b)\Lambda_F(c)A_F\left(a,1,\ldots,1\right),$$

$$\Lambda_{4,F}(n)=\sum_{\substack{abc=n\\ c>N^{\eta}\\ N^{\eta}<b\leqslant 2N^{1-\eta}}}\mu_F(b)\Lambda_F(c)A_F\left(a,1,\ldots,1\right).$$

Proof. We take $X=Y=N^{\eta}$ with $\eta\leqslant 1/3$ in Lemma 2.1 and decompose the second term in the right-hand side of (2.7) further. The partition according to the dichotomy of either $a\geqslant 2N^{1-\eta}$ or $a<2N^{1-\eta}$ is obvious. The extra condition that $N^{1-2\eta}<a$ is due to $b,c\leqslant N^{\eta}\Longrightarrow bc\leqslant N^{2\eta}$, together with $abc=n\leqslant 2N$, we have $N^{1-2\eta}<a$. Moreover, the extra condition in the fourth sum is apparent as

$$abc=n\leqslant 2N \text{ and } c>N^{\eta}\Longrightarrow ab\leqslant 2N^{1-\eta}\Longrightarrow b\leqslant 2N^{1-\eta}.$$

This completes the proof of Lemma 2.2. ■

2.2. The classical Vaughan identity for the Möbius function

Lemma 2.3. *Let $X\geqslant 1$ and $Y\geqslant 1$. Then for any $n>Y$ we have*

$$\mu(n)=-\sum\sum_{\substack{bc\mid n\\ b\leqslant X,c\leqslant Y}}\mu(b)\mu(c)+\sum\sum_{\substack{bc\mid n\\ b>X,c>Y}}\mu(b)\mu(c).$$

Proof. See [4, Proposition 13.5]. ■

Lemma 2.4. *For $N<n\leqslant 2N$, $n\geqslant N^{\eta}$ with $\eta\leqslant 1/3$, we have*

$$\mu(n)=-\sum_{\substack{abc=n\\ bc\leqslant N^{\eta}}}\mu(b)\mu(c)-\sum_{\substack{abc=n\\ b,c\leqslant N^{\eta}\\ N^{1-2\eta}<a<2N^{1-\eta}}}\mu(b)\mu(c)+\sum_{\substack{abc=n\\ c>N^{\eta}\\ N^{\eta}<b\leqslant 2N^{1-\eta}}}\mu(b)\mu(c).$$

Proof. Similar to the proof of Lemma 2.2. ■

2.3. Estimates of some arithmetic functions

All results in the following are due to Rankin-Selberg theory.

Lemma 2.5. *Let $F(z)$ be a Hecke-Maass form for $SL(m,\mathbb{Z})$. For any $\varepsilon > 0$, we have*

$$\sum_{n_1 n_2^2 \cdots n_{m-1}^{m-1} \leqslant x} |A_F(n_1, n_2, \ldots, n_{m-1})|^2 = cx + O_{F,\varepsilon}\left(x^{\frac{m^2-1}{m^2+1}+\varepsilon}\right). \tag{2.9}$$

where $c > 0$ is a constant which depends only on F.

Proof. Recall (see [2, Definition 12.1.2])

$$\sum_{n_1=1}^{\infty} \cdots \sum_{n_{m-1}=1}^{\infty} \frac{|A_F(n_1, n_2, \ldots, n_{m-1})|^2}{(n_1 n_2^2 \cdots n_{m-1}^{m-1})^s} = L(s, F \times F)\zeta(ms).$$

By [2, Theorem 12.4], we know that the Rankin-Selberg L-function $L(s, F \times F)$ has a meromorphic continuation to all $s \in \mathbb{C}$ with a pole at $s = 1$ only. Then the result follows by the refinement of Landau's Lemma [1, Theorem 3.2]. ■

Lemma 2.6. *Assume that Hypothesis H holds. Let $F(z)$ be a Hecke-Maass form for $SL(m,\mathbb{Z})$. Then for any $\varepsilon > 0$, we have*

$$\begin{aligned} \sum_{n \leqslant x} d(n)|\mu_F(n)|^2 &\ll x \log x, \\ \sum_{n \leqslant x} d(n)|A_F(n, 1, \cdots, 1)|^2 &\ll x \log x, \\ \sum_{n \leqslant x} d(n)|a_F(n)|^2 &\ll x \log x, \\ \sum_{n \leqslant x} |\mu_F(n)|^2 &\ll x, \end{aligned} \tag{2.10}$$

where the implied constants depend on F only.

Proof. To the arithmetic function $d(n)|\mu_F(n)|^2$, we attach the Dirichlet series

$$D(\mu_F, s) = \sum_{n=1}^{\infty} d(n)|\mu_F(n)|^2 n^{-s}. \tag{2.11}$$

we shall decompose the attached Dirichlet series $D(\mu_F, s)$ into some functions whose properties are well known. From the definition of $\mu_F(n)$, we know that it is multiplicative. Then Dirichlet series $D(\mu_F, s)$ has Euler product

$$D(\mu_F, s) = \prod_p \left(1 + \frac{2|\mu_F(p)|^2}{p^s} + \frac{3|\mu_F(p^2)|^2}{p^{2s}} + \cdots + \frac{(m+1)|\mu_F(p^m)|^2}{p^{ms}}\right).$$

Here $\mu_F(p) = -\sum_{j=1}^{m} \alpha_F(p, j)$. We can treat this infinite series as a rational function in p^{-s}. In particular, the coefficient of p^{-s} is

$$2\sum_{j=1}^{m}\sum_{i=1}^{m} \alpha_F(p, j)\overline{\alpha_F(p, i)}.$$

Recall that

$$L(s,F\times F)=\prod_p\prod_{1\leqslant j\leqslant m}\prod_{1\leqslant i\leqslant m}\left(1-\frac{\alpha_F(p,j)\overline{\alpha_F(p,i)}}{p^s}\right)^{-1}.$$

Then we see that $D(\mu_F,s)$ has the same coefficients of p^{-s} as $L^2(s,F\times F)$. Write

$$D(\mu_F,s)=L^2(s,F\times F)U(s). \tag{2.12}$$

Then a straightforward calculation shows that $U(s)=\prod_p U_p(s)$ where

$$U_p(s)=1+O\left(\sum_{\nu\geqslant 2}\frac{|a_F(p^\nu)|^2}{p^{\nu\sigma}}\right). \tag{2.13}$$

Put $\eta_m:=\frac{1}{2}(1-2\theta_m-\varepsilon)>0$, where θ_m is given by (2.5) and $\varepsilon>0$ is sufficiently small. In view of (2.6), we see that

$$|a_F(p^\nu)|\leqslant mp^{\theta_m\nu}$$

for all primes p and integers $\nu\geqslant 1$. From this we deduce that, for any $\sigma\geqslant 1-\varepsilon$,

$$\begin{aligned}\sum_{\nu\geqslant[1/(2\eta_m)]+2}\sum_p\frac{|a_F(p^\nu)|^2}{p^{\nu\sigma}}&\ll\sum_p\sum_{\nu\geqslant[1/(2\eta_m)]+2}\frac{1}{p^{\nu(1-2\theta_m-\varepsilon)}}\\&\ll\sum_p\sum_{\nu\geqslant[1/(2\eta_m)]+2}\frac{1}{p^{2\eta_m\nu}}\\&\ll\sum_p\frac{1}{p^{1+2\eta_m}}\\&\ll 1.\end{aligned} \tag{2.14}$$

Further, we derive

$$\begin{aligned}\log|U(s)|&\ll\sum_p\log|U_p(s)|\\&\ll\sum_p\sum_{\nu\geqslant 2}\frac{|a_F(p^\nu)|^2}{p^{\nu\sigma}}\\&\ll\sum_{2\leqslant\nu\leqslant[1/(2\eta_m)]+2}\sum_p\frac{|a_F(p^\nu)|^2}{p^{\nu\sigma}}+1\\&\ll 1\end{aligned} \tag{2.15}$$

providing $\sigma=1$ under Hypothesis H. Thus $U(s)$ converges absolutely in $\mathrm{Re}s\geqslant 1$. By a standard use of the Wiener-Ikehara Theorem, we have

$$\sum_{n\leqslant x}d(n)|\mu_F(n)|^2\ll x\log x. \tag{2.16}$$

Similar to the first assertions, we can prove the others in the same way. So we omit here. ■

2.4. Bilinear forms and Fourier coefficients with exponentials

We shall need to estimate certain bilinear forms with exponentials. The following lemmas suffice for our enterprise.

Lemma 2.7. *Let $h(x), g(x)$ be a real-valued functions on $[M, 2M]$, $[N, 2N]$ respectively, such that $h \ll H$, $g \ll G$ and $|h'| \geqslant HM^{-1}$, $|g'| \geqslant GN^{-1}$. Then for any complex number β_m, γ_n we have*

$$\sum_m \sum_n \beta_m \gamma_n e\left(h(m)g(n)\right) \ll (HG)^{-\frac{1}{2}}(HG+M)^{\frac{1}{2}}(HG+N)^{\frac{1}{2}}\|\beta\|\|\gamma\|,$$

where $\|\beta\|^2 = \sum |\beta_m|^2$ and $\|\gamma\|^2 = \sum |\gamma_n|^2$.

Proof. The proof can be found in [4, Corollary 7.4]. ∎

Lemma 2.8. *Let F be a Hecke-Maass form on $SL(m, Z)$. Then we have, for $\theta \neq 1/m$,*

$$\sum_{X<n\leqslant 2X} A_F(n,1,\cdots,1)e\left(\alpha n^{\theta}\right) \ll (\alpha X^{\theta})^{m/2} + X^{\sigma_m+\varepsilon}, \tag{2.17}$$

and for $\theta = 1/m$,

$$\begin{aligned}\sum_{X<n\leqslant 2X} & A_F(n,1,\cdots,1)e\left(\alpha n^{1/m}\right)\\ &= -\Upsilon_{\alpha,\theta,X}\sqrt{m}X^{1/(2m)+1/2}(-i)^{(m-1)/2}I(m,\alpha,X)\frac{A_F(1,\cdots,1,n_\alpha)}{n_\alpha^{1/2-1/(2m)}}\\ &\quad + O(\alpha^{\delta_m}X^{1/2-1/(2m)+\varepsilon}) + O(X^{\sigma_m+\varepsilon}),\end{aligned} \tag{2.18}$$

where n_α is the integer satisfying $(\alpha/m)^m - n_\alpha \in (-1/2, 1/2]$,

$$I(m,\alpha,X) = \int_1^{2^{1/m}} t^{m/2-1/2}e\big((\alpha - mn_\alpha^{1/m})X^{1/m}t\big)dt,$$

$$\Upsilon_{\alpha,\theta,X} = \begin{cases} 0, & \textit{if} \quad 2\max\{1, 2^{\theta-1/m}\}(\alpha\theta)^m > X^{1-\theta m},\\ 1, & \textit{otherwise,}\end{cases}$$

$$\delta_m = \begin{cases} 23/32, & \textit{if} \quad m=2,\\ 29/14, & \textit{if} \quad m=3,\\ m-1/2, & \textit{if} \quad m\geqslant 4,\end{cases}$$

and

$$\sigma_m = (m^2-m)/(m^2+1).$$

Remark 5. The previous values of σ_m in [12, 11, 14] are

$$\sigma_m = \min\{(m-1)(1+\theta_m)/(m+1), 1-1/m\},$$

where θ_m are as in (2.5). Clearly, we improve the results before. In particular, for fixed α and large X, (2.18) is an asymptotic formula for $m \leqslant 3$.

Proof. Ren-Ye [12, 11] and Sun-Wu [14] need to estimate the short interval sum of $A_F(n,1,\cdots,1)$ in their proofs, that is

$$\sum_{X<n\leqslant X+X/\Delta} |A_F(n,1,\cdots,1)|,$$

where Δ is some parameter less than X. By applying the Cauchy-Schwarz inequality, we obtain

$$\sum_{X<n\leqslant X+X/\Delta} |A_F(n,1,\cdots,1)| \ll \left(\sum_{X<n\leqslant X+X/\Delta} |A_F(n,1,\cdots,1)|^2\right)^{1/2} \left(\frac{X}{\Delta}\right)^{1/2}.$$

By inserting the estimates in Lemma 2.5, then we have

$$\sum_{X<n\leqslant X+X/\Delta} |A_F(n,1,\cdots,1)| \ll \frac{X}{\Delta} + \frac{X^{1-1/(m^2+1)+\varepsilon}}{\Delta^{1/2}}.$$

We use these bounds instead of the previous results, namely,

$$\sum_{X<n\leqslant X+X/\Delta} |A_F(n,1,\cdots,1)| \ll \min\left\{X^{1+\theta_m+\varepsilon}/\Delta, X/\Delta^{1/2}\right\}.$$

Finally, we obtain that for $\theta \neq 1/m$,

$$\sum_{X<n\leqslant 2X} A_F(n,1,\cdots,1)e(\alpha n^\theta) \ll (\alpha X^\theta)^{m/2} + X^{(m^2-m)/(m^2+1)+\varepsilon}.$$

Moreover, we have the corresponding results for $\theta = 1/m$ by applying the same idea. ■

3. Proof of Theorem 1

It follows from the definition of $\Lambda_F(n)$ that

$$\sum_{n\leqslant N} \Lambda(n)A_F(n,1,\ldots,1)e(\alpha n^\theta) = \sum_{n\leqslant N} \Lambda_F(n)e(\alpha n^\theta) + O\left(\log N \sum_{\substack{p^k\leqslant N\\ k\geqslant 2}} \left(|A(p^k,1,\cdots,1)| + |a_F(p^k)|\right)\right).$$

We obtain from (2.2) that

$$\begin{aligned} A(p^2,1,\cdots,1) &= A(p,1,\cdots,1)^2 - A(1,p,\cdots,1),\\ a_F(p^2) &= A(p,1,\cdots,1)^2 - 2A(1,p,\cdots,1). \end{aligned}$$

Moreover, we know that

$$\sum_{p^2\leqslant N}|A(p,1,\cdots,1)|^2 \ll \sum_{n\leqslant N^{1/2}}|A(n,1,\cdots,1)|^2 \ll N^{\frac{1}{2}},$$

and

$$\sum_{p^2\leqslant N}|A(1,p,\cdots,1)| \ll N^{\frac{1}{2}}\left(\sum_{n\leqslant N^{1/2}}|A(1,n,\cdots,1)|^2\right)^{1/2} \ll N^{\frac{3}{4}}.$$

Thus, one derives from (2.4) that

$$\begin{aligned}\sum_{\substack{p^2\leqslant N\\ k\geqslant 2}}\big(|A(p^k,1,\cdots,1)|+|a_F(p^k)|\big) &\ll N^{\frac{3}{4}} + \sum_{\substack{p^k\leqslant N\\ k\geqslant 3}}\big(|A(p^k,1,\cdots,1)|+|a_F(p^k)|\big)\\ &\ll N^{\frac{3}{4}} + N^{\frac{1}{3}+\theta_m+\varepsilon}.\end{aligned}$$

Collecting together these estimates above, we find

$$\sum_{n\leqslant N}\Lambda(n)A_F(n,1,\ldots,1)e(\alpha n^\theta) = \sum_{n\leqslant N}\Lambda_F(n)e(\alpha n^\theta) + O\left(N^{\frac{3}{4}} + N^{\frac{1}{3}+\theta_m+\varepsilon}\right). \tag{3.1}$$

Now the sum of our interest in (1.9) can be transformed into

$$\sum_{n\leqslant N}\Lambda_F(n)e(\alpha n^\theta).$$

On applying Vaughan's identity in Lemma 2.2, it can be decomposed and it suffices to estimate each individual component. We have

$$\left|\sum_{N<n\leqslant 2N}\Lambda_F(n)e(\alpha n^\theta)\right| \leqslant \sum_{i=1}^{4}|S_{F,i}(N)|, \tag{3.2}$$

where $S_{i,F}(N)=\sum_{N<n\leqslant 2N}\Lambda_{i,F}(n)e(\alpha n^\theta)$, and $\Lambda_{i,F}(n)$ are defined in Lemma 2.2.

3.1. Bilinear forms treatment

The last two sums of (3.2) are similar and can be disposed using similar means. Toward that end, we have

Lemma 3.1. *With $S_{F,i}(N)$ defined as before, we have*

$$|S_{F,3}(N)|+|S_{F,4}(N)| \ll \left(N^{\frac{1+\max\{2\eta,\theta\}}{2}} + N^{\frac{2-\min\{\eta,\theta\}}{2}}\right)\log^{\frac{9}{2}} N,$$

where the implied constant depends on α, θ and F.

Proof. Breaking the summations into dyadic intervals, it suffices to estimate, for arithmetic functions $\beta(m)$ and $\gamma(l)$, sums of the following shape,

$$\sum_{M<m\leqslant 2M} \beta(m) \sum_{L<l\leqslant 2L} \gamma(l) e\left(\alpha(lm)^{\theta}\right), \tag{3.3}$$

and with $N^{\eta} \leqslant M \leqslant N^{2\eta}$, $N^{1-2\eta} \leqslant L \leqslant N^{1-\eta}$ for $S_{F,3}(N)$, $N^{\eta} \leqslant M \leqslant N^{1-\eta}$, $N^{\eta} \leqslant L \leqslant N^{1-\eta}$ for $S_{F,4}(N)$ and $ML = N$. Here the value of $\eta \leqslant 1/3$ shall be chosen at the end of this paper, $\beta(m)$ is

$$\sum_{\substack{bc=m \\ b,c\leqslant N^{\eta}}} \mu_F(b)\Lambda_F(c) \text{ and } \sum_{\substack{ac=m \\ c>N^{\eta}}} \Lambda_F(c) A_F(a,1,\ldots,1), \tag{3.4}$$

and $\gamma(l)$ is

$$A_F(l,1,\ldots,1) \text{ and } \mu_F(l)$$

for $S_{F,3}(N)$ and $S_{F,4}(N)$, respectively. We can estimate the square moments of $\beta(m)$ by Cauchy's inequality and Lemma 2.6

$$\begin{aligned}
\sum_{M<m\leqslant 2M} \left| \sum_{\substack{bc=m \\ b,c\leqslant N^{\eta}}} \mu_F(b)\Lambda_F(c) \right|^2 &\ll \sum_{M<m\leqslant 2M} \sum_{bc=m} |\mu_F(b)|^2 |\Lambda_F(c)|^2 d(m) \\
&\ll \log^2 M \sum_{b\leqslant 2M} d(b)|\mu_F(b)|^2 \\
&\quad \times \sum_{M/b<c\leqslant 2M/b} d(c)|a_F(c)|^2 \\
&\ll M \log^5 M,
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
\sum_{m\leqslant x} \left| \sum_{\substack{ac=m \\ c>N^{\eta}}} \Lambda_F(c) A_F(a,1,\ldots,1) \right|^2 &\ll \sum_{M<m\leqslant 2M} \sum_{ac=m} |A_F(a,1,\ldots,1)|^2 |\Lambda_F(c)|^2 d(m) \\
&\ll \log^2 M \sum_{a\leqslant 2M} d(a)|A_F(a,1,\ldots,1)|^2 \\
&\quad \times \sum_{M/b<c\leqslant 2M/b} d(c)|a_F(c)|^2 \\
&\ll M \log^5 M.
\end{aligned} \tag{3.6}$$

Recall that

$$\sum_{L<l\leqslant 2L} |A_F(l,1,\ldots,1)|^2 \ll L \qquad \text{and} \qquad \sum_{L<l\leqslant 2L} |\mu_F(l)|^2 \ll L. \tag{3.7}$$

On using Lemma 2.7 with $h(x) = g(x) = \alpha^{1/2}x^\theta$, we find that the bilinear sum (3.3) is bounded by

$$\left(N^{\frac{\theta}{2}} + M^{\frac{1}{2}} + L^{\frac{1}{2}} + N^{\frac{1-\theta}{2}}\right)\left(\sum_{M<m\leqslant 2M} |\beta(m)|^2\right)^{\frac{1}{2}}\left(\sum_{L<l\leqslant 2L} |\gamma(l)|^2\right)^{\frac{1}{2}}$$
$$\ll \left(N^{\frac{1+\theta}{2}} + N^{\frac{1}{2}}M^{\frac{1}{2}} + N^{\frac{1}{2}}L^{\frac{1}{2}} + N^{\frac{2-\theta}{2}}\right)\log^{\frac{5}{2}} N. \quad (3.8)$$

On summing up all the dyadic intervals, then Lemma 3.1 follows. ■

3.2. Linear sums

It still remains to estimate the other terms in (3.2) which are similar.

Lemma 3.2. *With $S_{i,F}(N)$ defined as before, we have, if $\theta \neq 1/m$,*

$$|S_{F,1}(N)| + |S_{F,2}(N)| \ll N^{\frac{m\theta}{2}+\eta}\log^{\frac{9}{2}} N + N^{\sigma_m+\eta(1-\sigma_m)+\varepsilon},$$

and if $\theta = 1/m$,

$$|S_{F,1}(N)| + |S_{F,2}(N)| \ll \left(N^{\frac{1}{2}+\frac{1}{2m}} + N^{\frac{1}{2}-\frac{1}{2m}+\frac{\eta(m+2\delta_m+1)}{2m}}\right)\log^{\frac{9}{2}} N$$
$$+ N^{\sigma_m+\eta(1-\sigma_m)+\varepsilon},$$

where the implied constant depends on α, θ and F.

Proof. By partial summation, $S_{1,F}(N)$ and $S_{2,F}(N)$ become the same type. Similar to the beginning for the proof of bilinear forms above, we need to estimate the following sum

$$\sum_{Q<q\leqslant 2Q} |\beta(q)| \left|\sum_{L<l\leqslant 2L} A_F(a,1,\ldots,1)e\left(\alpha(lq)^\theta\right)\right|, \quad (3.9)$$

with $Q \leqslant N^\eta$, $QL = N$. Here $\beta(q)$ is

$$\mu_F(q) \qquad \text{and} \qquad \sum_{bc=q} \mu_F(b)\Lambda_F(c), \quad (3.10)$$

for $S_{F,1}(N)$ and $S_{F,2}(N)$ respectively. Note that there is a factor $\log N$ for bounding the sum $S_{1,F}(N)$, which appears because of partial summation.

By inserting the estimates in Lemma 2.8, we obtain that, for $\theta \neq 1/m$, (3.9) is majorized by

$$\sum_{Q<q\leqslant 2Q} |\beta(q)| \left((qL)^{\frac{m\theta}{2}} + L^{\sigma_m+\varepsilon}\right) \ll N^{\frac{m\theta}{2}+\eta}\log^{\frac{5}{2}} N + N^{\sigma_m+\eta(1-\sigma_m)+\varepsilon}, \quad (3.11)$$

and for $\theta = 1/m$, (3.9) is bounded by

$$\sum_{Q<q\leqslant 2Q} |\beta(q)| \left(\frac{|A_F(1,\cdots,1,q)|}{q^{1/2-1/(2m)}} L^{\frac{1}{2}+\frac{1}{2m}} + q^{\frac{\delta_m}{m}} L^{\frac{1}{2}-\frac{1}{2m}} + L^{\sigma_m+\varepsilon} \right)$$
$$\ll N^{\frac{1}{2}+\frac{1}{2m}} \log^{\frac{5}{2}} N + N^{\frac{1}{2}-\frac{1}{2m}+\frac{\eta(m+2\delta_m+1)}{2m}} \log^{\frac{5}{2}} N + N^{\sigma_m+\eta(1-\sigma_m)+\varepsilon}, \quad (3.12)$$

where the notation σ_m, δ_m is defined as in Lemma (2.8). We use Cauchy's inequality, (3.5) and (3.7) in the last step of (3.11) and (3.12).

■

3.3. Choosing for the parameter η

Combining Lemma 3.1 and Lemma 3.2, we have the best choice for the parameter η in the following. We take $\eta = (2-m\theta)/3$ if $\theta > 1/m$, and $\eta = 1/3$ if $\theta \leqslant 1/m$. Finally, we have

$$\left| \sum_{N<n\leqslant 2N} \Lambda_F(n) e(\alpha n^\theta) \right| \ll \left(N^{\frac{5}{6}} + N^{\frac{4+m\theta}{6}} + N^{\frac{2-\theta}{2}} \right) \log^{\frac{9}{2}} N + N^{1-\frac{2(m+1)}{3(m^2+1)}+\varepsilon}. \quad (3.13)$$

After adding up the sum over all the dyadic intervals and combining (3.1), we finish the proof of Theorem 1.

4. Proof of Theorem 2

Let

$$T'_\theta(F,N) = \sum_{N<n\leqslant 2N} \mu(n) A_F(n) e(\alpha n^\theta). \quad (4.1)$$

After applying Vaughan identity in Lemma 2.4, we have

$$\begin{aligned} &T'_\theta(F,N) \\ &= \sum_{N<n\leqslant 2N} \left(-\sum_{\substack{abc=n \\ bc\leqslant N^\eta}} \mu(b)\mu(c) - \sum_{\substack{abc=n \\ b,c\leqslant N^\eta \\ N^{1-2\eta}<a<2N^{1-\eta}}} \mu(b)\mu(c) + \sum_{\substack{abc=n \\ c>N^\eta \\ N^\eta<b\leqslant 2N^{1-\eta}}} \mu(b)\mu(c) \right) \\ &\quad \times A_F(n) e(\alpha n^\theta) \\ &= T'_{1,\theta}(F,N) + T'_{2,\theta}(F,N) + T'_{3,\theta}(F,N), \end{aligned} \quad (4.2)$$

respectively. First, we go to estimate the sum $T_{1,\theta}(F,N)$. Introducing the notation

$$\alpha_e = \sum_{bc=e} \mu(b)\mu(c).$$

Clearly, we see that

$$T'_{1,\theta}(F,N) = \sum_{e \leqslant N^\eta} \alpha_e \sum_{a \sim N/e} A_F\,(ae)\,e\left(\alpha(ae)^\theta\right)$$

and α_e satisfies the bound

$$|\alpha_e| \leqslant d(e).$$

By multiplicative property of Hecke eigenvalues

$$A_F(mn) = \sum_{d \mid \gcd(m,n)} \mu(d) A_F\left(\frac{m}{d}\right) A_F\left(\frac{n}{d}\right), \tag{4.3}$$

we obtain that

$$\begin{aligned} T'_{1,\theta}(F,N) &= \sum_{e \leqslant N^\eta} \alpha_e \sum_{l \mid e} \mu(l) A_F\left(\frac{e}{l}\right) \sum_{a \sim N/(el)} A_F\,(a)\,e\left(\alpha(ael)^\theta\right) \\ &= \sum_{e \leqslant N^\eta} d(e) \sum_{l \mid e} \left|A_F\left(\frac{e}{l}\right)\right| \left|\sum_{a \sim N/(el)} A_F\,(a)\,e\left(\alpha(ael)^\theta\right)\right|. \end{aligned} \tag{4.4}$$

After appealing to the estimate in Lemma 2.8, the sum is

$$\begin{aligned} &\ll \sum_{e \leqslant N^\eta} d(e) \sum_{l \mid e} |A_F\,(l)| \left(N^\theta + \left(\frac{N}{el}\right)^{\sigma_m+\varepsilon}\right) \\ &\ll N^{\sigma_m+\varepsilon} + N^\theta \sum_{e \leqslant N^\eta} d(e) \sum_{l \mid e} |A_F\,(l)| \end{aligned} \tag{4.5}$$

providing $\theta \neq 1/2$. Due to the inequality $d(el) \leqslant d(e)d(l)$, the classical result $\sum_{n \leqslant N} d(n) \ll N \log N$ and Lemma 2.6 , we have

$$\begin{aligned} T'_{1,\theta}(F,N) &\ll N^{\sigma_m+\varepsilon} + N^\theta \sum_{e \leqslant N^\eta} d(e) \sum_{l \leqslant N^\eta/e} d(l)\,|A_F\,(l)| \\ &\ll N^{\sigma_m+\varepsilon} + N^\theta \sum_{e \leqslant N^\eta} d(e) \left(\sum_{l \leqslant N^\eta/e} d(l)\right)^{\frac{1}{2}} \left(\sum_{l \leqslant N^\eta/e} d(l)\,|A_F\,(l)|^2\right)^{\frac{1}{2}} \\ &\ll N^{\sigma_m+\varepsilon} + N^{\theta+\eta} \log^3 N. \end{aligned} \tag{4.6}$$

When $\theta = 1/2$, we similarly obtain that

$$T'_{1,\theta}(F,N) \ll N^{\frac{3}{4}+\eta\theta_2+\varepsilon} + N^{1+\frac{\eta(3+2\delta_2)}{4}} \log^3 N + N^{\sigma_2+\eta(1-\sigma_2)+\varepsilon}. \tag{4.7}$$

For the two other sums, namely $T'_{2,\theta}(F,N)$ and $T'_{3,\theta}(F,N)$, it suffices to estimate, for arithmetic functions $\beta(m)$ and $\gamma(l)$, sums of the following shape,

$$\sum_{M < m \leqslant 2M} \beta(m) \sum_{L < l \leqslant 2L} \gamma(l) A_F\,(ml)\,e\left(\alpha(ml)^\theta\right), \tag{4.8}$$

with $N^\eta \leqslant M \leqslant N^{2\eta}$, $N^{1-2\eta} \leqslant L \leqslant N^{1-\eta}$ for $T'_{2,\theta}(j,N)$, $N^\eta \leqslant M \leqslant N^{1-\eta}$, $N^\eta \leqslant L \leqslant N^{1-\eta}$ for $T'_{3,\theta}(j,N)$ and $ML = N$. Here the value of η shall be chosen at the end of this paper, $\beta(m)$ is

$$\sum_{\substack{bc=m\\ b,c\leqslant N^\eta}} \mu(b)\mu(c) \text{ and } \sum_{\substack{ac=m\\ c>N^\eta}} \mu(c), \tag{4.9}$$

and $\gamma(l)$ is

$$1 \text{ and } \mu(l)$$

for $T'_{2,\theta}(F,N)$ and $T'_{F,\theta}(j,N)$ respectively. Note that

$$|\beta(m)| \leqslant d(m), \quad |\gamma(l)| \leqslant 1.$$

By (4.3), (4.8) is equal to

$$\begin{aligned}
&\sum_{M<m\leqslant 2M} \beta(m) \sum_{d|m} \mu(d) A_F\left(\frac{m}{d}\right) \sum_{l\sim L/d} \gamma(ld) A_F(l)\, e\left(\alpha(mld)^\theta\right)\\
&\quad \ll \sum_{d\ll \min\{M,L\}} |\mu(d)| \left| \sum_{m\sim M/d} \beta(md) A_F(m) \sum_{l\sim L/d} \gamma(ld) A_F(l)\, e\left(\alpha(mld)^\theta\right)\right|\\
&\quad \ll \sum_{d\ll \min\{M,L\}} \left(N^{\frac{\theta}{2}} d^{-\frac{\theta}{2}} + L^{\frac12} d^{-\frac12} + M^{\frac12} d^{-\frac12} + N^{\frac12-\frac{\theta}{2}} d^{-1+\frac{\theta}{2}}\right) |\mu(d)| d(d)\\
&\qquad \times \left(\sum_{m\sim M/d} |d(m) A_F(m)|^2\right)^{\frac12} \left(\sum_{l\sim L/d} |A_F(l)|^2\right)^{\frac12}\\
&\quad \ll \left(N^{\frac{1+\eta}{2}} + N^{\frac{2-\eta}{2}} + N^{\frac{1+\theta}{2}} + N^{\frac{2-\theta}{2}}\right) \log^{\frac32} N.
\end{aligned} \tag{4.10}$$

Here the last step is due to the estimates

$$\sum_{m\sim M} |d(m) A_F(m)|^2 \ll x \log^3 x, \qquad \sum_{m\sim M} |A_F(m)|^2 \ll x.$$

The former appeals the fact that

$$\sum_{m\geqslant 1} \frac{|d(m) A_F(m)|^2}{m^s} = L^4(s, F\times F) U(s),$$

where $U(s)$ converges absolutely in $\mathrm{Re} s > 1/2 + \varepsilon$. The latter follows from Lemma 2.5. Thus, we see that $|T'_{2,\theta}(F,N)| + |T'_{F,\theta}(j,N)|$ is

$$\ll \left(N^{1+\frac{\eta}{2}} + N^{\frac{2-\eta}{2}} + N^{\frac{1+\theta}{2}} + N^{\frac{2-\theta}{2}}\right) \log^{\frac52} N. \tag{4.11}$$

On taking the same value for the parameter η as in Section 3.3 when $m = 2$, the conclusion follows, that is to say,

$$T_\theta(F,N) \ll \left(N^{\frac56} + N^{\frac{2+\theta}{3}} + N^{\frac{2-\theta}{2}}\right) \log^4 N. \tag{4.12}$$

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