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SOME PROPERTIES OF THE GENERALIZED FAVARD-DURRMEYER OPERATORS

GRZEGORZ NOWAK & PAULINA PYCH-TABERSKA

Abstract: The Durrmeyer modification $\widetilde{F}_n f$ of the generalized Favard operators in some weighted function spaces are considered. The rate of convergence of $\widetilde{F}_n f(x)$ at the Lebesgue points x of f is estimated. In particular, a corresponding estimate in the class of functions f of bounded p-th power variation is deduced.

Keywords: Favard-Durrmeyer operator, rate of convergence, Lebesgue point, p-th power variation

1. Preliminaries

Let $X_\sigma(R)$ be the space of all measurable real-valued functions f on the real line $R := (-\infty, \infty)$, with the norm

$$\|f\|_\sigma := \sup_{x\in R} |f(x)\exp(-\sigma x^2)| < \infty,$$

where $\sigma > 0$. For functions $f \in X_\sigma(R)$ consider first the generalized Favard operators defined by

$$F_n f(x) := \sum_{k=-\infty}^{\infty} f(k/n) p_{n,k}(x;\gamma),$$

where $x \in R, n \in N$,

$$p_{n,k}(x;\gamma) := \frac{1}{n\gamma_n\sqrt{2\pi}} \exp\Big(-\frac{1}{2\gamma_n^2}\Big(\frac{k}{n} - x\Big)^2\Big)$$

and $\gamma = (\gamma_n)_{n=1}^{\infty}$ is a positive sequence convergent to 0 (see [4]). In the special case where $\gamma_n^2 = \kappa/2n$ with a positive constant κ, F_n become the known discrete

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Favard operators introduced in [3]. Some properties of operators $F_n f$ for continuous functions on R can be found e.g. in [4] and [5]. In this paper we deal with the Durrmeyer type modification of the operators F_n, defined by

$$\widetilde{F}_n f(x) := n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \int_{-\infty}^{\infty} f(t) p_{n,k}(t;\gamma) dt. \tag{1.1}$$

We will examine the rate of convergence of $\widetilde{F}_n f(x)$, mainly, at those points $x \in R$ at which

$$\lim_{h \to 0} \frac{1}{h} \int_0^h (f(x+t) - f(x)) dt = 0.$$

The general estimate will be expressed in terms of the quantity

$$w_x(\delta; f) := \sup_{0<|h|\leq\delta} \left| \frac{1}{h} \int_0^h (f(x+t) - f(x)) dt \right|, \quad (\delta > 0).$$

Some analogous results for the generalized Favard-Kantorovich operators are presented in [6].

Throughout the paper, the symbols $K(\cdots), K_j(\cdots)$ $(j = 1, 2, \ldots)$ will mean some positive constants depending only on the parameters indicated in parentheses.

2. Main result

As is known ([2], pp. 126, 204; [4], p. 388), for all $n \in N, x \in R$,

$$\sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) = 1 + S_n(x), \tag{2.1}$$

where

$$S_n(x) := 2 \sum_{j=1}^{\infty} \exp(-2\pi^2 j^2 n^2 \gamma_n^2) \cos(2\pi n j x) \tag{2.2}$$

and

$$|S_n(x)| \leq (\pi n \gamma_n)^{-2}. \tag{2.3}$$

It is easy to see that for all $n \in N, \nu \in N, k \in Z := \{0, \pm1, \pm2, \ldots\}$,

$$\int_{-\infty}^{\infty} p_{n,k}(t;\gamma) dt = \frac{1}{n}, \tag{2.4}$$

$$\int_{-\infty}^{\infty} \left(\frac{k}{n} - t\right)^{2\nu} p_{n,k}(t;\gamma) dt = \frac{(2\nu-1)!!}{n} \gamma_n^{2\nu}$$

and, by the Schwarz inequality,

$$\int_{-\infty}^{\infty} \left|\frac{k}{n} - t\right|^{\nu} p_{n,k}(t;\gamma)dt \le \frac{\sqrt{(2\nu-1)!!}}{n}\gamma_n^{\nu}. \tag{2.5}$$

In view of Lemma 2.1 in [5], if $\nu \in N$ and $n\gamma_n^2 \ge c$ for all $n \in N$, with a positive absolute constant c, then

$$\left|\sum_{k=-\infty}^{\infty} \left(\frac{k}{n} - x\right)^{\nu} p_{n,k}(x;\gamma)\right| \le 75A_c(2/e)^{\nu/2}\nu!\gamma_n^{\nu},$$

where $A_c = \max\{1, (2c\pi^2)^{-1}\}$. Further, from (2.1) and (2.3) it follows that

$$\sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \le 3A_c \quad \text{for all} \quad n \in N, x \in R, \tag{2.6}$$

and consequently,

$$\sum_{k=-\infty}^{\infty} \left|\frac{k}{n} - x\right|^{\nu} p_{n,k}(x;\gamma) \le 15A_c(2/e)^{\nu/2}\sqrt{(2\nu)!}\gamma_n^{\nu}. \tag{2.7}$$

Let $f \in X_\sigma(R)$ with some $\sigma > 0$. Then the operators (1.1) are well defined for all $x \in R$ and $n \in N$ such that $16\sigma\gamma_n^2 \le 1$. Indeed, using the obvious inequality $(a+b)^2 \le 2(a^2+b^2)$ and denoting by $\sqrt{2}\gamma$ the sequence $(\sqrt{2}\gamma_n)_{n=1}^{\infty}$ we easily observe that

$$\begin{aligned} p_{n,k}(t;\gamma)\exp(\sigma t^2) &\le \frac{1}{n\gamma_n\sqrt{2\pi}}\exp(2\sigma k^2/n^2)\exp\Big(-\Big(\frac{1}{2\gamma_n^2} - 2\sigma\Big)\Big(\frac{k}{n} - t\Big)^2\Big) \\ &\le \sqrt{2}\exp(2\sigma k^2/n^2)p_{n,k}(t;\sqrt{2}\gamma) \end{aligned}$$

and

$$p_{n,k}(t;\gamma)\exp(2\sigma k^2/n^2) \le \sqrt{2}\exp(4\sigma x^2)p_{n,k}(x;\sqrt{2}\gamma).$$

Hence, in view of (2.1) – (2.4)

$$\begin{aligned} |\widetilde{F}_n f(x)| &\le 2n\|f\|_\sigma \exp(4\sigma x^2) \sum_{k=-\infty}^{\infty} p_{n,k}(x;\sqrt{2}\gamma) \int_{-\infty}^{\infty} p_{n,k}(t;\sqrt{2}\gamma)dt \\ &\le 2(1 + (\pi n\gamma_n)^{-2})\|f\|_\sigma \exp(4\sigma x^2). \end{aligned}$$

Theorem 2.1. *Let $f \in X_\sigma(R), \sigma > 0$ and let $\gamma = (\gamma_n)_{n=1}^{\infty}$ be a positive sequence convergent to 0 and satisfying the condition $n\gamma_n^2 \ge c$, where c is a positive absolute constant. Then, given any numbers $q \in N, \varrho \ge 0$, we have*

$$|\widetilde{F}_n f(x) - f(x)| \le K(q,c)\sum_{r=0}^{\infty} \frac{w_x((r+1)\gamma_n; f)}{(r+1)^q \exp(\varrho r^2\gamma_n^2)} + \frac{|f(x)|}{(\pi n\gamma_n)^2}$$

for all $x \in R$ and $n \in N$ such that $16\sigma\gamma_n^2 \le 1$, $8\varrho\gamma_n^2 \le 1$.

Proof. In view of (2.1) and (2.4)

$$\widetilde{F}_n f(x) - f(x) = n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \int_{-\infty}^{\infty} (f(t) - f(x)) p_{n,k}(t;\gamma) dt + f(x) S_n(x).$$

Clearly, if $f \in X_\sigma(R)$ and $t \in R$, then

$$\left| \int_x^t (f(u) - f(x)) du \right| \leq |t-x| \, \|f\|_\sigma (2\exp(\sigma x^2) + \exp(\sigma t^2))$$

and, under the assumption $16\sigma\gamma_n^2 \leq 1$,

$$\exp\Big(\sigma t^2 - \frac{1}{2\gamma_n^2}\Big(\frac{k}{n} - t\Big)^2\Big) \leq \exp\Big(2\sigma\Big(\frac{k}{n}\Big)^2 - \frac{3}{8\gamma_n^2}\Big(\frac{k}{n} - t\Big)^2\Big).$$

Hence, from the definition of $p_{n,k}(x;\gamma)$ it follows at once that for any $k \in Z$,

$$\lim_{t \to \pm\infty} p_{n,k}(t;\gamma) \int_x^t (f(u) - f(x)) du = 0.$$

Consequently, integration by parts gives

$$\begin{aligned}
&\widetilde{F}_n f(x) - f(x) \\
&= -n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \int_{-\infty}^{\infty} \Big(\int_x^t (f(u) - f(x)) du \Big) p'_{n,k}(t;\gamma) dt + f(x) S_n(x).
\end{aligned}$$

Observing that $p'_{n,k}(t;\gamma) = p_{n,k}(t;\gamma)\left(\frac{k}{n} - t\right)/\gamma_n^2$, applying (2.3) and the definition of $w_x(\delta; f)$ we get

$$\begin{aligned}
&|\widetilde{F}_n f(x) - f(x)| \\
&\leq \frac{n}{\gamma_n^2} \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \int_{-\infty}^{\infty} \Big|\frac{k}{n} - t\Big| p_{n,k}(t;\gamma) |t-x| w_x(|t-x|) dt + \frac{|f(x)|}{(\pi n \gamma_n)^2} \\
&= \frac{n}{\gamma_n^2} \sum_{r=0}^{\infty} Z_r(\lambda) + |f(x)| (\pi n \gamma_n)^{-2},
\end{aligned} \tag{2.8}$$

where λ is an arbitrary positive number,

$$Z_r(\lambda) := \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \int_{I_{r,\lambda}(x)} \Big|\frac{k}{n} - t\Big| p_{n,k}(t;\gamma) |t-x| w_x(|t-x|) dt,$$

$I_{r,\lambda}(x) = \{t \in R\colon r\lambda \leq |t-x| < (r+1)\lambda\}$ and $w_x(\delta) \equiv w_x(\delta; f)$.

Using (2.6) and (2.5) with $\nu = 1$ we obtain $Z_0(\lambda) \le 3A_c\lambda w_x(\lambda)n^{-1}\gamma_n$. Given any $\varrho \ge 0, q \in N$ we have for $r \ge 1$

$$Z_r(\lambda) \le \frac{w_x((r+1)\lambda)}{r^q\lambda^q \exp(\varrho r^2\lambda^2)} \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma)$$
$$\cdot \int_{-\infty}^{\infty} \left|\frac{k}{n} - t\right| p_{n,k}(t;\gamma)|t-x|^{q+1}\exp(\varrho(t-x)^2)dt.$$

Clearly,

$$|t-x|^{q+1} \le 2^q\left(\left|\frac{k}{n}-x\right|^{q+1} + \left|\frac{k}{n}-t\right|^{q+1}\right)$$

and

$$\exp(\varrho(t-x)^2) \le \exp\left(2\varrho\left(\frac{k}{n}-x\right)^2\right)\exp\left(2\varrho\left(\frac{k}{n}-t\right)^2\right).$$

Moreover, if $8\varrho\gamma_n^2 \le 1$, then

$$\begin{aligned} p_{n,k}(x;\gamma)\exp\left(2\varrho\left(\frac{k}{n}-x\right)^2\right) &= \frac{1}{n\gamma_n\sqrt{2\pi}}\exp\left(-\left(\frac{1}{2\gamma_n^2}-2\varrho\right)\left(\frac{k}{n}-x\right)^2\right) \\ &\le \frac{1}{n\gamma_n\sqrt{2\pi}}\exp\left(-\frac{1}{4\gamma_n^2}\left(\frac{k}{n}-x\right)^2\right) \\ &= \sqrt{2}p_{n,k}(x;\sqrt{2}\gamma). \end{aligned}$$

From the above inequalities and the estimates (2.5), (2.6), (2.7) it follows that

$$\sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma)\int_{-\infty}^{\infty}\left|\frac{k}{n}-t\right|p_{n,k}(t;\gamma)|t-x|^{q+1}\exp(\varrho(t-x)^2)dt$$
$$\le \frac{2^{q+1}}{n}\sum_{k=-\infty}^{\infty} p_{n,k}(x;\sqrt{2}\gamma)\left(\sqrt{(2q+3)!!}(\sqrt{2}\gamma_n)^{q+2} + \left|\frac{k}{n}-x\right|^{q+1}\sqrt{2}\gamma_n\right)$$
$$\le \frac{2^{q+1}}{n}\gamma_n^{q+2}(3A_c\sqrt{(2q+3)!!}(\sqrt{2})^{q+2} + 15A_c\sqrt{2}\sqrt{(2q+2)!}(2/\mathrm{e})^{(q+1)/2}).$$

Hence

$$Z_r(\lambda) \le K_1(q,c)\frac{w_x((r+1)\lambda)}{r^q\lambda^q\exp(\varrho r^2\lambda^2)}\gamma_n^{q+2}n^{-1} \quad \text{for } r > 0.$$

Choosing $\lambda = \gamma_n$ and using (2.8) we get the desired result immediately. ■

It is easy to see that under the assumptions $f \in X_\sigma(R)$ and $\delta > 0$ we have

$$w_x(\delta; f) \le (\exp(\sigma x^2) + \exp(2\sigma x^2))\exp(2\sigma\delta^2)\|f\|_\sigma$$

(see e.g. [6]). Consequently,

$$w_x((r+1)\gamma_n; f) \le \exp(2\sigma x^2)(1+\exp(4\sigma r^2\gamma_n^2))\exp(4\sigma\gamma_n^2)\|f\|_\sigma.$$

From this inequality it follows at once that the right-hand side of the estimate given in Theorem 2.1 (with $q \geq 2, \varrho \geq 4\sigma$) converges to 0 as $n \to \infty$, at every Lebesgue point x of f.

3. Corollaries

Let $f \in X_\sigma(R)$ be continuous on R and let $\Omega(\delta; f)_\sigma$ be its weighted modulus of continuity defined by

$$\Omega(\delta; f)_\sigma := \sup_{|h| \leq \delta} \|f(\cdot + h) - f(\cdot)\|_\sigma \quad (\delta > 0).$$

Then, given any $x \in R$ and $r \in N$ we have

$$w_x(r\delta; f) \leq r \exp(2\sigma x^2 + 2\sigma(r-1)^2\delta^2)\Omega(\delta; f)_\sigma$$

(see [6], p. 149). This inequality and Theorem 2.1 with $q = 3, \varrho = 2\sigma$ lead to

Corollary 3.1. *If the sequence $\gamma = (\gamma_n)_{n=1}^\infty$ satisfies the conditions of Theorem 2.1 and if $f \in X_\sigma(R)$ is continuous on R, then*

$$\|\widetilde{F}_n f - f\|_{2\sigma} \leq K(\mathrm{c})\Big(\Omega(\gamma_n; f)_\sigma + \frac{1}{n}\|f\|_{2\sigma}\Big)$$

for all $n \in N$ such that $16\sigma\gamma_n^2 \leq 1$.

For some $m \in N_0$ let $C_m(R)$ be the space of all continuous functions f on R such that

$$\|f\|_m^\circ := \sup_{x \in R} |f(x)(1 + x^{2m})^{-1}| < \infty.$$

Clearly, $C_m(R) \subset X_\sigma(R)$ for arbitrary $m \in N_0$, $\sigma > 0$. Moreover, for any $x \in R$ and $r \in N_0$ there holds the inequality

$$w_x((r+1)\delta; f) \leq (1 + (2x)^{2m} + (2r\delta)^{2m})(r+1)\omega(\delta; f)_m,$$

(see [6]), where

$$\omega(\delta; f)_m := \sup_{|h| \leq \delta} \|f(\cdot + h) - f(\cdot)\|_m^\circ.$$

Consequently, from Theorem 2.1 (with $\varrho = 0$, $q = 2m + 3$) it follows

Corollary 3.2. *If the sequence $\gamma = (\gamma_n)_{n=1}^\infty$ satisfies the conditions of Theorem 2.1 and if $f \in C_m(R)$, then*

$$\|\widetilde{F}_n f - f\|_m^\circ \leq K(\mathrm{c}, m)\Big(\omega(\gamma_n; f)_m + \frac{1}{n}\|f\|_m^\circ\Big)$$

for all $n \in N$.

Finally, let us suppose that $f \in X_\sigma(R)$ with some $\sigma > 0$ and that at a fixed point $x \in R$ the one-sided limits $f(x+), f(x-)$ exist. Introduce the functions

$$g_x(t) := \begin{cases} f(t) - f(x+) & \text{if } t > x, \\ 0 & \text{if } t = x, \\ f(t) - f(x-) & \text{if } t < x, \end{cases} \qquad \operatorname{sgn}_x(t) := \begin{cases} 1 & \text{if } t > x, \\ 0 & \text{if } t = x, \\ -1 & \text{if } t < x. \end{cases}$$

Then, it is easy to verify that

$$\begin{aligned} f(t) &= \frac{1}{2}(f(x+) + f(x-)) + g_x(t) + \frac{1}{2}(f(x+) - f(x-)) \operatorname{sgn}_x(t) \\ &\quad + \Big(f(x) - \frac{1}{2}f(x+) - \frac{1}{2}f(x-)\Big)\delta_x(t), \end{aligned}$$

where $\delta_x(x) = 1$ and $\delta_x(t) = 0$ if $t \neq x$. Hence

$$\begin{aligned} \widetilde{F}_n f(x) &= \frac{1}{2}(f(x+) + f(x-))(1 + S_n(x)) + \widetilde{F}_n g_x(x) \\ &\quad + \frac{1}{2}(f(x+) - f(x-))\widetilde{F}_n \operatorname{sgn}_x(x), \end{aligned}$$

where $S_n(x)$ is defined by (2.2) and estimated in (2.3). As is shown in [1] (p. 104),

$$|\widetilde{F}_n \operatorname{sgn}_x(x)| = n\Big|\sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma)\Big(\int_x^{\infty} - \int_{-\infty}^{x}\Big)p_{n,k}(t;\gamma)dt\Big| \leq \frac{4}{n\gamma_n}.$$

In order to estimate $|\widetilde{F}_n g_x(x)| = |\widetilde{F}_n g_x(x) - g_x(x)|$ we use Theorem 2.1 (with $\varrho = 0$ and $q \geq 2$). Consequently, by a simple calculation (cf. e.g. [6], p. 150) we obtain that under the assumptions of Theorem 2.1,

$$\begin{aligned} &\Big|\widetilde{F}_n f(x) - \frac{1}{2}(f(x+) + f(x-))\Big| \\ &\leq K_2(q,c)\gamma_n^{q-1}\int_0^{1/\gamma_n} t^{q-2} w_x(1/t; g_x)dt \\ &+ \frac{1}{2}|f(x+) + f(x-)|(\pi n\gamma_n)^{-2} + 2|f(x+) - f(x-)|(n\gamma_n)^{-1} \end{aligned}$$

for all $n \in N$ such that $16\sigma\gamma_n^2 \leq 1$.

In particular, let us consider the class $BV_p(R)$ of all functions f of bounded p-th power variation on R and let us denote by $V_p(f; I)$ or $V_p(f; a, b)$ the total p-th variation of f on the interval $I = [a, b]$ (defined as in [6]). The obvious inequality

$$w_x(\delta; f) \leq V_p(f; x - \delta, x + \delta) \quad (\delta > 0)$$

and some easy computations lead to

Corollary 3.3. *If the sequence $\gamma = (\gamma_n)_{n=1}^{\infty}$ satisfies the conditions of Theorem 2.1 and if $f \in BV_p(R)$, $p \geq 1$, then for all $x \in R$ and $n \in N$,*

$$\begin{aligned}&\left|\widetilde{F}_n f(x) - \frac{1}{2}(f(x+) + f(x-))\right| \\ &\leq K_3(q,c)\gamma_n^{q-1}\sum_{k=0}^{\mu} k^{q-2} V_p(g_x; I_k) \\ &+ \frac{1}{2}|f(x+) + f(x-)|(nc)^{-1} + 2|f(x+) - f(x-)|(n\gamma_n)^{-1},\end{aligned}$$

where $q \geq 2, \mu = [1/\gamma_n], I_0 = R, I_k = [x - 1/k, x + 1/k]$ if $k = 1, \ldots, \mu$.

Clearly, in view of the continuity of g_x at x, the right-hand side of the inequality given in Corollary 3.3 converges to 0 as $n \to \infty$. Moreover, in some classes of functions this inequality cannot be essentially improved. To see this, let us first mention some properties of the functions

$$H_{n,r}(x) := n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \int_{-\infty}^{\infty} (x-t)^r p_{n,k}(t;\gamma)\,dt,$$

where $r \in N_0, n \in N, x \in R$. It is easy to verify that with $S_n(x)$ as defined by (2.2) we have the recursion formula

$$\begin{gathered}H_{n,0}(x) = 1 + S_n(x), \quad H_{n,1} = \gamma_n^2 S_n'(x), \\ H_{n,r+1}(x) = \gamma_n^2 H_{n,r}'(x) - 2r\gamma_n^2 H_{n,r-1}(x).\end{gathered}$$

From this formula, by the method of induction, it follows the representation

$$\begin{aligned}&H_{n,2r}(x) \\ &= \gamma_n^{2r}\left(d_{0,r}(1 + S_n(x)) + \sum_{l=1}^{r-1} d_{l,r}\gamma_n^{2l}(S_n^{(2l-1)}(x) + S_n^{(2l)}(x)) + \gamma_n^{2r} S_n^{(2r-1)}(x)\right),\end{aligned}$$

where $d_{l,r}$ $(l = 0, \ldots, r-1)$ are real numbers independent of n and x. Moreover, under the assumption $n\gamma_n^2 \geq c$ for all $n \in N$, the functions $S_n^{(\nu)}(x)$ $(\nu = 0, 1, \ldots, 2r-1)$ are bounded uniformly in $x \in R$ and $n \in N$. Consequently,

$$H_{n,2r}(x) \leq K_4(c,r)\gamma_n^{2r} \quad \text{for all } x \in R,\ n \in N \tag{3.1}$$

and

$$\lim_{n\to\infty} \gamma_n^{-2r} H_{n,2r}(x) = d_{0,r} \tag{3.2}$$

uniformly in $x \in R$.

Now, let us fix a point x_0 and a positive number α and let us denote by $U(\alpha, x_0)$ the class of all functions $f \in BV_p(R)$, continuous at x_0 and such that

$V_p(f; x_0-\delta, x_0+\delta) \le \delta^\alpha$ for $\delta \in (0,1]$. From Corollary 3.3 with $q > \max(2, \alpha+1)$ it follows that for $f \in U(\alpha, x_0)$ and $n \in N$,

$$|\widetilde{F}_n f(x_0) - f(x_0)| \le K_5(\alpha, c)(1 + V_p(f; R))\gamma_n^\alpha + \frac{|f(x_0)|}{nc}. \tag{3.3}$$

On the other hand, the function $f_\alpha(t) := \frac{1}{2}|t-x_0|^\alpha$ if $|t-x_0| \le 1, f_\alpha(t) = 1/2$ otherwise on R, belongs to $U(\alpha, x_0)$ and

$$\widetilde{F}_n f_\alpha(x_0) - f_\alpha(x_0) \ge \frac{1}{2} n \sum_{k=-\infty}^{\infty} p_{n,k}(x_0;\gamma) \int_{x_0-\delta}^{x_0+\delta} |t-x_0|^\alpha p_{n,k}(t;\gamma)dt$$

for any $\delta \in (0,1]$. Let τ be a positive number such that $\tau + \alpha = 2r$, where $r \in N$. Then

$$\begin{aligned} &\widetilde{F}_n f_\alpha(x_0) - f_\alpha(x_0) \\ &\ge \frac{1}{2} n\delta^{-\tau} \sum_{k=-\infty}^{\infty} p_{n,k}(x_0;\gamma) \int_{x_0-\delta}^{x_0+\delta} (t-x_0)^{2r} p_{n,k}(t;\gamma)dt \\ &= \frac{1}{2}\delta^{\alpha-2r}\Big(H_{n,2r}(x_0) - n \sum_{k=-\infty}^{\infty} p_{n,k}(x_0;\gamma) \int_{|t-x_0|>\delta} (t-x_0)^{2r} p_{n,k}(t;\gamma)dt\Big) \\ &\ge \frac{1}{2}\delta^{\alpha-2r}(H_{n,2r}(x_0) - \delta^{-2} H_{n,2r+2}(x_0)). \end{aligned}$$

From (3.2) it follows that

$$H_{n,2r}(x_0) \ge \frac{1}{2} d_{0,r}\gamma_n^{2r}$$

for sufficiently large n. Applying inequality (3.1) for $H_{n,2r+2}(x_0)$ and putting $\delta = 2\gamma_n^2 \sqrt{K_4(c, r+1)} \Big/ \sqrt{d_{0,r}}$ we obtain that

$$\widetilde{F}_n f_\alpha(x_0) - f_\alpha(x_0) \ge 2^{\alpha-2r-3}(K_4(c, r+1))^{-r+\alpha/2}(d_{0,r})^{1+r-\alpha/2}\gamma_n^\alpha \tag{3.4}$$

for sufficiently large n. Inequalities (3.3) and (3.4) ensure that in the classes $U(\alpha, x_0)$ with $0 < \alpha \le 2$, or $\overline{U}(\alpha, x_0) := \{f \in U(\alpha, x_0)\colon f(x_0) = 0\}$ with arbitrary $\alpha > 0$, the estimate given in Corollary 3.3 is the best concerning the order.

References

[1] G. Anioł and P. Pych-Taberska, *On the rate of pointwise convergence of the Durrmeyer type operators*, Approximation Theory and its Applications, **11** No 2 (1995), 94–105.

[2] P.L. Butzer, R.J. Nessel, *Fourier Analysis and Approximation*, Vol. I, Academic Press, New York and London, 1971.

[3] J. Favard, *Sur les multiplicateurs d'interpolation*, J. Math. Pures Appl. **23** (1944), 219–247.

[4] W. Gawronski, U. Stadtmüller, *Approximation of continuous functions by generalized Favard operators*, J. Approx. Theory **34** (1982), 384–396.

[5] P. Pych-Taberska, *On the generalized Favard operators*, Funct. Approximatio, Comment. Math. **26** (1998), 256–273.

[6] P. Pych-Taberska and G. Nowak, *Approximation properties of the generalized Favard-Kantorovich operators*, Comment. Math. **39** (1999), 139–152.

Address: Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Matejki 48/49, 60-769 Poznań, Poland

E-mail: grzegnow@amu.edu.pl; ppych@amu.edu.pl

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