Functiones et Approximatio
41.2 (2009), 113–128

ON A HYBRID MEAN VALUE OF THE CHARACTER SUMS OVER SHORT INTERVALS

Zhefeng Xu, Wenpeng Zhang

Abstract: The main purpose of this paper is to study the hybrid mean value of the character sums over short interval and $B(\chi)$, by using the mean value theorems of the Dirichlet L-functions, and give an interesting asymptotic formula.

Keywords: Character sums, short intervals, hybrid mean value, asymptotic formula.

1. Introduction

Let $q \geqslant 3$ be an integer, χ be a Dirichlet character modulo q. The character sums

$$\sum_{a=N+1}^{N+H} \chi(a)$$

play an important role in analytic number theory. Pólya [1] and Vinogradov [2] given the first fundamental estimate

$$\left| \sum_{a=1}^{x} \chi(a) \right| \leqslant c\sqrt{p}\ln p,$$

where c is a constant. Actually, one can establish the above inequality with the constant $c = 1$. For a primitive character χ modulo q, A.V.Sokolovskiĭ [3] proved the existence of x with

$$\left| \sum_{n=x}^{x+\left[\frac{q}{2}\right]} \chi(n) \right| > \sqrt{1 - \frac{8\ln q}{q}} \cdot \frac{1}{2\sqrt{2}} \cdot \sqrt{q},$$

This work is supported by China Postdoctoral Science Foundation funded project (20080430202, 200902598) and P.N.S.F of Shaanxi (SJ08A04).

2000 Mathematics Subject Classification: 11L40 where $[y]$ denotes the greatest integer which less than or equal to y. For any positive integer h, D. A. Burgess [4] summed the mean value over all primitive characters and obtained

$$\sum_{\chi \bmod q}^{*} \sum_{n=1}^{q} \left| \sum_{m=1}^{h} \chi(n+m) \right|^4 \leqslant 8\tau^7(q) q^2 h^2,$$

where $\sum\limits_{\chi \bmod q}^{*}$ denotes the summation over all primitive characters modulo q and $\tau(n)$ is the Dirichlet divisor function.

In order to obtain an asymptotic formula for the higher moments of character sums, the authors [5] studied the character sum over the interval $\left[1, \frac{q}{4}\right)$:

$$S\left(\frac{q}{4}, \chi\right) = \sum_{a=1}^{\left[\frac{q}{4}\right]} \chi(a),$$

and proved

$$\sum_{\chi(-1)=1}^{*} \left| S\left(\frac{q}{4}, \chi\right) \right|^{2k} = \frac{J(q) q^k}{16} \left(\frac{\pi}{8}\right)^{2k-2} \prod_{p \mid q} \left(1 - \frac{1}{p^2}\right)^{2k-1} \prod_{p \nmid 2q} \left(1 - \frac{1 - C_{2k-2}^{k-1}}{p^2}\right) + O\left(q^{k+\epsilon}\right),$$

where $\sum\limits_{\chi(-1)=1}^{*}$ denotes the summation over all primitive characters modulo q such that $\chi(-1) = 1$, ϵ is any fixed positive number, $J(q)$ the number of all primitive characters modulo q and $\prod\limits_{p \mid q}$ the product over all prime divisors p of q, and $C_m^n = \frac{m!}{n!(m-n)!}$.

In the present work, we will study the hybrid mean square value of $S\left(\frac{q}{4}, \chi\right)$ and $B(\chi)$ which emerges in the zero expansion of logarithmic differentiation of Dirichlet L-function $L(s, \chi)$:

$$\frac{L'(s, \chi)}{L(s, \chi)} = -\frac{1}{2} \ln \frac{q}{\pi} - \frac{1}{2} \frac{\Gamma'\left(\frac{1}{2}s + \frac{1}{2}a\right)}{\Gamma\left(\frac{1}{2}s + \frac{1}{2}a\right)} + B(\chi) + \sum_{\rho} \left(\frac{1}{1-\rho} + \frac{1}{\rho}\right),$$

where $\sum\limits_{\rho}$ denotes the summation over all nontrivial zero ρ of $L(s, \chi)$ and

$$a = \begin{cases} 0, & \text{if } \chi(-1) = 1; \\ 1, & \text{if } \chi(-1) = -1. \end{cases}$$

About the arithmetic properties of $B(\chi)$, none had studied it before. It seems to be very difficult to estimate $B(\chi)$ at all. As claimed in [6], the difficulty of estimating $B(\chi)$ is connected with the fact that $L(s, \chi)$ may have a zero near to $s = 0$.

Throughout this paper,

- $J(q)$ denotes the number of all primitive characters modulo q,
- $\Gamma(s)$ is the Gamma function, $\Lambda(n)$ is the Mangoldt function,
- χ_4 denotes the primitive character modulo 4,
- $r(n) = \sum_{d|n} \chi_4(d)\Lambda\left(\frac{n}{d}\right)$,
- $\Re s$ denotes the real part of complex number s,
- ϵ is any fixed positive integer.

Now we give the main result of this paper.

Theorem. *Let $q \geqslant 5$ be an odd integer. Then we have the asymptotic formula*

$$\sideset{}{^*}\sum_{\chi \bmod q} \left|B(\chi)S\left(\frac{q}{4},\chi\right)\right|^2 = CqJ(q) + O(q^{1+\epsilon}),$$

where

$$C = \frac{c_1}{2\pi^2} + \frac{C_1c_2}{\pi^2} + \frac{3c_3}{4\pi^2} - \left(\frac{\ln 2}{8\pi^2} - \frac{3C_2}{2\pi^2}\right)c_4 + \left(\frac{C_1^2}{16} - \frac{C_2\ln 2}{48} + \frac{C_2^2}{8}\right)\prod_{p|q}\left(1-\frac{1}{p^2}\right)$$

and

$$c_1 = \sum_{\substack{n=1\\(n,q)=1}}^{\infty} \frac{r^2(n)}{n^2}, \qquad c_2 = \sum_{\substack{n=1\\(n,q)=1}}^{\infty} \frac{\chi_4(n)r(n)}{n^2},$$

$$c_3 = \sum_{\substack{n=1\\(n,q)=1}}^{\infty} \frac{\ln^2 n}{n^2}, \qquad c_4 = \sum_{\substack{n=1\\(n,q)=1}}^{\infty} \frac{\ln n}{n^2},$$

$$C_1 = \frac{1}{2}\ln\frac{q}{\pi} + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right), \qquad C_2 = \frac{1}{2}\ln\frac{q}{\pi} + \frac{1}{2}\frac{\Gamma'}{\Gamma}(1).$$

In fact, our method also works for the higher moment as

$$\sideset{}{^*}\sum_{\chi \bmod q} \left|B(\chi)S\left(\frac{q}{4},\chi\right)\right|^{2k}$$

and

$$\sideset{}{^*}\sum_{\chi \bmod q} \left|B(\chi)S\left(\frac{q}{8},\chi\right)\right|^{2k}.$$

2. Some Lemmas

To prove the theorem, we need the following lemmas.

Lemma 1. *Let χ be a primitive character modulo q. Then we have*

$$|B(\chi)| = \left| \frac{L'(1,\chi)}{L(1,\chi)} + C(\chi) \right|$$

where

$$C(\chi) = \begin{cases} C_1 = \frac{1}{2}\ln\frac{q}{\pi} + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right), & \text{if } \chi(-1) = 1; \\ C_2 = \frac{1}{2}\ln\frac{q}{\pi} + \frac{1}{2}\frac{\Gamma'}{\Gamma}(1), & \text{if } \chi(-1) = -1. \end{cases}$$

Proof. From the definition of $B(\chi)$, we have

$$\frac{L'(1,\chi)}{L(1,\chi)} = -C(\chi) + B(\chi) + \sum_\rho \left(\frac{1}{1-\rho} + \frac{1}{\rho} \right),$$

where $\sum_\rho$ denotes the summation over all zeros ρ of $L(s,\chi)$ and

$$C(\chi) = \begin{cases} C_1 = \frac{1}{2}\ln\frac{q}{\pi} + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right), & \text{if } \chi(-1) = 1; \\ C_2 = \frac{1}{2}\ln\frac{q}{\pi} + \frac{1}{2}\frac{\Gamma'}{\Gamma}(1), & \text{if } \chi(-1) = -1. \end{cases}$$

From the functional equation of Dirichlet L-function, we know that if ρ is a zero of $L(s,\chi)$ then $1-\overline{\rho}$ is also a zero of $L(s,\chi)$. This implies

$$\sum_\rho \left(\Re\frac{1}{1-\bar\rho} + \Re\frac{1}{\bar\rho} \right) = \sum_\rho \left(\frac{1}{1-\rho} + \frac{1}{\rho} \right).$$

Noting that

$$2\Re B(\chi) = -\sum_\rho \left(\Re\frac{1}{1-\bar\rho} + \Re\frac{1}{\bar\rho} \right),$$

see reference [6], we can write

$$\begin{aligned} |B(\chi)| = |B(\chi) - 2\Re B(\chi)| &= \left| B(\chi) + \sum_\rho \left(\Re\frac{1}{1-\bar\rho} + \Re\frac{1}{\bar\rho} \right) \right| \\ &= \left| B(\chi) + \sum_\rho \left(\frac{1}{1-\rho} + \frac{1}{\rho} \right) \right| \\ &= \left| \frac{L'(1,\chi)}{L(1,\chi)} + C(\chi) \right|. \end{aligned}$$

This proves Lemma 1. ■

Lemma 2. *Let χ be a primitive character modulo m with $\chi(-1) = -1$. Then we have*

$$\frac{1}{m}\sum_{b=1}^{m} b\chi(b) = \frac{i}{\pi}\tau(\chi)L(1,\bar{\chi}),$$

where $\tau(\chi) = \sum_{a=1}^{m}\chi(a)e\left(\frac{a}{q}\right)$ *is the Gauss sum,* $e(y) = e^{2\pi i y}$*, and* $L(s,\chi)$ *denotes the Dirichlet L-function corresponding to* χ*.*

Proof. This can be easily deduced from Theorem 12.11 and Theorem 12.20 of [7]. ■

Lemma 3. *Let $q \geqslant 3$ be an odd number. For any nonprincipal character χ* mod *q, we have*

$$\sum_{a=1}^{q} a\chi(a) = \frac{\chi(2)q}{1-2\chi(2)}\sum_{a=1}^{\frac{q-1}{2}}\chi(a).$$

Proof. From the properties of Dirichlet character, we have

$$\begin{aligned}\sum_{a=1}^{q} 2a\chi(2a) &= \sum_{a=1}^{\frac{q-1}{2}} 2a\chi(2a) + \sum_{a=\frac{q+1}{2}}^{q} 2a\chi(2a)\\ &= \sum_{a=1}^{\frac{q-1}{2}} 2a\chi(2a) + \sum_{a=1}^{\frac{q+1}{2}}(2a-1)\chi(q+2a-1) + q\sum_{a=1}^{\frac{q+1}{2}}\chi(2a-1)\\ &= \sum_{a=1}^{q} a\chi(a) + q\sum_{a=1}^{\frac{q+1}{2}}\chi(2a-1).\end{aligned}$$

Noting that

$$\sum_{a=1}^{\frac{q+1}{2}}\chi(2a-1) + \sum_{a=1}^{\frac{q-1}{2}}\chi(2a) = \sum_{a=1}^{q}\chi(a) = 0,$$

we can write

$$(1-2\chi(2))\sum_{a=1}^{q} a\chi(a) = \sum_{a=1}^{q} a\chi(a) - \sum_{a=1}^{q} 2a\chi(2a) = q\sum_{a=1}^{\frac{q-1}{2}}\chi(2a) = \chi(2)q\sum_{a=1}^{\frac{q-1}{2}}\chi(a).$$

That is,

$$\sum_{a=1}^{q} a\chi(a) = \frac{\chi(2)q}{1-2\chi(2)}\sum_{a=1}^{\frac{q-1}{2}}\chi(a).$$

This proves Lemma 3. ■

Lemma 4. *Let q be an odd number and χ be a primitive Dirichlet character modulo q such that $\chi(-1) = -1$. Then we have*

$$S\left(\frac{q}{4}, \chi\right) = \frac{2 + \bar{\chi}(2) - \bar{\chi}(4)}{2i\pi}\tau(\chi)L(1, \bar{\chi}).$$

Proof. We separate q into two cases: $q \equiv 1(\bmod 4)$ and $q \equiv 3(\bmod 4)$. First, we suppose $q \equiv 1(\bmod 4)$. From the properties of the Dirichlet character modulo q, we can write

$$\begin{aligned}
4\chi(4)\sum_{a=1}^{q-1} a\chi(a) &= \sum_{a=1}^{\frac{q-1}{4}} 4a\chi(4a) + \sum_{a=\frac{q+3}{4}}^{\frac{2q-2}{4}} 4a\chi(4a) + \sum_{a=\frac{2q+2}{4}}^{\frac{3q-3}{4}} 4a\chi(4a) + \sum_{a=\frac{3q+1}{4}}^{q-1} 4a\chi(4a) \\
&= \sum_{a=1}^{\frac{q-1}{4}} 4a\chi(4a) + \sum_{a=1}^{\frac{q-1}{4}} (4a+q-1)\chi(4a-1) \\
&\quad + \sum_{a=1}^{\frac{q-1}{4}} (4a+2q-2)\chi(4a-2) + \sum_{a=1}^{\frac{q-1}{4}} (4a+3q-3)\chi(4a-3) \\
&= \sum_{a=1}^{q-1} a\chi(a) + \chi(4)q\sum_{a=1}^{\frac{q-1}{4}} \chi(a-\bar{4}) \\
&\quad + 2\chi(4)q\sum_{a=1}^{\frac{q-1}{4}} \chi(a - 2\cdot\bar{4}) + 3\chi(4)q\sum_{a=1}^{\frac{q-1}{4}} \chi(a - 3\cdot\bar{4}). \qquad (1)
\end{aligned}$$

Note that $\bar{4} \equiv \frac{3q+1}{4}(\bmod q)$ if $q \equiv 1(\bmod 4)$. So from (1), we have

$$\begin{aligned}
4\chi(4)\sum_{a=1}^{q-1} a\chi(a) &= \sum_{a=1}^{q-1} a\chi(a) - \chi(4)q\sum_{a=\frac{2q+2}{4}}^{\frac{3q-3}{4}} \chi(a) \\
&\quad - 2\chi(4)q\sum_{a=\frac{q+3}{4}}^{\frac{2q-2}{4}} \chi(a) - 3\chi(4)q\sum_{a=1}^{\frac{q-1}{4}} \chi(a) \\
&= \sum_{a=1}^{q-1} a\chi(a) - \chi(4)q\sum_{a=\frac{q+3}{4}}^{\frac{2q-2}{4}} \chi(a) - 3\chi(4)q\sum_{a=1}^{\frac{q-1}{4}} \chi(a) \\
&= \sum_{a=1}^{q-1} a\chi(a) - \chi(4)q\sum_{a=1}^{\frac{q-1}{2}} \chi(a) - 2\chi(4)q\sum_{a=1}^{\frac{q-1}{4}} \chi(a), \qquad (2)
\end{aligned}$$

where we used the fact $\chi(-1) = -1$ and

$$\sum_{a=\frac{q+3}{4}}^{\frac{2q-2}{4}} \chi(a) = -\sum_{a=\frac{2q+2}{4}}^{\frac{3q-3}{4}} \chi(a).$$

Now, from (2) and Lemma 3, we can get

$$4\chi(4)\sum_{a=1}^{q-1} a\chi(a) = \sum_{a=1}^{q-1} a\chi(a) - (\chi(2) - 2\chi(4))\sum_{a=1}^{q-1} a\chi(a) - 2\chi(4)q\sum_{a=1}^{\frac{q-1}{4}} \chi(a).$$

That is,

$$\sum_{a=1}^{\frac{q-1}{4}} \chi(a) = \frac{\bar{\chi}(4) - \bar{\chi}(2) - 2}{2q}\sum_{a=1}^{q-1} a\chi(a) = \frac{\bar{\chi}(4) - \bar{\chi}(2) - 2}{2q}\sum_{a=1}^{q} a\chi(a).$$

Then from Lemma 2, we have

$$\sum_{a=1}^{\frac{q-1}{4}} \chi(a) = \frac{2 + \bar{\chi}(2) - \bar{\chi}(4)}{2i\pi}\tau(\chi)L(1, \bar{\chi}). \tag{3}$$

This proves Lemma 4 in the case of $q \equiv 1(\mathrm{mod} 4)$. By the same method, we can also prove

$$\sum_{a=1}^{\frac{q-3}{4}} \chi(a) = \frac{2 + \bar{\chi}(2) - \bar{\chi}(4)}{2i\pi}\tau(\chi)L(1, \bar{\chi}), \tag{4}$$

if $q \equiv 3(\mathrm{mod} 4)$. Combining (3) and (4), we can immediately get

$$\sum_{a=1}^{[\frac{q}{4}]} \chi(a) = \frac{2 + \bar{\chi}(2) - \bar{\chi}(4)}{2i\pi}\tau(\chi)L(1, \bar{\chi}).$$

This completes the proof of Lemma 4. ∎

Lemma 5. *Let $q \geqslant 5$ be an odd integer and χ be a primitive Dirichlet character modulo q such that $\chi(-1) = 1$. Then we have*

$$S\left(\frac{q}{4}, \chi\right) = -\frac{i\bar{\chi}(4)}{2\pi}\tau(\chi\chi_4)L(1, \bar{\chi}\chi_4),$$

where χ_4 is the primitive Dirichlet character modulo 4.

Proof. See Lemma 2 of reference [5]. ∎

Lemma 6. *Let q and r be integers with $q \geqslant 2$ and $(r, q) = 1$, χ be a Dirichlet character modulo q. Then we have the identities*

$$\sum_{\chi \bmod q}^{*} \chi(r) = \sum_{d|(q,r-1)} \mu\left(\frac{q}{d}\right)\phi(d)$$

and

$$J(q) = \sum_{d|q} \mu(d)\phi\left(\frac{q}{d}\right),$$

where $\sum\limits_{\chi \bmod q}^{*}$ *denotes the summation over all primitive characters modulo* q*, and* $J(q)$ *denotes the number of all primitive characters modulo* q*.*

Proof. See Lemma 4 of reference [8]. ∎

Lemma 7. *Let* q *be any odd integer with* $q > 2$*,* χ *be the Dirichlet character modulo* q*. Then we have the following asymptotic formulae:*

$$\sum_{\chi(-1)=1}^{*} \left| \frac{L'(1,\chi)}{L(1,\chi)} \right|^2 |L(1,\bar{\chi}\chi_4)|^2 = \frac{1}{2} J(q) \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{r^2(n)}{n^2} + O(q^{\epsilon})$$

and

$$\sum_{\chi(-1)=1}^{*} \frac{L'(1,\chi)}{L(1,\chi)} |L(1,\bar{\chi}\chi_4)|^2 = \frac{1}{2} J(q) \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{\chi_4(n) r(n)}{n^2} + O(q^{\epsilon}).$$

Proof. We only prove the first formula. By the same argument we can get the second one. From

$$\frac{L'(1,\chi)}{L(1,\chi)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)}{n}$$

and

$$L(1,\chi\chi_4) = \sum_{n=1}^{\infty} \frac{\chi\chi_4(n)}{n},$$

we can write

$$\frac{L'(1,\chi)}{L(1,\chi)} L(1,\chi\chi_4) = \sum_{n=1}^{\infty} \frac{\chi(n) r(n)}{n},$$

where $r(n) = \sum\limits_{d|n} \Lambda(d)\chi\left(\frac{n}{d}\right)$. Then from Abel's identity we have

$$\frac{L'(1,\chi)}{L(1,\chi)} L(1,\chi\chi_4) = \sum_{1 \leqslant n \leqslant N} \frac{\chi(n) r(n)}{n} + \int_N^{\infty} \frac{A(y,\chi)}{y^2} dy,$$

where

$$A(y,\chi) = \sum_{N < n \leqslant y} \chi(n) r(n)$$

and N is a parameter with $q \leqslant N < q^4$. Similarly,

$$\frac{L'(1,\bar{\chi})}{L(1,\bar{\chi})} L(1,\bar{\chi}\chi_4) = \sum_{1 \leqslant n \leqslant N} \frac{\bar{\chi}(n) r(n)}{n} + \int_N^{\infty} \frac{A(y,\bar{\chi})}{y^2} dy.$$

So we can write

$$\begin{aligned}
&\sum_{\chi(-1)=1}^{*}\left|\frac{L'(1,\chi)}{L(1,\chi)}\right|^2|L(1,\bar{\chi}\chi_4)|^2\\
&\quad=\sum_{\chi(-1)=1}^{*}\left(\sum_{1\leqslant n_1\leqslant N}\frac{\chi(n_1)r(n_1)}{n_1}\right)\left(\sum_{1\leqslant n_2\leqslant N}\frac{\bar{\chi}(n_2)r(n_2)}{n_2}\right)\\
&\qquad+\sum_{\chi(-1)=1}^{*}\left(\sum_{1\leqslant n_1\leqslant N}\frac{\chi(n_1)r(n_1)}{n_1}\right)\left(\int_N^{\infty}\frac{A(y,\bar{\chi})}{y^2}dy\right) \qquad (5)\\
&\qquad+\sum_{\chi(-1)=1}^{*}\left(\sum_{1\leqslant n_2\leqslant N}\frac{\bar{\chi}(n_2)r(n_2)}{n_2}\right)\left(\int_N^{\infty}\frac{A(y,\chi)}{y^2}dy\right)\\
&\qquad+\sum_{\chi(-1)=1}^{*}\left(\int_N^{\infty}\frac{A(y,\bar{\chi})}{y^2}dy\right)\left(\int_N^{\infty}\frac{A(y,\chi)}{y^2}dy\right)=:M_1+M_2+M_3+M_4.
\end{aligned}$$

Now we calculate each term in the expression (5).

(i) From Lemma 6 we have

$$\begin{aligned}
\sum_{\chi(-1)=1}^{*}\chi(a)&=\frac{1}{2}\sum_{\chi \bmod q}^{*}(1+\chi(-1))\chi(a)=\frac{1}{2}\sum_{\chi \bmod q}^{*}\chi(a)+\frac{1}{2}\sum_{\chi \bmod q}^{*}\chi(-a)\\
&=\frac{1}{2}\sum_{d|(q,a-1)}\mu\left(\frac{q}{d}\right)\phi(d)+\frac{1}{2}\sum_{d|(q,a+1)}\mu\left(\frac{q}{d}\right)\phi(d).
\end{aligned}$$

Hence we can write

$$\begin{aligned}
M_1&=\sum_{\chi(-1)=1}^{*}\left(\sum_{1\leqslant n_1\leqslant N}\frac{\chi(n_1)r(n_1)}{n_1}\right)\left(\sum_{1\leqslant n_2\leqslant N}\frac{\bar{\chi}(n_2)r(n_2)}{n_2}\right)\\
&=\frac{1}{2}\sum_{1\leqslant n_1\leqslant N}{}'\sum_{1\leqslant n_2\leqslant N}{}'\frac{r(n_1)r(n_2)}{n_1n_2}\sum_{d|(q,\bar{n}_2n_1-1)}\mu\left(\frac{q}{d}\right)\phi(d)\\
&\quad+\frac{1}{2}\sum_{1\leqslant n_1\leqslant N}{}'\sum_{1\leqslant n_2\leqslant N}{}'\frac{r(n_1)r(n_2)}{n_1n_2}\sum_{d|(q,\bar{n}_2n_1+1)}\mu\left(\frac{q}{d}\right)\phi(d)\\
&=\frac{1}{2}\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sum_{\substack{1\leqslant n_1\leqslant N\\ n_2\equiv n_1(\bmod d)}}{}'\sum_{1\leqslant n_2\leqslant N}{}'\frac{r(n_1)r(n_2)}{n_1n_2}\\
&\quad-\frac{1}{2}\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sum_{\substack{1\leqslant n_1\leqslant N\\ n_2\equiv -n_1(\bmod d)}}{}'\sum_{1\leqslant n_2\leqslant N}{}'\frac{r(n_1)r(n_2)}{n_1n_2}, \qquad (6)
\end{aligned}$$

where $\sum_{1\leqslant n\leqslant N}{}'$ denotes the summation over n from 1 to N such that $(n,q)=1$.

Now we calculate the first summation. For convenience, we split the sum over n_1 or n_2 into following cases: i) $d \leqslant n_1 \leqslant N, d \leqslant n_2 \leqslant N$; ii) $d \leqslant n_1 \leqslant N, 1 \leqslant n_2 \leqslant d-1$; iii) $1 \leqslant n_1 \leqslant d-1, d \leqslant n_2 \leqslant N$; iv) $1 \leqslant n_1 \leqslant d-1, 1 \leqslant n_2 \leqslant d$. So we have

$$\begin{aligned}
&\sum_{d|q} \mu\left(\frac{q}{d}\right) \phi(d) \sideset{}{'}\sum_{\substack{d \leqslant n_1 \leqslant N \\ n_2 \equiv n_1 (\bmod d)}} \sideset{}{'}\sum_{d \leqslant n_2 \leqslant N} \frac{r(n_1) r(n_2)}{n_1 n_2} \\
&\qquad \ll \sum_{d|q} \phi(d) \sum_{1 \leqslant r_1 \leqslant \frac{N}{d}} \sum_{1 \leqslant r_2 \leqslant \frac{N}{d}} \sideset{}{'}\sum_{\substack{l_1=1 \\ l_2 \equiv l_1 (\bmod d)}}^{d-1} \sideset{}{'}\sum_{l_2=1}^{d-1} \frac{r(r_1 d + l_1) r(r_2 d + l_2)}{(r_1 d + l_1)(r_2 d + l_2)} \\
&\qquad \ll \sum_{d|q} \phi(d) \sum_{1 \leqslant r_1 \leqslant \frac{N}{d}} \sum_{1 \leqslant r_2 \leqslant \frac{N}{d}} \sideset{}{'}\sum_{l_1=1}^{d-1} \frac{[(r_1 d + l_1)(r_2 d + l_1)]^{\epsilon}}{(r_1 d + l_1)(r_2 d + l_1)} \\
&\qquad \ll \sum_{d|q} \frac{\phi(d)}{d} \sum_{1 \leqslant r_1 \leqslant \frac{N}{d}} \sum_{1 \leqslant r_2 \leqslant \frac{N}{d}} \frac{[(r_1 d + 1)(r_2 d + 1)]^{\epsilon}}{r_1 r_2} \\
&\qquad \ll q^{\epsilon},
\end{aligned}$$

$$\begin{aligned}
&\sum_{d|q} \mu\left(\frac{q}{d}\right) \phi(d) \sideset{}{'}\sum_{\substack{d \leqslant n_1 \leqslant N \\ n_2 \equiv n_1 (\bmod d)}} \sideset{}{'}\sum_{1 \leqslant n_2 \leqslant d-1} \frac{r(n_1) r(n_2)}{n_1 n_2} \\
&\qquad \ll \sum_{d|q} \phi(d) \sum_{1 \leqslant r_1 \leqslant \frac{N}{d}} \sum_{1 \leqslant n_2 \leqslant d-1} (r_1 n_2 d)^{\epsilon-1} \\
&\qquad \ll q^{\epsilon}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{d|q} \mu\left(\frac{q}{d}\right) \phi(d) \sideset{}{'}\sum_{\substack{1 \leqslant n_1 \leqslant d-1 \\ n_2 \equiv n_1 (\bmod d)}} \sideset{}{'}\sum_{d \leqslant n_2 \leqslant N} \frac{r(n_1) r(n_2)}{n_1 n_2} \\
&\qquad \ll \sum_{d|q} \phi(d) \sum_{1 \leqslant n_1 \leqslant d-1} \sum_{1 \leqslant r_2 \leqslant \frac{N}{d}} (n_1 r_2 d)^{\epsilon-1} \\
&\qquad \ll q^{\epsilon},
\end{aligned}$$

where we have used the estimate $r(n) \ll n^{\epsilon}$.

For the case $1 \leqslant n_1 \leqslant d-1, 1 \leqslant n_2 \leqslant d-1$, the solution of the congruence $n_2 \equiv n_1 (\bmod d)$ is $n_2 = n_1$. Hence,

$$
\begin{aligned}
&\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sideset{}{'}\sum_{\substack{1\leqslant n_1\leqslant d-1\\ n_2\equiv n_1(\bmod d)}}\sideset{}{'}\sum_{1\leqslant n_2\leqslant d-1}\frac{r(n_1)r(n_2)}{n_1n_2}\\
&\qquad=\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sideset{}{'}\sum_{1\leqslant n_2\leqslant d-1}\frac{r^2(n_2)}{n_2^2} \qquad (7)\\
&\qquad=\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sideset{}{'}\sum_{n_2=1}^{\infty}\frac{r^2(n_2)}{n_2^2}+O(q^\epsilon)\\
&\qquad=\frac{1}{2}J(q)\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{r^2(n)}{n^2}+O(q^\epsilon).
\end{aligned}
$$

Similarly, we can also get the estimate

$$
\begin{aligned}
&\frac{1}{2}\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sideset{}{'}\sum_{\substack{1\leqslant n_1\leqslant N\\ n_2\equiv -n_1(\bmod d)}}\sideset{}{'}\sum_{1\leqslant n_2\leqslant N}\frac{r(n_1)r(n_2)}{n_1n_2}\\
&\qquad=\frac{1}{2}\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sideset{}{'}\sum_{\substack{1\leqslant n_1\leqslant N\\ n_2+n_1=d}}\sideset{}{'}\sum_{1\leqslant n_2\leqslant N}\frac{r(n_1)r(n_2)}{n_1n_2}\\
&\qquad\quad+\frac{1}{2}\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sideset{}{'}\sum_{\substack{1\leqslant n_1\leqslant N\\ n_2+n_1=ld,}}\sideset{}{'}\sum_{\substack{1\leqslant n_2\leqslant N\\ l\geqslant 2}}\frac{r(n_1)r(n_2)}{n_1n_2}\\
&\qquad\ll\sum_{d|q}\phi(d)\sum_{1\leqslant n\leqslant d-1}\frac{r(n)r(d-n)}{n(d-n)}\\
&\qquad\quad+\sum_{d|q}\phi(d)\sideset{}{'}\sum_{1\leqslant n_1\leqslant N}\sum_{l=\left[\frac{n_1}{d}\right]+2}^{\left[\frac{N+n_1}{d}\right]}\frac{r(n_1)r(ld-n_1)}{ldn_1-n_1^2}\\
&\qquad\ll\sum_{d|q}\frac{\phi(d)}{d}\sum_{1\leqslant n\leqslant d-1}\frac{r(n)r(d-n)}{n}\\
&\qquad\quad+\sum_{d|q}\frac{\phi(d)}{d}\sideset{}{'}\sum_{1\leqslant n_1\leqslant N}\sum_{l=\left[\frac{n_1}{d}\right]+2}^{\left[\frac{N+n_1}{d}\right]}\frac{n_1^\epsilon(ld-n_1)^{\epsilon_1}}{ln_1-\frac{n_1^2}{d}}\\
&\qquad\ll q^\epsilon+\sum_{d|q}\frac{\phi(d)d^{\epsilon_1}}{d}\sum_{n_1=1}^{N}\sum_{l=1}^{N}\frac{n_1^\epsilon l^{\epsilon_1}}{ln_1}\\
&\qquad\ll q^\epsilon. \qquad (8)
\end{aligned}
$$

Then from (13), (15) and (16), we have

$$M_1 = \frac{1}{2}J(q) \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{r^2(n)}{n^2} + O(q^{\epsilon}). \tag{9}$$

(ii) From the partition identity

$$\begin{aligned} A(y,\chi) = {} & 2\sum_{n\leqslant\sqrt{y}} \Lambda(n)\chi(n) \sum_{m\leqslant\frac{y}{n}} \chi(m)\chi_4(m) - 2\sum_{n\leqslant\sqrt{N}} \Lambda(n)\chi(n) \sum_{m\leqslant\frac{N}{n}} \chi(m)\chi_4(m) \\ & - \left(\sum_{n\leqslant\sqrt{y}} \Lambda(n)\chi(n)\right)\left(\sum_{m\leqslant\sqrt{y}} \chi(m)\chi_4(m)\right) \\ & + \left(\sum_{n\leqslant\sqrt{N}} \Lambda(n)\chi(n)\right)\left(\sum_{m\leqslant\sqrt{N}} \chi(m)\chi_4(m)\right), \end{aligned}$$

and the Pólya-Vinogradov inequality

$$\left|\sum_{n=a}^{b} \chi(n)\right| \ll \sqrt{q}\ln q,$$

noting that $\Lambda(n) \leqslant \ln n$ we can easily get

$$|A(y,\chi)| \ll q^{\frac{1}{2}} y^{\frac{1}{2}} \ln y \ln q. \tag{10}$$

Then we have

$$\begin{aligned} M_2 &= \sideset{}{^*}\sum_{\chi(-1)=-1} \left(\sum_{1\leqslant n_1\leqslant N} \frac{\chi(n_1)r(n_1)}{n_1}\right)\left(\int_N^{\infty} \frac{A(y,\bar{\chi})}{y^2}dy\right) \\ &\ll \sum_{1\leqslant n_1\leqslant N} n_1^{\epsilon-1} \int_N^{\infty} \frac{1}{y^2}\left(\sum_{\chi(-1)=1} |A(y,\bar{\chi})|\right) dy \\ &\ll N^{\epsilon} \int_N^{\infty} \frac{q^{\frac{3}{2}} \ln q y^{\frac{1}{2}} \ln y}{y^2} dy \ll \frac{q^{\frac{3}{2}} \ln q}{N^{\frac{1}{2}-\epsilon}}. \end{aligned} \tag{11}$$

(iii) Similar to (ii), we can also get

$$M_3 \ll \frac{q^{\frac{3}{2}} \ln q}{N^{\frac{1}{2}-\epsilon}}. \tag{12}$$

(iv) From (18), we can get the estimate

$$\sum_{\chi(-1)=1} |A(y,\chi)|^2 \ll y^{1+\epsilon} q^2 \ln^2 q.$$

Hence, by the same argue in section (ii), we can write

$$\begin{aligned}
M_4 &= \sideset{}{^*}\sum_{\chi(-1)=1} \left(\int_N^\infty \frac{A(y,\bar\chi)}{y^2}dy\right)\left(\int_N^\infty \frac{A(y,\chi)}{y^2}dy\right) \\
&\leqslant \int_N^\infty \int_N^\infty \frac{1}{y^2z^2} \sideset{}{^*}\sum_{\chi(-1)=1} |A(y,\bar\chi)||A(z,\chi)|dydz \\
&\ll \int_N^\infty \frac{1}{y^2}\int_N^\infty \frac{1}{z^2}\left(\sum_{\chi(-1)=1}|A(y,\bar\chi)|^2\right)^{\frac12}\left(\sum_{\chi(-1)=1}|A(z,\chi)|^2\right)^{\frac12} dydz \\
&\ll \left(\int_N^\infty \frac{1}{y^2}\left(\sum_{\chi(-1)=1}|A(y,\chi)|^2\right)^{\frac12} dy\right)^2 \\
&\ll \left(\int_N^\infty \frac{q\ln q}{y^{\frac32-\epsilon}}dy\right)^2 \ll \frac{q^2\ln^2 q}{N^{1-\epsilon}}. \qquad (13)
\end{aligned}$$

Now, taking $N = q^3$, combining (13)-(21) we obtain the asymptotic formula

$$\sideset{}{^*}\sum_{\chi(-1)=1}\left|\frac{L'(1,\chi)}{L(1,\chi)}\right|^2 |L(1,\bar\chi\chi_4)|^2 = \frac12 J(q)\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{r^2(n)}{n^2} + O(q^\epsilon).$$

This proves Lemma 7. ■

Lemma 8. *Let q be any odd integer with $q > 2$, χ be the Dirichlet character modulo q. There holds:*

$$\sideset{}{^*}\sum_{\chi(-1)=1} |L(1,\bar\chi\chi_4)|^2 = \frac{\pi^2}{16}J(q)\prod_{p|q}\left(1-\frac{1}{p^2}\right) + O(q^\epsilon).$$

Proof. From the same method of proving lemma 7, we can also get

$$\sideset{}{^*}\sum_{\chi(-1)=1} |L(1,\bar\chi\chi_4)|^2 = \frac12 J(q)\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{\chi_4^2(n)}{n^2} + O(q^\epsilon).$$

Noting that

$$\chi_4^2(n) = \begin{cases} 1, & \text{if } 2\nmid n;\\ 0 & \text{if } 2|n,\end{cases}$$

we can write

$$\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{\chi_4^2(n)}{n^2}=\sum_{\substack{n=1\\(n,2q)=1}}^{\infty}\frac{1}{n^2}=\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{1}{n^2}-\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{1}{(2n)^2}=\frac{3}{4}\zeta(2)\prod_{p|q}\left(1-\frac{1}{p^2}\right)$$
$$=\frac{\pi^2}{8}\prod_{p|q}\left(1-\frac{1}{p^2}\right),$$

where we used the fact $\zeta(2)=\frac{\pi^2}{6}$. This proves lemma 8. ■

Lemma 9. *Let q be any odd integer with $q>2$, χ be the Dirichlet character modulo q and $m\geqslant 0$ be a fixed integer. Then we have the following asymptotic formulae:*

$$\sum_{\chi(-1)=-1}^{*}\chi(2^m)\left|\frac{L'(1,\chi)}{L(1,\chi)}\right|^2|L(1,\chi)|^2$$
$$=\frac{1}{2^{m+1}}J(q)\left(m\ln 2\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{\ln n}{n^2}+\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{\ln^2 n}{n^2}\right)+O(q^{\epsilon}),$$

$$\sum_{\chi(-1)=-1}^{*}\chi(2^m)\frac{L'(1,\chi)}{L(1,\chi)}|L(1,\chi)|^2=\frac{1}{2^{m+1}}J(q)\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{\ln n}{n^2}+O(q^{\epsilon}),$$

$$\sum_{\chi(-1)=-1}^{*}\bar{\chi}(2^m)\frac{L'(1,\chi)}{L(1,\chi)}|L(1,\chi)|^2$$
$$=\frac{1}{2^{m+1}}J(q)\left(\frac{m\pi^2\ln 2}{6}\prod_{p|q}\left(1-\frac{1}{p^2}\right)+\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{\ln n}{n^2}\right)+O(q^{\epsilon})$$

and

$$\sum_{\chi(-1)=-1}^{*}\chi(2^m)|L(1,\chi)|^2=\frac{\pi^2}{2^{m+2}\cdot 3}J(q)\prod_{p|q}\left(1-\frac{1}{p^2}\right)+O(q^{\epsilon}).$$

Proof. By using the same method as proving Lemma 7 and Lemma 8, we can also obtain these formulae. ■

3. Proof of the theorem

In this section we will complete the proof of the theorem. It is clear that $|\tau(\chi)|=\sqrt{q}$ if χ is a primitive character. So from Lemma 1, Lemma 4 and Lemma 5, we can write

$$
\begin{aligned}
&\sum_{\chi \bmod q}{}^{*} \left|B(\chi) S\left(\frac{q}{4}, \chi\right)\right|^2 \\
&\quad= \sum_{\chi(-1)=1}{}^{*} \left|B(\chi) S\left(\frac{q}{4}, \chi\right)\right|^2 + \sum_{\chi(-1)=-1}{}^{*} \left|B(\chi) S\left(\frac{q}{4}, \chi\right)\right|^2 \\
&\quad= \frac{q}{\pi^2} \sum_{\chi(-1)=1}{}^{*} \left|\frac{L'(1,\chi)}{L(1,\chi)}\right|^2 |L(1,\chi\chi_4)|^2 + \frac{C_1 q}{\pi^2} \sum_{\chi(-1)=1}{}^{*} \frac{L'(1,\chi)}{L(1,\chi)} |L(1,\chi\chi_4)|^2 \\
&\qquad+ \frac{C_1 q}{\pi^2} \sum_{\chi(-1)=1}{}^{*} \frac{L'(1,\bar{\chi})}{L(1,\bar{\chi})} |L(1,\chi\chi_4)|^2 + \frac{C_1^2 q}{\pi^2} \sum_{\chi(-1)=1}{}^{*} |L(1,\chi\chi_4)|^2 \\
&\qquad+ \frac{q}{4\pi^2} \sum_{\chi(-1)=-1}{}^{*} (6+\chi(2)-2\chi(4)+\bar{\chi}(2)-2\bar{\chi}(4)) \left|\frac{L'(1,\chi)}{L(1,\chi)}\right|^2 |L(1,\chi)|^2 \\
&\qquad+ \frac{C_2 q}{4\pi^2} \sum_{\chi(-1)=-1}{}^{*} (6+\chi(2)-2\chi(4)+\bar{\chi}(2)-2\bar{\chi}(4)) \frac{L'(1,\chi)}{L(1,\chi)} |L(1,\chi)|^2 \\
&\qquad+ \frac{C_2 q}{4\pi^2} \sum_{\chi(-1)=-1}{}^{*} (6+\chi(2)-2\chi(4)+\bar{\chi}(2)-2\bar{\chi}(4)) \frac{L'(1,\bar{\chi})}{L(1,\bar{\chi})} |L(1,\chi)|^2 \\
&\qquad+ \frac{C_2^2 q}{4\pi^2} \sum_{\chi(-1)=-1}{}^{*} (6+\chi(2)-2\chi(4)+\bar{\chi}(2)-2\bar{\chi}(4)) |L(1,\chi)|^2.
\end{aligned}
$$

Now applying Lemma 7, Lemma 8 and Lemma 9 we can easily get

$$
\sum_{\chi \bmod q}{}^{*} \left|B(\chi) S\left(\frac{q}{4}, \chi\right)\right|^2 = CqJ(q) + O(q^{1+\epsilon}),
$$

where

$$
\begin{aligned}
C = \frac{c_1}{2\pi^2} + \frac{C_1 c_2}{\pi^2} + \frac{3c_3}{4\pi^2} - \left(\frac{\ln 2}{8\pi^2} - \frac{3C_2}{2\pi^2}\right) c_4 \\
+ \left(\frac{C_1^2}{16} - \frac{C_2 \ln 2}{48} + \frac{C_2^2}{8}\right) \prod_{p|q} \left(1 - \frac{1}{p^2}\right),
\end{aligned}
$$

and

$$
c_1 = \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{r^2(n)}{n^2} \qquad c_2 = \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{\chi_4(n) r(n)}{n^2},
$$

$$
c_3 = \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{\ln^2 n}{n^2} \qquad c_4 = \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{\ln n}{n^2}.
$$

This completes the proof of Theorem.

References

[1] G. Pólya, *Über die Verteilung der quadratische Reste und Nichtreste*, Göttingen Nachrichten, 1918, 21–29.

[2] I.M. Vinogradov, *On the diswtribution of residues and non-residues of powers*, Journal of the Physico-Mathematical society of Perm. **1** (1918), 94–96.

[3] A.V. Sokolovskiĭ, *On a theorem of Sárkozy*, Acta Arithmetica **41** (1982), 27–31.

[4] D.A. Burgess, *Mean value of character sums II*, Mathematika **33** (1987), 1–7.

[5] Zhefeng Xu, Wenpeng Zhang, *On the $2k$-th power mean of the character sums over short intervals*, Acta Arithmetica **121.2** (2006), 149–160.

[6] H. Davenport, *Multiplicative Number Theory* (Second Edition), Springer-Verlag, New York 1980, 83.

[7] Tom M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York 1976.

[8] Wenpeng Zhang, *On a Cochrane sum and its hybrid mean value formula(II)*, Journal of Mathematical Analysis and Applications **276** (2002), 446–457.

[9] L.K. Hua, *Introduction to Number Theory*, Science Press, Beijing 1979.

Addresses: Zhefeng Xu: College of Science, Xi'an Jiaotong University, Xi'an, Shaanxi, P. R. China and Department of Mathematics, Northwest University, Xi'an, Shaanxi, P. R. China;
Wenpeng Zhang: Department of Mathematics, Northwest University, Xi'an, Shaanxi, P. R. China

E-mail: zfxu@nwu.edu.cn, wpzhang@nwu.edu.cn

Received: 29 September 2008