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ON SOME ARITHMETICAL MULTIPLICATIVE FUNCTIONS

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Dedicated to Professor Eduard Wirsing
on the occasion of his 75th birthday

Abstract: We characterize some non-negative multiplicative functions $f(n)$ such that $\lim_{x\to+\infty}\frac{1}{x}\sum_{\substack{1\leqslant n\leqslant x\\ n\in A}} f(n)$ exists and is positive, but there exists a subset $A(f)$ of N of density 1 such that $\lim_{x\to+\infty}\frac{1}{x}\sum_{\substack{1\leqslant n\leqslant x\\ n\in A(f)}} f(n)=0$. An application to the case of the Ramanujan τ-function is provided.

Keywords: mean-value, multiplicative functions.

1. Introduction

Denoting the set of positive integers by N, we recall that a complex-valued multiplicative arithmetical function is a function $f: N\to C$, such that $f(1)=1$ and $f(mn)=f(m)f(n)$ whenever $\gcd(m,n)=1$. Moreover, we denote by P the set of prime numbers.

1.1. A result of Wirsing. We recall the following result of Wirsing:

Theorem 1. *Let $f(n)$ be a non-negative multiplicative function such that for some constant $\tau>0$,*

$$\sum_{p\leqslant x} f(p)\log p = x(\tau+o(1)),\quad x\to+\infty,$$

$$\sum_{p\in P}\sum_{2\leqslant k} f(p^k)p^{-k}=o((\log x)^{-1}),$$

$$f(p)=O(p^{1-\delta})\quad \textit{for some } \delta<1.$$

Then, as $x\to+\infty$, the asymptotic formula

$$\sum_{1\leqslant n\leqslant x} f(n)=x\frac{(1+o(1))}{\Gamma(\tau)}\prod_{p\leqslant x}(1-p^{-1})\sum_{0\leqslant k} f(p^k)p^{-k}$$

holds, where Γ is the gamma-function.

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This result ([7] p.65 Theorem 4.1 and same page Remark 1) leads to closer consideration of the relationship between a property of the arithmetical meanvalue of a non-negative multiplicative function $f(n)$,

$$M(f) = \lim_{x\to+\infty} \frac{1}{x} \sum_{1\leqslant n\leqslant x} f(n)$$

and a specificity of the product

$$\prod_{p\leqslant x} (1-p^{-1}) \sum_{0\leqslant k} f(p^k)p^{-k}, \quad x\to+\infty$$

considered as associated to an underlying probability space.

1.2. Main result. We shall prove the following result:

Theorem 2. *Let $\mathcal{H}$ be the set of non-negative multiplicative functions $f(n)$ satisfying the following conditions: for any d in N, $\lim\limits_{x\to+\infty} \frac{1}{x} \sum\limits_{\substack{1\leqslant n\leqslant x\\ d \text{ divides } n}} f(n)$ exists; $\sum\limits_{p\in P} \sum\limits_{2\leqslant k} f(p^k)p^{-k}$ is finite.*
Let us assume now that f is an element of $\mathcal{H}$ and $\sum\limits_{p\in P} (1-f(p)^{1/2})^2/p$ is not finite. Then, there exists a subset A of N of density 1 such that $\lim\limits_{x\to+\infty} \frac{1}{x} \sum\limits_{\substack{1\leqslant n\leqslant x\\ n\in A}} f(n) = 0$.

Remark. Since f is non-negative, the existence of $\lim\limits_{x\to+\infty} \frac{1}{x} \sum\limits_{1\leqslant n\leqslant x} f(n)$ implies that $f(n) = O(n)$, and with the condition that $\sum\limits_{p\in P} \sum\limits_{2\leqslant k} f(p^k)p^{-k}$ is finite, this proves that there exists a positive number C such that for all p in P, the inequality $\sum\limits_{k\geqslant 0} f(p^k)p^{-k} \leqslant C$ holds.

1.3. Application. As an application of this result, we can consider the special case of the Ramanujan τ-function. We recall the following classical notation:
$n \mid m$ means n divides m,
$n \nmid m$ means n does not divide m,
and $p^k \parallel m$ means $p^k \mid m$, but $p^{k+1} \nmid m$.

Now, we show that in fact, the function $\left(\tau(n)n^{-11/2}\right)^2$ satisfies the hypothesis of the above theorem. We shall get some information on the properties of the function $\left(\tau(n)n^{-11/2}\right)^2$ essentially from [5] p.234 et seq., [6] p.357 et seq.

Let $\tau_0(n)$ be defined by $\tau_0(n) = \tau(n)n^{-11/2}$. By Deligne's theorem [1], we know that $\tau_0(p) = \varsigma_p + \varsigma_p^{-1}$, with $|\varsigma_p| = 1$. For $s = \sigma + it$, $\sigma > 1$, we have

$$\sum_{1\leqslant n} \tau_0(n)^2 n^{-s} = \prod_{p\in P} \left(\left(1+p^{-s}\right) \left((1-\varsigma_p^2 p^{-s})(1-\varsigma_p^{-2}p^{-s})(1-p^{-s}) \right)^{-1} \right),$$

and it is not difficult to prove that there exists some positive number C such that for all p in P, the inequality $\sum_{k\geqslant 0}\tau_0(p^k)^2p^{-k}\leqslant C$ holds, and that $\sum_{p\in P}\sum_{2\leqslant k}\tau_0(p^k)^2p^{-k}$ is finite.

Now, we remark that

$$\sum_{\substack{1\leqslant n\\ m|n}}\tau_0(n)^2n^{-s}=\left(\prod_{p^\alpha\ \|\ m}\sum_{k\geqslant 0}\tau_0(p^{\alpha+k})^2p^{-(\alpha+k)s}\right)\\ \times\left(\prod_{\substack{p\in P\\ p\nmid m}}\left(\left(1+p^{-s}\right)\left((1-\varsigma_p^2p^{-s})(1-\varsigma_p^{-2}p^{-s})(1-p^{-s})\right)^{-1}\right)\right)$$

and we write it as

$$\left(\prod_{p^\alpha\|m}\left(\left(\sum_{k\geqslant 0}\tau_0(p^{\alpha+k})^2p^{-(\alpha+k)s}\right)\right.\right.\\ \left.\left.\times\left(\left(1+p^{-s}\right)^{-1}(1-\varsigma_p^2p^{-s})(1-\varsigma_p^{-2}p^{-s})(1-p^{-s})\right)\right)\right)\\ \times\left(\prod_{p\in P}\left(\left(1+p^{-s}\right)\left((1-\varsigma_p^2p^{-s})(1-\varsigma_p^{-2}p^{-s})(1-p^{-s})\right)^{-1}\right)\right),$$

i.e.

$$\sum_{\substack{1\leqslant n\\ m|n}}\tau_0(n)^2n^{-s}=\left(\prod_{p^\alpha\|m}\left(\left(\sum_{k\geqslant 0}\tau_0(p^{\alpha+k})^2p^{-(\alpha+k)s}\right)\right.\right.\\ \left.\left.\times\left(\left(1+p^{-s}\right)^{-1}(1-\varsigma_p^2p^{-s})(1-\varsigma_p^{-2}p^{-s})(1-p^{-s})\right)\right)\right)\\ \times\sum_{1\leqslant n}\tau_0(n)^2n^{-s}.$$

The first term of this product is analytic in a neighborhood of $\sigma\geqslant 1$. As a consequence of the famous result of Rankin on the analytic properties of the series $\sum_{1\leqslant n}\tau_0(n)^2n^{-s}$ ([6] p.360, Th.3), we can use the Ikehara's theorem [7, p.322, Theorem A.4.3] and get that the limit $\lim_{x\to+\infty}\frac{1}{x}\sum_{\substack{1\leqslant n\leqslant x\\ m|n}}\tau_0(n)^2$ exists.

Now, the fact that $\sum_{p\in P}(1-\tau_0(p))^2/p$ is not finite is a simple consequence of the analytic properties of the Dirichlet series with coefficients $\tau_0(n)^2$ and $\tau_0(n)^4$, and this has been already considered elsewhere ([4] p.146).

This allows us to formulate the following result:

Proposition 3. *There exists a subset A of N of density 1 such that*

$$\lim_{x\to+\infty}\frac{1}{x}\sum_{\substack{1\leqslant n\leqslant x\\ n\in A}}\Big(\tau(n)n^{-11/2}\Big)^2=0.$$

2. Proof of the main result

2.1. Notation. If d is in N, the arithmetical function $I'_d(n)$ is defined by $I'_d(n)=1$ if $d\mid n$, and 0 otherwise.

E_p is the discrete set $(1,p,p^2,...)$ and $\overline{E_p}=E_p\cup\{p^\infty\}$ is the Alexandroff one-point compactification of the infinite discrete, hence locally compact but not compact space E_p [7, p. 145]

$$\overline{E}=\prod_{p\in P}\overline{E_p},\qquad E=\prod_{p\in P}E_p,\qquad E_{y-}=\prod_{\substack{p\in P\\ p\leqslant y-1}}E_p,$$

$$E_{y+}=\prod_{\substack{p\in P\\ y\leqslant p}}E_p,\qquad \overline{E}_{y-}=\prod_{\substack{p\in P\\ p\leqslant y-1}}\overline{E}_p,\qquad \overline{E}_{y+}=\prod_{\substack{p\in P\\ y\leqslant p}}\overline{E}_p.$$

An element t of $\overline{E}$ can be viewed as a sequence $\big(p^{v_p(t)}\big)_{p\in P}$, where $v_p(t)$ is either a non-negative integer, or ∞.

If t is in $\overline{E}$, we denote the finite sequence $t_{y-}=\big\{p^{v_p(t)}\big\}_{p\leqslant y-1}$ by t_{y-}.

The product space $\overline{E}=\prod_{p\in P}\overline{E}_p$, equipped with the product topology, is a compact space.

We say that a subset of $\overline{E}$ is elementary (resp. almost elementary) if it can be written as $\{\theta_{y-}\}\times\overline{E}_{y+}$, where θ_{y-} is in E_{y-} (resp. θ_{y-} is in $\overline{E}_{y-}$). An elementary set is open.

2.2. Some lemmas. First,we have the following result:

Lemma 4. *Let ν be a probability Borel measure on $\overline{E}$ such that for any p in P, $\nu(\{p^\infty\}\prod_{\substack{q\in P\\ q\neq p}}\overline{E}_q)=0$. Then, given any open set O of $\overline{E}$, there exists an open subset O^* of O such that $\nu(O)=\nu(O^*)$ and O^* can be written as a disjoint and at most countable union of elementary subsets of $\overline{E}$.*

Proof. Let t be in O. We remark that the sequence of almost elementary subsets $O_{t,k}=\Big(\prod_{p\leqslant k-1}\big\{p^{v_p(t)}\big\}\Big)\times\overline{E}_{k+}$ is ordered by inclusion, since we have $O_{t,k+1}\subset O_{t,k}$ for any $k\geqslant 1$. Now, due to the topology of $\overline{E}$, O contains an almost elementary subset $O_{t,y}=\Big(\prod_{p\leqslant y-1}\big\{p^{v_p(t)}\big\}\Big)\times\overline{E}_{y+}$. We define $y(t)$ by $y(t)=min\big\{y\geqslant 3; t_{y-}\overline{E}_{y+}\subseteq O\big\}$ and denote by O_t the almost elementary subset $O_{t,y(t)}=\Big(\prod_{p\leqslant y(t)-1}\big\{p^{v_p(t)}\big\}\Big)\times\overline{E}_{y(t)+}$, which is the maximal element of the decreasing sequence $O_{t,k}$, $k \geqslant 3$. (We require the index $y(t)$ to be greater than 2 only to avoid the trivial case $O = \overline{E} = \prod_{p\in P} \overline{E}_p$). If for some $p \leqslant y(t)-1$, $v_p(t) = \infty$, then t is in $\{p^\infty\}\prod_{\substack{q\in P\\ q\neq p}} \overline{E_q}$. Otherwise, O_t is an elementary set $t_{y(t)-}\overline{E}_{y(t)+}$, and we define O^* as the union of these sets. O^* is an open subset of O, and $O - O^*$ is contained in $\bigcup_{p\in P}\{p^\infty\}\prod_{\substack{q\in P\\ q\neq p}} \overline{E_q}$, which is of ν-measure 0, since it is a countable union of sets of ν-measure 0. Now, we remark that if the intersection of two elementary sets is not empty, then one of them is contained in the other. Hence we get that the set O^* can be written as $O^* = \bigcup_{t_{y-}\in A(O)} t_{y-}\overline{E}_{y+}$, where $A(O)$ is at most countable, and the union is disjoint. ■

Lemma 5. *Let ν be a probability Borel measure on $\overline{E}$ such that for any p in P, $\nu(\{p^\infty\}\prod_{\substack{q\in P\\ q\neq p}} \overline{E_q}) = 0$. Then, given any open set O of $\overline{E}$, and any positive ε, there exists an open subset O' of O such that $\nu(O) - \nu(O') \leqslant \varepsilon$, and O' can be written as a finite union of disjoint elementary subsets.*

Proof. Using Lemma 4, we replace O with O^*,where $O^* = \bigcup_{t_{y-}\in A(O)} t_{y-}\overline{E}_{y+}$, and $A(O)$ is at most countable, the union being disjoint. We remark that an elementary set $t_{y-}\overline{E}_{y+}$ is characterized by the integer $N(t_{y-}) = \prod_{p\leqslant y-1} p^{v_p(t_{y-})+1}$. For given such an integer, we know its greatest prime divisor will give the value of y, and t_{y-} will be given by $v_p(t_{y-}) = v_p(N(t_{y-})/\prod_{p\leqslant y-1} p)$.

Now, we remark that

$$\nu(O^*) = \lim_{k\to+\infty} \sum_{t_{y-}\in A(O), N(t_{y-})\leqslant k} \nu(t_{y-}\overline{E}_{y+}).$$

Since all the terms of this sum are non-negative, there exists an index $K(\varepsilon)$ such that

$$\nu(O^*) - \left(\sum_{t_{y-}\in A(O), N(t_{y-})\leqslant K(\varepsilon)} \nu(t_{y-}\overline{E}_{y+})\right) \leqslant \varepsilon$$

and so, we define O' by its characteristic function

$$I_{O'} = \sum_{t_{y-}\in A(O), N(t_{y-})\leqslant K(\varepsilon)} I_{t_{y-}}.$$ ■

Lemma 6. *Let O_1 and O_2 be two open sets, both being finite disjoint unions of elementary subsets of $\overline{E}$. Then, $O_1 \cup O_2$ can be written as a finite disjoint union of elementary subsets of $\overline{E}$.*

Proof. This is a simple consequence of the fact that if the intersection of two elementary sets is not empty, then one of them is contained in the other one. ■

Lemma 7. *Let $f(n)$ be a non-negative multiplicative function such that for any d in N, $\lim_{x\to+\infty} \frac{1}{x}\sum_{1\leqslant n\leqslant x} I'_d(n)f(n)$ exists. Then, for any subset $t_{y-}\overline{E}_{y+}$ with characteristic function $I_{t_{y-}}$,*

$$\lim_{x\to+\infty} \frac{1}{x} \sum_{1\leqslant n\leqslant x} I_{t_{y-}}(n)f(n)$$

exists and is equal to

$$\frac{f(t_{y-})}{t_{y-}}\left(\prod_{p\leqslant y-1}\sum_{0\leqslant k} f(p^k)p^{-k}\right)^{-1} M(f),$$

where $M(f)=\lim_{x\to+\infty}\frac{1}{x}\sum_{1\leqslant n\leqslant x} f(n)$.

Proof. Since $I_{t_{y-}}(n)=\prod_{p\leqslant y-1} J_{p^{v_p(t_{y-})}}(n)$, with $J_{p^\alpha}(n)=I'_{p^\alpha}(n)-I'_{p^{\alpha+1}}(n)$, due to our hypothesis , $M(I_{t_{y-}}f)=\lim_{x\to+\infty}\frac{1}{x}\sum_{1\leqslant n\leqslant x} I_{t_{y-}}(n)f(n)$ exists. Hence, by partial summation [7, p.54, lines 8-12], we have

$$\sum_{1\leqslant n} I_{t_{y-}}(n)f(n)n^{-\sigma} \backsim M(I_{t_{y-}}f)(\sigma-1)^{-1},$$

as $\sigma\to 1+$, and so,

$$\begin{aligned}
&M(I_{t_{y-}}f)\\
&=\lim_{\sigma\to1^+}(\sigma-1)\sum_{1\leqslant n} I_{t_{y-}}(n)f(n)n^{-\sigma}\\
&=\lim_{\sigma\to1^+}(\sigma-1)f(t_{y-})t_{y-}^{-\sigma}\sum_{\gcd(n,\prod_{p\leqslant y-1}p)=1} f(n)n^{-\sigma}\\
&=\lim_{\sigma\to1^+}(\sigma-1)f(t_{y-})t_{y-}^{-\sigma}\left(\prod_{p\leqslant y-1}\sum_{k\geqslant0} f(p^k)p^{-k\sigma}\right)^{-1}\left(\prod_{p\in P}\sum_{k\geqslant0} f(p^k)p^{-k\sigma}\right)\\
&=f(t_{y-})t_{y-}^{-1}\left(\prod_{p\leqslant y-1}\sum_{k\geqslant0} f(p^k)p^{-k1}\right)^{-1}\left(\lim_{\sigma\to1^+}(\sigma-1)\left(\prod_{p\in P}\sum_{k\geqslant0} f(p^k)p^{-k\sigma}\right)\right)\\
&=f(t_{y-})t_{y-}^{-1}\left(\prod_{p\leqslant y-1}\sum_{k\geqslant0} f(p^k)p^{-k}\right)^{-1} M(f). \qquad\blacksquare
\end{aligned}$$

Lemma 8. *Let $f(n)$ be a non-negative multiplicative function such that for any d in N, $\lim_{x\to+\infty}\frac{1}{x}\sum_{1\leqslant n\leqslant x} I'_d(n)f(n)=M(I'_df)$ exists. Then, there exists a unique probability Borel measure ν_f on $\overline{E}$ such that for any elementary subset $t_{y-}\overline{E}_{y+}$ of $\overline{E}$, the equality $\nu_f(t_{y-}\overline{E}_{y+})=f(t_{y-})t_{y-}^{-1}\left(\prod_{p\leqslant y-1}\sum_{k\geqslant0} f(p^k)p^{-k}\right)^{-1}$ holds.*

Proof. The set A of arithmetical functions which can be written as $h(n)=\sum l_d I'_d(n)$, where the sum is finite and the l_d are real numbers, is the restriction to

N of an algebra of continuous functions defined on $\overline{E}$. By the Stone-Weierstrass Theorem ([2], p. 101, note 1.a), it is dense in the set of the real-valued functions continuous on $\overline{E}$, equipped with the uniform topology. Now, by Lemma 7, the linear form $\langle f,h\rangle = M(fh)/M(f)$ is well defined, and it satisfies the relation $|\langle f,h\rangle| \leqslant \max(|h(n)|, n \in N)$. As a consequence, this linear continuous form can be extended to the whole set of real-valued functions continuous on $\overline{E}$. The Riesz representation Theorem ([2], p. 129, (11.37)) shows that $\langle f,\cdot\rangle$ defines a Borel measure ν_f on $\overline{E}$. It is clearly of mass 1. The equality $\nu_f(t_{y-}\overline{E}_{y+}) = f(t_{y-})t_{y-}^{-1}\left(\prod_{p\leqslant y-1}\sum_{k\geqslant 0} f(p^k)p^{-k}\right)^{-1}$ is the immediate consequence of Lemma 7. ■

Until this point, we remained in a rather general setting. From now on, we shall specialize to the set $\mathcal{H}$.

We begin by remarking that the multiplicative function $1(n)$ defined by $1(n) = 1$ for all n is an element of $\mathcal{H}$. So, the measure ν_1 is well-defined. Now, we prove the following lemma:

Lemma 9. *Given f, an element of $\mathcal{H}$, the measure ν_f is orthogonal to ν_1 if and only if $\sum_{p\in P}(1-f(p)^{1/2})^2/p$ is not finite.*

Proof. 1) First, we prove that the measure ν_f is orthogonal to ν_1 if and only if

$$\lim_{y\to+\infty}\sum_{p\leqslant y-1}\left(\left((1-p^{-1})\sum_{k\geqslant 0} f(p^k)p^{-k}\right) - \left((1-p^{-1})\sum_{0\leqslant k} f(p^k)^{1/2}p^{-k}\right)^2\right) = +\infty.$$

Proof. Since the multiplicative function $1(n)$ defined by $1(n) = 1$ for all n is an element of $\mathcal{H}$, the measure ν_1 is well-defined and by Lemma 9, we have

$$\nu_1(t_{y-}\overline{E}_{y+}) = t_{y-}^{-1}\left(\prod_{p\leqslant y-1}\sum_{0\leqslant k} p^{-k}\right)^{-1}.$$

Since $\nu_1(\{p^k\}) = p^{-k}(\sum_{0\leqslant r} p^{-r})^{-1} = p^{-k}(1-p^{-1})$, we can write $\nu_f(\{p^k\})$ as

$$\nu_f(\{p^k\}) = \nu_1(\{p^k\})\omega_p(\{p^k\}), \text{ where } \omega_p(\{p^k\}) = f(p^k)\left((1-p^{-1})\sum_{r\geqslant 0} f(p^r)p^{-r}\right)^{-1}.$$

The Kakutani Theorem ([8], p. 109) shows that the measure ν_f is orthogonal to ν_1 if and only if

$$\lim_{y\to+\infty}\prod_{p\leqslant y-1}\int_{\overline{E}_p}\omega_p^{1/2}d\nu_1 = 0,$$

i.e.

$$\lim_{y\to+\infty}\prod_{p\leqslant y-1}\left(\left((1-p^{-1})\sum_{0\leqslant k} f(p^k)^{1/2}p^{-k}\right) / \left((1-p^{-1})\sum_{k\geqslant 0} f(p^k)p^{-k}\right)^{1/2}\right) = 0$$

since by a direct computation, we obtain

$$\int_{\overline{E}_p} \omega_p^{1/2} d\nu_1 = \left((1-p^{-1}) \sum_{0 \leqslant k} f(p^k)^{1/2} p^{-k} \right) \left((1-p^{-1}) \sum_{k \geqslant 0} f(p^k) p^{-k} \right)^{-1/2} .$$

This is equivalent to

$$\lim_{y \to +\infty} \prod_{p \leqslant y-1} \left(\left((1-p^{-1}) \sum_{0 \leqslant k} f(p^k)^{1/2} p^{-k} \right) / \left((1-p^{-1}) \sum_{k \geqslant 0} f(p^k) p^{-k} \right)^{1/2} \right)^2 = 0,$$

i.e.

$$\lim_{y \to +\infty} \prod_{p \leqslant y-1} \left(\left((1-p^{-1}) \sum_{0 \leqslant k} f(p^k)^{1/2} p^{-k} \right)^2 / \left((1-p^{-1}) \sum_{k \geqslant 0} f(p^k) p^{-k} \right) \right) = 0,$$

and again, since

$$\left((1-p^{-1}) \sum_{0 \leqslant k} f(p^k)^{1/2} p^{-k} \right) / \left((1-p^{-1}) \sum_{k \geqslant 0} f(p^k) p^{-k} \right)^{1/2} \leqslant 1,$$

this is equivalent to

$$\lim_{y \to +\infty} \sum_{p \leqslant y-1} \left(1 - \left(\left((1-p^{-1}) \sum_{0 \leqslant k} f(p^k)^{1/2} p^{-k} \right)^2 / \left((1-p^{-1}) \sum_{k \geqslant 0} f(p^k) p^{-k} \right) \right) \right) = +\infty.$$

Since $f(1) = 1$ and $p \geqslant 2$, we have

$$\frac{1}{2} \leqslant (1-p^{-1}) \sum_{k \geqslant 0} f(p^k) p^{-k} \leqslant C,$$

and so, the above condition can be written as

$$\lim_{y \to +\infty} \sum_{p \leqslant y-1} \left(\left((1-p^{-1}) \sum_{k \geqslant 0} f(p^k) p^{-k} \right) - \left((1-p^{-1}) \sum_{0 \leqslant k} f(p^k)^{1/2} p^{-k} \right)^2 \right) = +\infty,$$

all the terms being non-negative. ■

2) The second step is to prove that

$$((1-p^{-1}) \sum_{0 \leqslant k} f(p^k)^{1/2} p^{-k})^2 - (1-p^{-1})(1+f(p)^{1/2} p^{-1})^2 = O(p^{-2} + \sum_{2 \leqslant k} f(p^k) p^{-k}),$$

with a uniform $O()$.

Proof. Since

$$((1-p^{-1})\sum_{0\leqslant k} f(p^k)^{1/2}p^{-k}) - (1-p^{-1})(1+f(p)^{1/2}p^{-1}) = (1-p^{-1})\sum_{2\leqslant k} f(p^k)^{1/2}p^{-k},$$

we have

$$\left((1-p^{-1})\sum_{0\leqslant k} f(p^k)^{1/2}p^{-k}\right)^2 - \Big((1-p^{-1})(1+f(p)^{1/2}p^{-1})\Big)^2$$
$$= \left((1-p^{-1})\sum_{2\leqslant k} f(p^k)^{1/2}p^{-k}\right)$$
$$\times\left(((1-p^{-1})\sum_{0\leqslant k} f(p^k)^{1/2}p^{-k}) + (1-p^{-1})(1+f(p)^{1/2}p^{-1})\right).$$

We have

$$f(p^k)^{1/2}p^{-k} = \Big(f(p^k)^{1/2}p^{-k/2}\Big)\times\Big(p^{-k/2}\Big) \leqslant (1/2)\left(\Big(f(p^k)^{1/2}p^{-k/2}\Big)^2 + \Big(p^{-k/2}\Big)^2\right),$$

and so, we get that

$$f(p^k)^{1/2}p^{-k} \leqslant (1/2)\left(f(p^k)p^{-k} + p^{-k}\right),$$

which implies that

$$(1-p^{-1})\sum_{2\leqslant k} f(p^k)^{1/2}p^{-k} \leqslant (1/2)\,(1-p^{-1})(\sum_{2\leqslant k} f(p^k)p^{-k} + \sum_{2\leqslant k} p^{-k})$$
$$\leqslant \sum_{2\leqslant k} f(p^k)p^{-k} + p^{-2}$$

since $\sum_{2\leqslant k} p^{-k} \leqslant 2p^{-2}$.

Now, we remark that

$$((1-p^{-1})\sum_{0\leqslant k} f(p^k)^{1/2}p^{-k}) + (1-p^{-1})(1+f(p)^{1/2}p^{-1}) \leqslant 2((1-p^{-1})\sum_{0\leqslant k} f(p^k)^{1/2}p^{-k}),$$

and using Cauchy inequality, we have

$$2(1-p^{-1})\sum_{0\leqslant k} f(p^k)^{1/2}p^{-k} \leqslant 2(1-p^{-1})\sqrt{\left(\sum_{0\leqslant k} f(p^k)p^{-k}\right)\left(\sum_{0\leqslant k} p^{-k}\right)}$$
$$= 2(1-p^{-1})\sqrt{\left(\sum_{0\leqslant k} f(p^k)p^{-k}\right)(1-p^{-1})^{-1}}$$
$$= 2\sqrt{\left(\sum_{0\leqslant k} f(p^k)p^{-k}\right)(1-p^{-1})} \leqslant 2\sqrt{C}.$$

As a consequence, we obtain that

$$\left((1-p^{-1})\sum_{0\leqslant k} f(p^k)^{1/2}p^{-k}\right)^2 - \Big((1-p^{-1})(1+f(p)^{1/2}p^{-1})\Big)^2$$
$$\leqslant \left(2\sqrt{C}\right)\left(\sum_{2\leqslant k} f(p^k)p^{-k}+p^{-2}\right). \qquad \blacksquare$$

3) We end the proof of the Lemma.

Proof. From 1), we know that the measure ν_f is orthogonal to ν_1 if and only if

$$\lim_{y\to+\infty}\sum_{p\leqslant y-1}\left(\left((1-p^{-1})\sum_{k\geqslant 0} f(p^k)p^{-k}\right) - \left((1-p^{-1})\sum_{0\leqslant k} f(p^k)^{1/2}p^{-k}\right)^2\right) = +\infty.$$

Now, by 2), we have

$$\left((1-p^{-1})\sum_{0\leqslant k} f(p^k)^{1/2}p^{-k}\right)^2 = \Big((1-p^{-1})(1+f(p)^{1/2}p^{-1})\Big)^2$$
$$+\,O\left(\sum_{2\leqslant k} f(p^k)p^{-k}+p^{-2}\right),$$

and as a consequence, we obtain

$$\left(\left((1-p^{-1})\sum_{k\geqslant 0} f(p^k)p^{-k}\right) - \left((1-p^{-1})\sum_{0\leqslant k} f(p^k)^{1/2}p^{-k}\right)^2\right)$$
$$= \Big((1-p^{-1})(1+f(p)p^{-1}) - ((1-p^{-1})(1+f(p)^{1/2}p^{-1}))^2\Big) + \left(\sum_{2\leqslant k} f(p^k)p^{-k}\right)$$
$$+\,O\left(\sum_{2\leqslant k} f(p^k)p^{-k}+p^{-2}\right)$$
$$= \Big((1-p^{-1})(1+f(p)p^{-1}) - ((1-p^{-1})(1+f(p)^{1/2}p^{-1}))^2\Big)$$
$$+\,O\left(\sum_{2\leqslant k} f(p^k)p^{-k}+p^{-2}\right).$$

Recalling that $\sum_{p\in P}\sum_{k\geqslant 2} f(p^k)p^{-k}$ and $\sum_{p\in P} p^{-2}$ are finite, we get

$$\lim_{y\to+\infty}\sum_{p\leqslant y-1}\left(\left((1-p^{-1})\sum_{k\geqslant 0} f(p^k)p^{-k}\right)-\left((1-p^{-1})\sum_{0\leqslant k} f(p^k)^{1/2}p^{-k}\right)^2\right)=+\infty$$

if and only if

$$\lim_{y\to+\infty}\sum_{p\leqslant y-1}\left|(1-p^{-1})(1+f(p)p^{-1})-\Big((1-p^{-1})(1+f(p)^{1/2}p^{-1})\Big)^2\right|=+\infty,$$

i.e.

$$\lim_{y\to+\infty}\sum_{p\leqslant y-1}\left|(1+f(p)p^{-1})-(1-p^{-1})(1+f(p)^{1/2}p^{-1})^2\right|=+\infty.$$

Now, we have

$$\begin{aligned}
&\left|(1+f(p)p^{-1})-(1-p^{-1})(1+f(p)^{1/2}p^{-1})^2\right|\\
&=\left|(1+f(p)p^{-1})-(1-p^{-1})(1+2f(p)^{1/2}p^{-1}+f(p)p^{-2})\right|\\
&=\left|(1+f(p)p^{-1})-(1+2f(p)^{1/2}p^{-1}+f(p)p^{-2})+(1+2f(p)^{1/2}p^{-1}+f(p)p^{-2})p^{-1}\right|\\
&=\left|1+f(p)p^{-1}-1-2f(p)^{1/2}p^{-1}-f(p)p^{-2}+p^{-1}+2f(p)^{1/2}p^{-2}+f(p)p^{-3}\right|\\
&=\left|f(p)p^{-1}-2f(p)^{1/2}p^{-1}+p^{-1}-f(p)p^{-2}+2f(p)^{1/2}p^{-2}+f(p)p^{-3}\right|\\
&=\left|(1-f(p)^{1/2})^2p^{-1}+2f(p)^{1/2}p^{-2}-f(p)p^{-2}+f(p)p^{-3}\right|\\
&=\left|(1-f(p)^{1/2})^2p^{-1}+(-p^{-2}+2f(p)^{1/2}p^{-2}-f(p)p^{-2})+(p^{-2}+f(p)p^{-3})\right|\\
&=\left|(1-f(p)^{1/2})^2p^{-1}-(1-f(p)^{1/2})^2p^{-2}+(p^{-2}+f(p)p^{-3})\right|\\
&=(1-f(p)^{1/2})^2p^{-1}(1-p^{-1})+O(p^{-2}+f(p)p^{-3}).
\end{aligned}$$

$\sum_{p\in P} p^{-2}$ is finite, and since $f(p)=O(p)$, $\sum_{p\in P} f(p)p^{-3}$ also is finite.

As a consequence, we get that

$$\lim_{y\to+\infty}\sum_{p\leqslant y-1}\left|((1+f(p)p^{-1})-(1-p^{-1})(1+f(p)^{1/2}p^{-1})^2\right|=+\infty$$

if and only if

$$\lim_{y\to+\infty}\sum_{p\leqslant y-1}(1-f(p)^{1/2})^2p^{-1}=+\infty.$$

This ends the proof of the Lemma. ∎

Proposition 10. Let A be a Borel set, f in $\mathcal{H}$. Then, there exists a subset C of N of density $\nu_1(A)$ such that

$$\nu_f(A)M(f) = \lim_{x\to+\infty} x^{-1} \sum_{\substack{1\leqslant n\leqslant x\\ n\in C}} f(n).$$

Proof. 1) First of all, we prove shall that there exists a sequence X_k of elements of N and a sequence of subsets O_k of N, $k \in N$, such that if $X \geqslant X_k$, the inequalities

$$\left| X^{-1} \sum_{n\leqslant X} I_{O_k}(n) - \nu_1(A)\right| \leqslant (2k)^{-2}$$

and

$$\left| X^{-1} \sum_{n\leqslant X} f(n) I_{O_k}(n) - M(f)\nu_f(A)\right| \leqslant (2k)^{-2})$$

hold. We begin with the simple remark that for any p in P, $\nu_1(\{p^\infty\}\prod_{\substack{q\in P\\ q\neq p}} \overline{E}_q)$ and $\nu_f(\{p^\infty\}\prod_{\substack{q\in P\\ q\neq p}} \overline{E}_q)$ are equal to zero.

Now, since A is a Borel set, by Lusin criterion ([3], p. 68, (vii)), there exists a sequence $A_{k,1}$ (resp. $A_{k,f}$) of open sets of $\bar{E}$ such that $A \subseteq A_{k,1}$ (resp. $A \subseteq A_{k,f}$) and $0 \leqslant \nu_1(A_{k,1}) - \nu_1(A) \leqslant (4k)^{-2}$ (resp. $0 \leqslant M(f)\,(\nu_f(A_{k,f}) - \nu_f(A)) \leqslant (4k)^{-2}$, for $0 \leqslant M(f) < +\infty$). Since $A \subseteq A_{k,1}$ and $A \subseteq A_{k,f}$, we know that $A \subseteq A_k$, where A_k is defined as $A_k = A_{k,1} \cap A_{k,\,f}$ and moreover, that $0 \leqslant \nu_1(A_k) - \nu_1(A) \leqslant (4k)^{-2}$ (resp. $0 \leqslant M(f)\,(\nu_f(A_k) - \nu_f(A)) \leqslant (4k)^{-2}$). Now, by Lemma 5 , there exists A'_k (resp. A''_k) a finite union of elementary subsets of $\overline{E}$ such that $A'_k \subseteq A_k$ (resp. $A''_k \subseteq A_k$) and $0 \leqslant \nu_1(A_k) - \nu_1(A'_k) \leqslant (4k)^{-2}$ (resp. $0 \leqslant M(f)\,(\nu_f(A_k) - \nu_f(A''_k)) \leqslant (4k)^{-2}$). Let O_k be defined by $O_k = A'_k \cup A''_k$. By Lemma 6, we know that O_k is a finite disjoint union of elementary subsets of $\overline{E}$, and that $A'_k \subseteq O_k \subseteq A_k$, which implies that $0 \leqslant \nu_1(A_k) - \nu_1(O_k) \leqslant (4k)^{-2}$, and similarly, $0 \leqslant M(f)\,(\nu_f(A_k) - \nu_f(O_k)) \leqslant (4k)^{-2}$. Since $0 \leqslant \nu_1(A_k) - \nu_1(A) \leqslant (4k)^{-2}$ and $0 \leqslant M(f)\,(\nu_f(A_k) - \nu_f(A)) \leqslant (4k)^{-2}$, we deduce that $|\nu_1(O_k) - \nu_1(A)| \leqslant 2^{-3}k^{-2}$ and $M(f)\,|\nu_f(O_k) - \nu_f(A)| \leqslant 2^{-3}k^{-2}$.

Now, since O_k is a finite disjoint union of elementary subsets of $\overline{E}$, by Lemma 7, $I_{O_k}(n)$ and $f(n)I_{O_k}(n)$ have a meanvalue, respectively $\nu_1(O_k)$ and $M(f)\nu_f(O_k)$, and moreover, there exists an integer $X_k(1)$ (resp. $X_k(f)$) such that for any $X \geqslant X_k(1)$ (resp. $X \geqslant X_k(f)$), the inequalities

$$\left| X^{-1} \sum_{n\leqslant X} I_{O_k}(n) - \nu_1(O_k)\right| \leqslant (4k)^{-2}$$

$$\left| X^{-1} \sum_{n\leqslant X} f(n) I_{O_k}(n) - M(f)\nu_f(O_k)\right| \leqslant (4k)^{-2}$$

hold. Now, since $0 \leqslant |\nu_1(A) - \nu_1(O_k)| \leqslant 2^{-1}(2k)^{-2}$ and $0 \leqslant M(f)\,|\nu_f(O_k) - \nu_f(A)| \leqslant 2^{-1}(2k)^{-2}$, we get that

$$\left| X^{-1} \sum_{n \leqslant X} I_{O_k}(n) - \nu_1(A) \right| \leqslant (2k)^{-2}$$

and

$$\left| X^{-1} \sum_{n \leqslant X} f(n) I_{O_k}(n) - M(f)\nu_f(A) \right| \leqslant (2k)^{-2}.$$

These inequalities hold a fortiori if $X \geqslant X_k$, where X_k is defined as $X_k = Max(X_k(1), X_k(f))$.

2) End of the proof of the Proposition.

Let Y_k, $k \geqslant 1$, be defined by $Y_k = \sum_{1 \leqslant r \leqslant k+1} X_k$, and C be a subset of N with characteristic function $I(n)$ defined by

$$I(n) = I_{O_k}(n) \quad \text{if } Y_{k-1} + 1 \leqslant n \leqslant Y_k,\ k \geqslant 2,$$
$$I(n) = 1 \quad \text{if } 1 \leqslant n \leqslant Y_1.$$

We shall prove that $M(I) = \nu_1(A)$ and $M(If) = \nu_f(A)M(f)$.

Let x be a positive integer, and let $k(x) = \max\{k \geqslant 0; Y_k + 1 \leqslant x \leqslant Y_{k+1}\}$. We denote by $g(n)$ any of the functions $f(n)$ or 1. We have

$$\begin{aligned} \sum_{n \leqslant x} I(n)g(n) &= \sum_{0 \leqslant k \leqslant k(x)-1} \; \sum_{Y_k+1 \leqslant n \leqslant Y_{k+1}} I(n)g(n) + \sum_{Y_{k(x)}+1 \leqslant n \leqslant x} I(n)g(n) \\ &= \sum_{0 \leqslant k \leqslant k(x)-1} \; \sum_{Y_k+1 \leqslant n \leqslant Y_{k+1}} I_{O_k}(n)g(n) + \sum_{Y_{k(x)}+1 \leqslant n \leqslant x} I_{O_{k(x)}}(n)g(n). \end{aligned}$$

Since $Y_k \geqslant X_k$, we obtain using 1),

$$\left| \sum_{1 \leqslant n \leqslant Y_{k+1}} I(n)g(n) - Y_{k+1} M(g)\nu_g(A) \right| \leqslant Y_{k+1}(2k)^{-2},$$
$$\left| \sum_{1 \leqslant n \leqslant Y_k} I(n)g(n) - Y_k M(g)\nu_g(A) \right| \leqslant Y_k(2k)^{-2},$$

and as a consequence, since

$$\sum_{Y_k+1 \leqslant n \leqslant Y_{k+1}} I(n)g(n) = \sum_{1 \leqslant n \leqslant Y_{k+1}} I(n)g(n) - \sum_{1 \leqslant n \leqslant Y_k} I(n)g(n),$$

we get

$$\left|\sum_{Y_k+1\leqslant n\leqslant Y_{k+1}} I(n)g(n)-(Y_{k+1}-Y_k)M(g)\nu_g(A)\right|\leqslant (Y_{k+1}+Y_k)(2k)^{-2}\leqslant 2Y_{k+1}(2k)^{-2}$$
$$\leqslant Y_{k+1}(k)^{-2}.$$

And similarly,

$$\left|\sum_{Y_{k(x)}+1\leqslant n\leqslant x} I(n)g(n)-(x-Y_{k(x)})M(g)\nu_g(A)\right|\leqslant (x+Y_{k(x)})(2k(x))^{-2}\leqslant 2x(2k(x))^{-2}$$
$$\leqslant xk(x)^{-2}.$$

Hence, we get that

$$\left|\sum_{n\leqslant x} I(n)g(n)-(\sum_{0\leqslant k\leqslant k(x)-1}(Y_{k+1}-Y_k)M(g)\nu_g(A)+(x-Y_{k(x)})M(g)\nu_g(A))\right|$$
$$\leqslant \sum_{1\leqslant k\leqslant k(x)-1} Y_{k+1}k^{-2}+xk(x)^{-2},$$

i.e.

$$\left|\sum_{n\leqslant x} I(n)g(n)-xM(g)\nu_g(A))\right|\leqslant \sum_{1\leqslant k\leqslant k(x)-1} Y_{k+1}k^{-2}+xk(x)^{-2}.$$

We remark that the right hand side of this inequality can be written as

$$\sum_{1\leqslant k\leqslant k(x)-1} Y_{k+1}k^{-2}+xk(x)^{-2}$$
$$=\sum_{1\leqslant k\leqslant k(\sqrt{x})-1} Y_{k+1}k^{-2}+\sum_{k(\sqrt{x})\leqslant k\leqslant k(x)-1} Y_{k+1}k^{-2}+xk(x)^{-2}.$$

Since for $k\leqslant k(\sqrt{x})-1$, we have $Y_{k+1}\leqslant\sqrt{x}$, we get

$$\sum_{1\leqslant k\leqslant k(\sqrt{x})-1} Y_{k+1}k^{-2}\leqslant \sum_{1\leqslant k\leqslant k(\sqrt{x})-1}\sqrt{x}k^{-2}\leqslant\sqrt{x}\sum_{1\leqslant k}k^{-2},$$

and since $Y_{k+1}\leqslant x$, we have

$$\sum_{k(\sqrt{x})\leqslant k\leqslant k(x)-1} Y_{k+1}k^{-2}+xk(x)^{-2}\leqslant \sum_{k(\sqrt{x})\leqslant k\leqslant k(x)-1} xk^{-2}+xk(x)^{-2}$$
$$\leqslant x\sum_{k(\sqrt{x})\leqslant k\leqslant k(x)} k^{-2}.$$

Hence, we get that

$$\left|\sum_{n\leqslant x} I(n)g(n) - xM(g)\nu_g(A))\right| \leqslant \sqrt{x}\sum_{1\leqslant k} k^{-2} + x \sum_{k(\sqrt{x})\leqslant k\leqslant k(x)} k^{-2}$$
$$= O(\sqrt{x}) + xo(1), \qquad x \to +\infty,$$

and so, we obtain

$$\sum_{n\leqslant x} I(n)g(n) - xM(g)\nu_g(A)) = o(x), x \to +\infty.$$

Substituting $g(n)$ with 1 and then with $f(n)$, we see that the set C with characteristic function $I(n)$ as defined above, fulfils the conditions of the Proposition. ■

2.3. Conclusion of the proof of the main result. By Lemma 9, the measure ν_f is orthogonal to ν_1. So, there exists a Borel set A such that $\nu_f(A) = 1$ and $\nu_1(A) = 0$. Proposition10 gives the conclusion. ■

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