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POSITIVELY CURVED SHRINKING RICCI SOLITONS ARE COMPACT

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Abstract

We show that a shrinking Ricci soliton with positive sectional curvature must be compact. This extends a result of Perelman in dimension three and improves a result of Naber in dimension four, respectively.

1. Introduction

In studying the singularities of Ricci flows, Hamilton [**13**] has introduced the concept of Ricci solitons and signified their importance. The importance was further demonstrated in the work of Perelman [**21**, **22**], where his classification result of three dimensional shrinking gradient Ricci solitons played a crucial role in the affirmative resolution of the Poincaré conjecture. Recall that a complete Riemannian manifold (M, g) is a shrinking gradient Ricci soliton if the equation

$$\mathrm{Ric} + \mathrm{Hess}\,(f) = \frac{1}{2}\, g$$

holds for some function f. Here, Ric is the Ricci curvature of (M, g) and $\mathrm{Hess}\,(f)$ the Hessian of f.

The importance of shrinking Ricci solitons can be partially seen through a conjecture attributed to Hamilton. The conjecture asserts that the blow-ups around a type-I singularity point of a Ricci flow (see [**9**] for definition) always converge to (nontrivial) gradient shrinking Ricci solitons. Important progress toward the conjecture was made in [**18**, **24**], where it was shown that blow-up limits must be gradient shrinking Ricci solitons. The nontriviality issue, which was explicitly raised by Cao [**3**], was later taken up by Enders, Müller and Topping [**12**]. See also [**7**].

Two dimensional shrinking Ricci solitons have been classified by Hamilton [**13**]. For the three dimensional case, Perelman [**22**] made the breakthrough and concluded that a noncollapsing shrinking gradient Ricci soliton with bounded curvature must be a quotient of the

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sphere $\mathbb{S}^3$, or $\mathbb{R}^3$, or $\mathbb{S}^2 \times \mathbb{R}$. The result was later shown to be true without any extra assumptions through the effort of [**5**, **20**] by a different approach. See also [**18**, **23**] for related works.

For a three dimensional shrinking gradient Ricci soliton with bounded curvature, according to Ivey [**16**] and Hamilton [**13**], its sectional curvature is necessarily nonnegative (see [**8**] for a more general result). On the other hand, the strong maximum principle in [**15**] implies that the universal cover of such a soliton must split off a line if its curvature is not strictly positive. Together with the fact that a compact three dimensional gradient shrinking Ricci soliton is necessarily covered by the standard sphere [**14**, **16**] (see [**11**] for a different proof), one sees that the classification result of Perelman can be deduced from the following theorem.

Theorem 1 (Perelman [**22**])**.** *Let (M, g) be a three dimensional noncollapsing gradient shrinking Ricci soliton with positive and bounded sectional curvature. Then (M, g) must be compact.*

A natural question, as has been raised by Cao [**4**], is whether the preceding theorem actually holds true in full generality for all dimensions. The main purpose of this short note is to provide a complete answer to this question.

Theorem 2. *Let (M, g) be an n dimensional gradient shrinking Ricci soliton with nonnegative sectional curvature and positive Ricci curvature. Then (M, g) must be compact.*

Notice that we do not require any extra assumptions such as the manifold is noncollapsing or the curvature is bounded. Recall that a simply connected shrinking gradient Ricci soliton M with nonnegative sectional curvature is necessarily of the form $M = \mathbb{R}^k \times N$ with N having positive Ricci curvature. This was stated as Corollary 4 in [**23**] under an integrability assumption on the Ricci curvature. However, according to [**17**], this assumption automatically holds true for all shrinking gradient Ricci solitons. Combining this with Theorem 2, we arrived at the following conclusion.

Corollary 3. *Let (M, g) be an n dimensional shrinking gradient Ricci soliton with nonnegative sectional curvature. Then (M, g) must be compact, or a quotient of $\mathbb{R}^n$ or of the product $\mathbb{R}^k \times N^{n-k}$ with $1 \leq k \leq n-2$, where N is a compact simply connected shrinking Ricci soliton of dimension $n - k$ with positive Ricci curvature.*

Since a compact simply connected shrinking gradient Ricci soliton with positive curvature operator must be the round sphere by Böhm and Wilking [**1**], the following corollary is immediate in view of Theorem 7.34 in [**9**]. This also answers a question stated in [**9**] on page 389.

Corollary 4. *Let* (M, g) *be an* n *dimensional shrinking gradient Ricci soliton with nonnegative curvature operator. Then* (M, g) *must be a quotient of a closed symmetric space* N^n*, or* $\mathbb{R}^n$*, or their product* $\mathbb{R}^k \times N^{n-k}$ *with* $1 \leq k \leq n-2$*.*

In [**18**], Naber was able to obtain a partial extension of Perelman's result to four dimensional shrinking gradient Ricci solitons with positive curvature operator. As a consequence, he also proved Corollary 4 for the case $n = 4$ under the further condition that the curvature is bounded. However, in both [**22**] and [**18**], the proofs seem to be dimension specific and it is not at all clear to us if they may be generalized to deal with arbitrary dimension case. Our proof is very much inspired by [**10**], where the authors obtained a quadratic decay positive lower bound for the scalar curvature of nontrivial, noncompact shrinking Ricci solitons. Here, we managed to obtain a similar estimate for the Ricci curvature, which in turn implies that the scalar curvature must increase and become unbounded at infinity. This last fact leads to a contradiction and the desired conclusion.

There exist quite a few related results in literature. Apart from those already been mentioned, the work of Ni [**19**] provides a classification of shrinking Kähler Ricci solitons with nonnegative bisectional curvature. In a recent work of Cai [**2**], he established a classification result for shrinking Ricci solitons under the assumption that the sectional curvature is nonnegative and bounded, and the covariant derivative of the Ricci curvature tensor decays exponentially.

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2. Proof of Theorem 2

Proof. Let us assume for the sake of contradiction that M is noncompact. We begin by recalling some important features of gradient shrinking Ricci solitons. It is known [**13**] that $S + |\nabla f|^2 - f$ is constant on M, where S is the scalar curvature of M. So, by adding a constant to f if necessary, we may normalize the soliton such that

$$S + |\nabla f|^2 = f. \tag{1}$$

Moreover, according to [**8**], $S > 0$ unless M is flat. Concerning the potential function f, Cao and Zhou [**6**] have proved that

$$\left(\frac{1}{2} r(x) - c\right)^2 \leq f(x) \leq \left(\frac{1}{2} r(x) + c\right)^2, \tag{2}$$

for all $r(x) \geq r_0$ with c and r_0 both only depending on n. Here $r(x) := d(p, x)$ is the distance of x to p, a minimum point of f on M, which always exists. Also, the Ricci curvature R_{ij} of (M, g) verifies the following differential equations.

$$\Delta_f R_{ij} = R_{ij} - 2R_{ikjl}R_{kl}, \tag{3}$$

where $\Delta_f := \Delta - \langle \nabla f, \rangle$ is the so-called weighted Laplacian and R_{ikjl} the curvature tensor. Let $\lambda(x) > 0$ be the smallest eigenvalue of Ric. We note that if v is an eigenvector corresponding to λ, then

$$R_{ikjl}R_{kl}v_iv_j = \mathrm{Rm}\,(v, e_k, v, e_l)\, R_{kl},$$

where Rm denotes the Riemann curvature tensor. Diagonalizing the Ricci tensor so that $R_{kl} = \lambda_k \delta_{kl}$, it follows from above that

$$R_{ikjl}R_{kl}v_iv_j = K\,(v, e_l)\,\lambda_l \geq 0, \tag{4}$$

where $K\,(X, Y)$ is the sectional curvature of the plane spanned by X and Y. The inequality above is true because we assumed $K\,(X, Y) \geq 0$, for any X, Y. From (3) and (4) it follows that λ satisfies the following differential inequality in the sense of barriers.

$$\Delta_f \lambda \leq \lambda. \tag{5}$$

For a fixed geodesic ball $B_p(r_0)$ of radius r_0 large enough, let

$$a := \inf_{\partial B_p(r_0)} \lambda > 0. \tag{6}$$

We now adapt an idea from [**10**] to obtain a global lower bound for λ. The argument uses maximum principle applied to (5).

Define the function

$$u := \lambda - af^{-1} - naf^{-2}. \tag{7}$$

By (6) and (7), it follows that if r_0 is large enough depending on dimension,

$$u > 0 \quad \text{on} \quad \partial B_p(r_0). \tag{8}$$

On $M \backslash B_p(r_0)$ we have

$$\begin{aligned} \Delta_f\left(f^{-1}\right) &= -\Delta_f(f)\, f^{-2} + 2\left|\nabla f\right|^2 f^{-3} \\ &\geq \left(f - \frac{n}{2}\right) f^{-2} + 2f^{-2} \\ &= f^{-1} - \frac{n}{2} f^{-2}, \end{aligned}$$

and

$$\begin{aligned} \Delta_f\left(f^{-2}\right) &= 2\left(f - \frac{n}{2}\right) f^{-3} + 6\left|\nabla f\right|^2 f^{-4} \\ &\geq \frac{3}{2} f^{-2}. \end{aligned}$$

Here we have used the fact that $\Delta_f\, f = \frac{n}{2} - f$.

Therefore, using this in (7) and combining with (5) imply

$$\begin{aligned}\Delta_f u &\leq \lambda - af^{-1} + a\frac{n}{2}f^{-2} - \frac{3}{2}naf^{-2} \\ &= u.\end{aligned}$$

We have thus established that

$$\Delta_f u \leq u \text{ on } M \backslash B_p(r_0). \tag{9}$$

We now claim that $u \geq 0$ on $M \backslash B_p(r_0)$. Suppose this is not true. Then there exists $x_0 \in M \backslash B_p(r_0)$ so that $u(x_0) < 0$. Since by (8) we have $u > 0$ on $\partial B_p(r_0)$ and u is obviously nonnegative at infinity, it follows that u achieves its minimum in the interior of $M \backslash B_p(r_0)$. Furthermore, $u < 0$ at this minimum point. By the maximum principle, this contradicts with (9). Thus, we conclude that $u \geq 0$ on $M \backslash B_p(r_0)$. So there exists some constant $0 < b \leq 1$ such that

$$\text{Ric} \geq \frac{b}{f} \text{ on } M. \tag{10}$$

Note that, as (M, g) is not flat, (1) implies that $f > 0$ on M. We use (10) to show that the scalar curvature on $M \backslash B_p(r_0)$ must satisfy

$$S > b \ln f. \tag{11}$$

Suppose by contradiction that there exists a point $x \in M \backslash B_p(r_0)$ where

$$S(x) \leq b \ln f(x). \tag{12}$$

Let us consider $\sigma(\eta)$, where $\eta \geq 0$, to be the integral curve of $-\frac{\nabla f}{|\nabla f|^2}$, such that $\sigma(0) = x$. Since $0 < b \leq 1$, by (1) and (12) we have

$$\begin{aligned}|\nabla f|^2(x) &= f(x) - S(x) \\ &\geq f(x) - b\ln f(x) \\ &\geq 1.\end{aligned} \tag{13}$$

Hence, the flow $\sigma(\eta)$ is defined at least in a neighborhood of x. Since $\frac{d}{d\eta}f(\sigma(\eta)) = -1$, it results that

$$f(\sigma(\eta)) = t - \eta,$$

where $t := f(x)$. Using (10), we have that

$$\begin{aligned}\frac{d}{d\eta}S(\sigma(\eta)) &= -\frac{\langle \nabla S, \nabla f\rangle}{|\nabla f|^2} \\ &= -2\text{Ric}\left(\frac{\nabla f}{|\nabla f|}, \frac{\nabla f}{|\nabla f|}\right) \\ &\leq -\frac{2b}{f(\sigma(\eta))} \\ &= -\frac{2b}{t-\eta},\end{aligned}$$

where we have used the known relation that $\nabla S = 2\mathrm{Ric}\,(\nabla f)$. Integrating this differential inequality, we find that

$$S\,(x) - S\,(\sigma\,(\eta)) \geq 2b\,\ln t - 2b\,\ln\,(t-\eta)\,. \tag{14}$$

Since initially $S\,(x) \leq b\,\ln t$, it follows that

$$\begin{aligned} S\,(\sigma\,(\eta)) &\leq 2b\,\ln\,(t-\eta) - b\,\ln t \\ &< b\,\ln\,(f\,(\sigma\,(\eta)))\,. \end{aligned}$$

Since $S\,(y) \leq b\,\ln f\,(y)$ holds true along $\sigma\,(\eta)$ and $b \leq 1$, we get as in (13) that $|\nabla f|\,(\sigma\,(\eta)) \geq 1$ for all $0 \leq \eta \leq t-1$. Hence, $\sigma\,(\eta)$ exists at least for $0 \leq \eta \leq t-1$, and

$$S\,(\sigma\,(t-1)) < b\,\ln f\,(\sigma(t-1)) = 0.$$

This is a contradiction. In conclusion, (11) is true for all x.

Now we recall that by [**6**], for all $r > 0$,

$$\int_{B_p(r)} S \leq c\,(n)\,\mathrm{Vol}\,(B_p\,(r))\,.$$

For any q with $d(p,q) = \frac{3}{4}\,r$, it follows from (11) and (2) that

$$\int_{B_p(r)} S \geq \int_{B_q\left(\frac{r}{4}\right)} S > b\,\ln(\frac{r}{4} - c)^2\,\mathrm{Vol}\left(B_q\left(\frac{r}{4}\right)\right). \tag{15}$$

Now Bishop–Gromov relative volume comparison implies that

$$c\,(n)\,\mathrm{Vol}\left(B_q\left(\frac{r}{4}\right)\right) \geq \mathrm{Vol}\,(B_q\,(2r)) \geq \mathrm{Vol}\,(B_p\,(r))\,. \tag{16}$$

By (15) and (16) we infer that $\ln r \leq \frac{c(n)}{b}$, which is a contradiction to M being noncompact.

The theorem is proved. q.e.d.

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