

ON NORMALIZED DIFFERENTIALS ON HYPERELLIPTIC CURVES OF INFINITE GENUS

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Abstract

We develop a new approach for constructing normalized differentials on hyperelliptic curves of infinite genus and obtain uniform asymptotic estimates for the distribution of their zeros.

1. Introduction

Much of the analysis of closed Riemann surfaces is based on the Riemann bilinear relation. Given a canonical basis $A_1, B_1, \dots, A_g, B_g$, of the homology group $H_1(X, \mathbb{Z})$ of a closed Riemann surface X of genus g , it reads

$$\int_X \omega \wedge \eta = \sum_{m=1}^g \left(\int_{A_m} \omega \int_{B_m} \eta - \int_{A_m} \eta \int_{B_m} \omega \right),$$

where ω and η are arbitrarily chosen smooth closed 1-forms on X . As a consequence one obtains the following vanishing theorem: for any holomorphic 1-form ω on X with vanishing A -periods, $\int_{A_m} \omega = 0$ $\forall 1 \leq m \leq g$, one has $-\frac{1}{2i} \int_X \omega \wedge \bar{\omega} = 0$, and hence $\omega = 0$. Furthermore, it follows from Hodge theory that the space of holomorphic differentials on X is a complex vector space of dimension g admitting a basis $\omega_1, \dots, \omega_g$ such that $\int_{A_m} \omega_n = \delta_{mn}$ for any $1 \leq m, n \leq g$. By the vanishing theorem, such a basis is unique. The period matrix $B_X = \left(\int_{B_m} \omega_n \right)_{1 \leq m, n \leq g}$, which enters the definition of the theta function associated to X , is known to be symmetric and has the property that $\text{Im } B_X$ is sign definite.

However, for many applications to integrable PDEs one needs to consider *open* Riemann surfaces of *infinite genus*, a subject pioneered by Ahlfors and Nevanlinna – see the monographs [2] and [4] as well as references therein. Unfortunately, it is not sufficiently developed for our purposes. In particular for applications to the focusing nonlinear Schrödinger (NLS) equation we need to establish a vanishing theorem for holomorphic 1-forms on two sheeted open Riemann surfaces of infinite genus which are not necessarily L^2 -integrable. In view of these

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applications we formulate our results for the specific Riemann surfaces involved. However, our method is quite general and can be directly applied for studying Riemann surfaces related to other non-linear equations.

Consider the NLS system of equations in one space dimension with periodic boundary conditions,

$$(1) \quad \begin{cases} i\partial_t\varphi_1 = -\partial_x^2\varphi_1 + 2\varphi_1^2\varphi_2, \\ -i\partial_t\varphi_2 = -\partial_x^2\varphi_2 + 2\varphi_2^2\varphi_1, \end{cases}$$

where $\varphi = (\varphi_1, \varphi_2)$ is in $L_c^2 := L^2(\mathbb{T}, \mathbb{C}) \times L^2(\mathbb{T}, \mathbb{C})$ and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. If $\varphi_2 = \overline{\varphi_1}$ [$\varphi_2 = -\overline{\varphi_1}$], the system is referred to as defocusing [focusing] NLS equation. By Zakharov and Shabat [21] the NLS system admits a Lax pair formulation, $\partial_t L = [B, L]$ where $L(\varphi)$ denotes the Zakharov–Shabat (ZS) operator

$$L(\varphi) := i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & \varphi_1 \\ \varphi_2 & 0 \end{pmatrix}.$$

Associated to this operator is the curve

$$\mathcal{C}_\varphi := \{(\lambda, w) \in \mathbb{C}^2 \mid w^2 = \Delta^2(\lambda, \varphi) - 4\},$$

where $\Delta(\lambda, \varphi)$ is the discriminant of $L(\varphi)$ (cf. Section 2). It is known (see, e.g., [5]) that for any given $\varphi \in L_c^2$, the entire function $\Delta^2(\lambda, \varphi) - 4$ vanishes at $\lambda \in \mathbb{C}$ iff λ is a periodic eigenvalue of $L(\varphi)$, i.e., an eigenvalue of $L(\varphi)$, considered on the interval $[0, 2]$ with periodic boundary conditions. In addition, the algebraic multiplicity of λ as a root of $\Delta(\lambda, \varphi)^2 - 4$ coincides with the algebraic multiplicity of it as a periodic eigenvalue of $L(\varphi)$ (see [10]). Note that $L(\varphi)$ has a compact resolvent and hence its spectrum is discrete. In this paper we consider potentials φ in L_c^2 so that each periodic eigenvalue λ of $L(\varphi)$ has algebraic multiplicity $m(\lambda)$ at most two (see Remark 1.5 for a discussion of the general case). Denote the subset of such elements by L_\bullet^2 . Then L_\bullet^2 is open, dense, and contains the zero potential ([10]). Note that in the case of the defocusing NLS equation, $L(\varphi)$ is self-adjoint and the algebraic multiplicity of an eigenvalue coincides with its geometric multiplicity which is at most two, and hence $\varphi \in L_\bullet^2$. For an arbitrary $\varphi \in L_\bullet^2$, the periodic eigenvalues of $L(\varphi)$ can be listed as a sequence of distinct pairs, $\lambda_k^-(\varphi)$, $\lambda_k^+(\varphi)$, $k \in \mathbb{Z}$, so that $\lambda_k^\pm(\varphi) = k\pi + \ell^2(k)$, i.e., $\sum_{k \in \mathbb{Z}} |\lambda_k^\pm(\varphi) - k\pi|^2 < \infty$, and $\lambda_k^- = \lambda_k^+$ iff λ_k^- has algebraic multiplicity two, $m(\lambda_k^-) = 2$ (Section 2).

Let Z_φ denote the subset in \mathbb{C} of periodic double eigenvalues of $L(\varphi)$,

$$Z_\varphi := \{\lambda \in \text{spec } L(\varphi) \mid m(\lambda) = 2\}.$$

Then

$$\mathcal{C}_\varphi^\bullet := \mathcal{C}_\varphi \setminus \{(\lambda, 0) \mid \lambda \in Z_\varphi\}$$

is a two sheeted open Riemann surface. Generically it is a surface of infinite genus ([10]). Our aim is to prove a vanishing theorem for holomorphic differentials on $\mathcal{C}_\varphi^\bullet$ which are not necessarily L^2 -integrable and to construct a family of normalized holomorphic differentials $\omega_n, n \in \mathbb{Z}$, on $\mathcal{C}_\varphi^\bullet$ with respect to an appropriately chosen infinite set of cycles on $\mathcal{C}_\varphi^\bullet$. In addition, we want to get asymptotic estimates of the zeros of these differentials. The cycles are defined as follows: for any given potential $\psi \in L^2_\bullet$ list its periodic eigenvalues in pairs, $\lambda_k^-, \lambda_k^+, k \in \mathbb{Z}$, as discussed above. It is shown in Section 2 that there exist an open neighborhood \mathcal{W} of ψ in L^2_\bullet and a family of simple, closed, smooth, counterclockwise oriented curves $\Gamma_m, m \in \mathbb{Z}$, so that the closures of the domains in \mathbb{C} , bounded by the Γ_m are pairwise disjoint and for any $\varphi \in \mathcal{W}$ and $m \in \mathbb{Z}$ the open domain bounded by Γ_m contains the pair $\lambda_m^-(\varphi), \lambda_m^+(\varphi)$ but no other periodic eigenvalues of $L(\varphi)$. Denote by A_m the cycle on the canonical sheet \mathcal{C}_φ^c of \mathcal{C}_φ (cf. Section 2) so that $\pi(A_m) = \Gamma_m$ where $\pi : \mathcal{C}_\varphi \rightarrow \mathbb{C}, (\lambda, w) \mapsto \lambda$. We are then looking for holomorphic differentials ω_n on $\mathcal{C}_\varphi^\bullet$ so that $\int_{A_m} \omega_n = 2\pi\delta_{mn}$ for any $m, n \in \mathbb{Z}$. In addition, we want to prove a vanishing theorem for holomorphic differentials on $\mathcal{C}_\varphi^\bullet$ with vanishing A -periods which are not necessarily L^2 -integrable. For an arbitrary entire function $\zeta : \mathbb{C} \rightarrow \mathbb{C}$ with $\zeta|_{Z_\varphi} = 0$, let

$$\omega_\zeta := \frac{\zeta(\lambda)}{\sqrt{\Delta^2(\lambda, \varphi) - 4}} d\lambda.$$

Then ω_ζ is a holomorphic 1-form on $\mathcal{C}_\varphi^\bullet$ which is locally square integrable. More precisely, for any $r > 0$,

$$V(r) := -\frac{1}{2i} \int_{X_r} \omega_\zeta \wedge \overline{\omega_\zeta} < \infty,$$

where

$$X_r := \pi^{-1}(\{\lambda \in \mathbb{C} \mid |\lambda| \leq r\}) \cap \mathcal{C}_\varphi^\bullet,$$

and where the orientation is induced by the complex structure on \mathbb{C} . Using polar coordinates (ρ, θ) on \mathbb{C} and taking into account that \mathcal{C}_φ is a two sheeted curve one has

$$V(r) = 2 \int_0^r \int_0^{2\pi} \left| \frac{\zeta(\rho e^{i\theta})}{\sqrt{\Delta^2(\rho e^{i\theta}) - 4}} \right|^2 \rho d\theta d\rho.$$

In particular, if $\zeta \not\equiv 0$ then $V(r) > 0$ for any $r > 0$ and $V(r)$ is strictly increasing. Note that ω_ζ might not be L^2 -integrable as it could happen that $\lim_{r \rightarrow \infty} V(r) = \infty$.

Theorem 1.1. *Let $\varphi \in \mathcal{W}$ with \mathcal{W} and $(A_m)_{m \in \mathbb{Z}}$ be given as above and let $\zeta : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function with $\zeta \not\equiv 0$ and $\zeta|_{Z_\varphi} = 0$. If $\int_{A_m} \omega_\zeta = 0$ for any $m \in \mathbb{Z}$, then there exists $C > 0$ so that*

$$V(r) \geq C r^{2/\pi},$$

for any $r \geq 1$.

Theorem 1.1 leads to the following vanishing theorem:

Theorem 1.2. *Let $\varphi \in \mathcal{W}$ with \mathcal{W} and $(A_m)_{m \in \mathbb{Z}}$ be given as above and let $\zeta : \mathbb{C} \rightarrow \mathbb{C}$ be entire with $\zeta|_{Z_\varphi} = 0$. If $\int_{A_m} \omega_\zeta = 0$ for any $m \in \mathbb{Z}$ and*

$$V(m\pi) = o(m^{2/\pi}) \text{ as } m \rightarrow \infty,$$

then $\zeta \equiv 0$ and hence $\omega_\zeta \equiv 0$.

Remark 1.1. The conclusion of Theorem 1.2 no longer holds when the assumption $V(m\pi) = o(m^{2/\pi})$ is dropped – see the form $\hat{\omega}$ defined in (4).

Remark 1.2. As $V(r)$ is increasing, the condition $V(m\pi) = o(m^{2/\pi})$ as $m \rightarrow \infty$ is equivalent to $V(r) = o(r^{2/\pi})$ as $r \rightarrow \infty$.

Next we state our result on normalized differentials on $\mathcal{C}_\varphi^\bullet$ and describe features of them needed in our studies of the focusing NLS equation.

Theorem 1.3. *For any $\psi \in L_\bullet^2$ there exist an open neighborhood \mathcal{W} of ψ in L_\bullet^2 , cycles A_m , $m \in \mathbb{Z}$, chosen as above, and analytic functions $\zeta_n : \mathbb{C} \times \mathcal{W} \rightarrow \mathbb{C}$, $n \in \mathbb{Z}$, so that for any $\varphi \in \mathcal{W}$ and $n \in \mathbb{Z}$, the holomorphic differential $\omega_n = \frac{\zeta_n(\lambda, \varphi)}{\sqrt{\Delta^2(\lambda, \varphi) - 4}} d\lambda$ on $\mathcal{C}_\varphi^\bullet$ satisfies*

$$\frac{1}{2\pi} \int_{A_m} \omega_n = \delta_{nm} \quad \forall m \in \mathbb{Z}.$$

In addition, there exists $N \geq 1$ such that for any $n \in \mathbb{Z}$ and for any $\varphi \in \mathcal{W}$ the entire function $\zeta_n(\cdot, \varphi)$ admits infinitely many zeros so that, when listed appropriately, these zeros σ_k^n , $k \in \mathbb{Z} \setminus \{n\}$, satisfy:

- (i) For any $|k| > N$, $k \neq n$, σ_k^n is the only zero of $\zeta_n(\cdot, \varphi)$ in the disk $D_k(\pi/4)$ of radius $\pi/4$ centered at $k\pi$ and the map $\sigma_k^n : \mathcal{W} \rightarrow D_k(\pi/4)$ is analytic. Furthermore, for any $|k| \leq N$, $k \neq n$, $\sigma_k^n \in D_0(N\pi + \pi/4)$. There are no other zeros of $\zeta_n(\cdot, \varphi)$ in \mathbb{C} .
- (ii) For any $|k| > N$, $k \neq n$,

$$(2) \quad \sigma_k^n = \frac{\lambda_k^- + \lambda_k^+}{2} + (\lambda_k^+ - \lambda_k^-)^2 \ell^2(k),$$

uniformly in $n \in \mathbb{Z}$ and locally uniformly in \mathcal{W} .

Moreover, $\zeta_n(\cdot, \varphi)$ admits the product representation

$$\zeta_n(\lambda, \varphi) = -\frac{2}{\pi_n} \prod_{k \neq n} \frac{\sigma_k^n - \lambda}{\pi_k},$$

where

$$\prod_{k \neq n} \frac{\sigma_k^n - \lambda}{\pi_k} := \lim_{K \rightarrow \infty} \prod_{|k| \leq K, k \neq n} \frac{\sigma_k^n - \lambda}{\pi_k},$$

and

$$\pi_k := \begin{cases} k\pi, & k \neq 0, \\ 1, & k = 0. \end{cases}$$

For any $\lambda_k^\pm \in Z_\varphi$ with $k \neq n$, $\zeta_n(\lambda_k^\pm, \varphi) = 0$, and ω_n is L^2 -integrable in sufficiently small punctured neighborhoods in $\mathcal{C}_\varphi^\bullet$ of the point $(\lambda_k^\pm, 0)$. If, however, $\lambda_n^\pm \in Z_\varphi$, then ω_n is not L^2 -integrable in any punctured neighborhood in $\mathcal{C}_\varphi^\bullet$ of the point $(\lambda_n^\pm, 0)$.

Remark 1.3. Using the product representation for $\zeta_n(\lambda)$, the asymptotic estimates for the σ_k^n 's, and estimates on infinite products in [5, Lemma C.5] one can show that for any $n \in \mathbb{Z}$ and $\varphi \in \mathcal{W}$ there exists $C > 0$ so that $-\frac{1}{2i} \int_{X_r \setminus X_{|n|\pi + \pi/4}} \omega_n \wedge \overline{\omega_n} \geq C \log r$ for any $r \geq |n|\pi + \pi/4$. Hence ω_n is never L^2 -integrable.

Remark 1.4. The uniformity statement in the asymptotic formula (2) means that for any $\varphi \in \mathcal{W}$ there is a neighborhood \mathcal{V} of φ in \mathcal{W} and a constant $C > 0$ so that for any $n \in \mathbb{Z}$ and for any $\nu \in \mathcal{V}$ there are constants $(c_k^n)_{|k| > N, k \neq n}$, $c_k^n \geq 0$, so that for any $|k| > N$, $k \neq n$,

$$\left| \sigma_k^n - \frac{\lambda_k^+ + \lambda_k^-}{2} \right| \leq |\lambda_k^+ - \lambda_k^-|^2 c_k^n,$$

and $\sum_{k \neq n} |c_k^n|^2 \leq C$.

Remark 1.5. We expect that our result extends to any potential in L_c^2 if the algebraic multiplicities of periodic eigenvalues are appropriately dealt with.

Let \mathcal{W} and A_m , $m \in \mathbb{Z}$, be as above. Consider the holomorphic 1-form on $\mathcal{C}_\varphi^\bullet$,

$$\Omega^* := \frac{\dot{\Delta}(\lambda)}{\sqrt{\Delta^2(\lambda) - 4}} d\lambda,$$

where $\dot{\Delta}(\lambda) := \partial_\lambda \Delta(\lambda, \varphi)$. By Lemma 2.2 in Section 2 below one can choose \mathcal{W} and $N \geq 1$ in Theorem 1.3 so that for any $|m| \geq N$,

$$(3) \quad \int_{A_m} \Omega^* = 0.$$

Hence,

$$(4) \quad \hat{\omega} := \Omega^* - \frac{1}{2\pi} \sum_{|m| < N} a_m \omega_m,$$

with $a_m := \int_{A_m} \Omega^*$ and ω_m given by Theorem 1.3, is a holomorphic 1-form on $\mathcal{C}_\varphi^\bullet$ satisfying $\int_{A_m} \hat{\omega} = 0$ for any $m \in \mathbb{Z}$. In particular, the conclusion of Theorem 1.2 does *not* hold if the growth condition is dropped.

Clearly, the differentials ω_n of Theorem 1.3 are unique within the class of holomorphic differentials obtained by perturbations of the type defined by Theorem 1.2. However, without some conditions on the behavior of the differentials near infinity, one cannot expect uniqueness. Indeed, any sequence of holomorphic differentials of the form $\tilde{\omega}_n := \omega_n + c_n \hat{\omega}$, $c_n \in \mathbb{C}$, satisfies $\frac{1}{2\pi} \int_{A_m} \tilde{\omega}_n = \delta_{mn}$ for any $m, n \in \mathbb{Z}$.

Besides Theorem 1.2, a key ingredient in the proof of Theorem 1.3 is a novel ansatz for the entire functions ζ_n , $n \in \mathbb{Z}$, leading to a *linear* system of equations. A detailed outline of the proof of Theorem 1.3 is given in Section 4.

Applications: In [8], Theorem 1.2 is used to construct action-angle variables for the focusing NLS equation, significantly extending previous our results obtained in [7] near the zero potential. See [1] for related results for 1-gap and 2-gap potentials and [3], [6], [18], [20] for finite gap potentials. Such coordinates allow to obtain various results concerning well-posedness for these equations and study their (Hamiltonian) perturbations – see, e.g., [11], [13], where results in this direction have been obtained for the KdV equation.

Related results: (1) In [5] (cf. also [16]) the case of the defocusing NLS equation is treated, i.e., the case where $\varphi = (\varphi_1, \varphi_2)$ satisfies $\varphi_2 = \overline{\varphi_1}$. In this case $\varphi \in L^2_\bullet$ and $L(\varphi)$ is self-adjoint, hence its periodic spectrum is real. More precisely the eigenvalues can be listed in such a way that

$$\dots < \lambda_k^- \leq \lambda_k^+ < \lambda_{k+1}^- \leq \lambda_{k+1}^+ < \dots \quad \text{and} \quad \lambda_k^\pm = k\pi + \ell^2(k).$$

It then follows that zeros of ζ_n are confined to the closed gaps $[\lambda_k^-, \lambda_k^+]$ with $k \neq n$. More precisely, as $\Delta(\lambda)$ is real valued and $|\Delta(\lambda)| > 2$ on the open gaps $(\lambda_k^-, \lambda_k^+)$, $k \in \mathbb{Z}$, one easily deduces that for any $n \in \mathbb{Z}$, $\zeta_n(\lambda)$ must have a zero σ_k^n in any closed gap $[\lambda_k^-, \lambda_k^+]$ with $k \neq n$. Using the implicit function theorem it is shown in [5] that normalized differentials can be constructed in a sufficiently small neighborhood $W \subseteq L^2_\bullet$ of such a φ . Note that if $\varphi \in L^2_\bullet$ is arbitrary we have no a priori knowledge on the zeros of ζ_k and hence one *cannot* apply the implicit function theorem approach in [5] as well as the method in [16].

In [9] we consider finite gap potentials $\varphi \in L^2_\bullet$ so that the ZS operator $L(\varphi)$ is *not* necessarily self-adjoint. Listing the periodic eigenvalues of $L(\varphi)$ in pairs $\lambda_k^-(\varphi), \lambda_k^+(\varphi)$, $k \in \mathbb{Z}$, as above it means that the subset J_φ consisting of all $k \in \mathbb{Z}$ with $\lambda_k^-(\varphi) \neq \lambda_k^+(\varphi)$ is finite. Using again the implicit function theorem it is shown in [9] that normalized differentials can be constructed in a sufficiently small neighborhood of such a finite gap potential. Unfortunately, the method in [9] does *not* work if φ is an arbitrary not necessarily finite gap potential in L^2_\bullet .

(2) In the case of the KdV equation on the circle, the relevant operator in the Lax pair is the Hill operator. For potentials with sufficiently small imaginary part, normalized differentials have been constructed, e.g., in

[11], using the implicit function theorem and the same method as in [5]. In the case where the Hill operator has simple periodic spectrum, the corresponding spectral curve is a Riemann surface of infinite genus and the existence of normalized holomorphic differentials can be proved by Hodge theory using the fact that in this case these differentials are L^2 -integrable – see [4], [15]. Note, however, that these arguments do *not* provide the asymptotic estimates of the zeros nor the analytic dependence on the potential. But even for the existence part this approach would not work for the ZS operator as the differentials of Theorem 1.3 are never L^2 -integrable on $\mathcal{C}_\varphi^\bullet$.

Organization: In the preliminary Section 2 we introduce additional notation and review the spectral properties of ZS operators needed throughout the paper. In Section 3 we prove Theorem 1.1 and Theorem 1.2 whereas in Section 4 we give an outline of the proof of Theorem 1.3. Its details are then presented in the remaining sections.

2. Preliminaries

In this section, we introduce some more notation and review properties of the Zakharov–Shabat operator $L(\varphi)$, introduced in Section 1. For $\varphi \in L^2_c$ and $\lambda \in \mathbb{C}$, let $M(x) \equiv M(x, \lambda, \varphi)$ denote the fundamental 2×2 matrix of $L(\varphi)$, $L(\varphi)M(x) = \lambda M(x)$, satisfying the initial condition $M(0, \lambda, \varphi) = Id_{2 \times 2}$. Let $\mathbb{R}_{\geq 0} := \{x \in \mathbb{R} \mid x \geq 0\}$. It is well known that $M : \mathbb{R}_{\geq 0} \times \mathbb{C} \times L^2_c \rightarrow \mathbb{C}^{2 \times 2}$ is continuous and for any x fixed, $M(x, \cdot, \cdot) : \mathbb{C} \times L^2_c \rightarrow \mathbb{C}^{2 \times 2}$ is analytic – see, e.g., [5], Chapter I.

Periodic spectrum: Denote by $\text{spec } L(\varphi)$ the spectrum of $L(\varphi)$ with domain

$$\text{dom}_{\text{per}} L(\varphi) := \{F \in H^1_{\text{loc}} \times H^1_{\text{loc}} : F(2) = F(0)\},$$

where $H^1_{\text{loc}} \equiv H^1_{\text{loc}}(\mathbb{R}, \mathbb{C})$. As $L(\varphi)$ has a compact resolvent, the periodic spectrum of $L(\varphi)$ is discrete. It has been analyzed in great detail.

The discriminant $\Delta(\lambda) \equiv \Delta(\lambda, \varphi)$ of $L(\varphi)$ is defined to be the trace of $M(1, \lambda, \varphi)$, $\Delta(\lambda) = \text{tr } M(1, \lambda)$. It is straightforward to see that λ is a periodic eigenvalue of $L(\varphi)$ iff $\Delta^2(\lambda, \varphi) - 4 = 0$. Clearly, $\Delta : \mathbb{C} \times L^2_c \rightarrow \mathbb{C}$ is analytic.

We say that $a, b \in \mathbb{C}$ are lexicographically ordered, $a \preccurlyeq b$, iff $[\text{Re}(a) < \text{Re}(b)]$ or $[\text{Re}(a) = \text{Re}(b) \text{ and } \text{Im}(a) \leq \text{Im}(b)]$. Similarly, $a \prec b$ iff $a \preccurlyeq b$ and $a \neq b$. The following two propositions are well known – see, e.g., [5, Section 3] or references therein. For any $k \in \mathbb{Z}$ and $r > 0$ denote by $D_k(r)$ the disk

$$D_k(r) := \{\lambda \in \mathbb{C} \mid |\lambda - k\pi| < r\}.$$

Proposition 2.1. *For any $\psi \in L^2_c$ there exist an open neighborhood \mathcal{W} of ψ in L^2_c and an integer $N_0 \geq 1$ such that for any $\varphi \in \mathcal{W}$ the following hold:*

- (i) For any $k \in \mathbb{Z}$ with $|k| \geq N_0$, the disk $D_k(\pi/6)$ contains precisely two periodic eigenvalues $\lambda_k^-(\varphi) \preceq \lambda_k^+(\varphi)$ of $L(\varphi)$ and one zero $\dot{\lambda}_k(\varphi)$ of $\dot{\Delta}(\lambda, \varphi) = \partial_\lambda \Delta(\lambda, \varphi)$ (all counted with their algebraic multiplicities).
- (ii) The disk $D_0((N_0 - 3/4)\pi)$ contains precisely $4N_0 - 2$ periodic eigenvalues of $L(\varphi)$ and $2N_0 - 1$ zeros of $\dot{\Delta}(\lambda, \varphi)$ (all counted with their algebraic multiplicities).
- (iii) There are no other periodic eigenvalues of $L(\varphi)$ and no other zeros of $\dot{\Delta}(\lambda, \varphi)$ than the ones listed in (i) and (ii).

Proposition 2.2. *Let $\mathcal{W} \subseteq L_c^2$ be given by Proposition 2.1. For any $\varphi \in \mathcal{W}$, the periodic eigenvalues $(\lambda_k^\pm)_{|k| \geq N_0}$ and the zeros $(\dot{\lambda}_k)_{|k| \geq N_0}$ satisfy the asymptotic estimates*

$$\lambda_k^\pm = k\pi + \ell^2(k) \text{ and } \dot{\lambda}_k = k\pi + \ell^2(k),$$

locally uniformly in \mathcal{W} . More precisely, it means that $\sum_{|k| \geq N_0} |\dot{\lambda}_k - k\pi|^2$ is locally bounded in \mathcal{W} . Similar statements hold for λ_k^\pm .

Take $\psi \in L_\bullet^2$ and construct a neighborhood $\mathcal{W} \subseteq L_\bullet^2$ of ψ in L_\bullet^2 so that the statements of Proposition 2.1 and Proposition 2.2 hold.¹ For any $\varphi \in \mathcal{W}$, in addition to the periodic eigenvalues $(\lambda_k^\pm)_{|k| \geq N_0}$ of the ZS operator $L(\varphi)$ there are $4N_0 - 2$ periodic eigenvalues in the disk $D_0((N_0 - 3/4)\pi)$. We list these eigenvalues in pairs $\lambda_k^-, \lambda_k^+, |k| < N_0$, in an arbitrary way except that any double eigenvalue is listed as a pair and for any $|k| < N_0$ the eigenvalues λ_k^- and λ_k^+ are lexicographically ordered $\lambda_k^- \preceq \lambda_k^+$. For all integers $|k| < N_0$, choose simple, closed smooth, counterclockwise oriented curves Γ_k contained in the disk $D_0((N_0 - 3/4)\pi)$ so that the closures of the (open) domains in \mathbb{C} , bounded by the Γ_k are pairwise disjoint and for any $|k| < N_0$, the domain bounded by Γ_k contains the pair λ_k^\pm , but no other periodic eigenvalue of $L(\varphi)$. For each $|k| < N_0$ choose a closed smooth curve Γ'_k in the interior of Γ_k so that $\text{dist}(\Gamma_k, \Gamma'_k) > 0$ and λ_k^\pm are inside Γ'_k . By shrinking the neighborhood \mathcal{W} of ψ in L_\bullet^2 if necessary, the choice of $\Gamma'_k, |k| < N_0$, can be done uniformly in \mathcal{W} , i.e., for any $\varphi \in \mathcal{W}$ and for any $|k| < N_0$ the domain bounded by Γ'_k contains precisely two periodic eigenvalues of $L(\varphi)$, $\lambda_k^-(\varphi) \preceq \lambda_k^+(\varphi)$. Furthermore, for any $\varphi \in \mathcal{W}$ and $|k| < N_0$ chose a continuously differentiable simple curve $G_k \equiv G_k(\varphi)$ inside Γ'_k connecting $\lambda_k^-(\varphi)$ with $\lambda_k^+(\varphi)$. In the case when $\lambda_k^-(\varphi) = \lambda_k^+(\varphi)$, $G_k(\varphi)$ is chosen to be the constant curve $\lambda_k^-(\varphi)$. For $|k| \geq N_0$, we choose Γ_k to be the counterclockwise oriented boundary of the disk $D_k(\pi/4)$ and G_k to be the straight line,

$$G_k : [0, 1] \rightarrow D_k(\pi/4), \quad t \mapsto (1 - t)\lambda_k^-(\varphi) + t\lambda_k^+(\varphi).$$

¹Recall that L_\bullet^2 is open and dense in L_c^2 .

Furthermore, for $k \in \mathbb{Z}$ and $\varphi \in \mathcal{W}$ we define

$$\tau_k(\varphi) := (\lambda_k^-(\varphi) + \lambda_k^+(\varphi))/2,$$

and

$$\gamma_k(\varphi) := \lambda_k^+(\varphi) - \lambda_k^-(\varphi).$$

Infinite products: We say that an infinite product $\prod_{k \in \mathbb{Z}}(1 + a_k)$ with $a_k \in \mathbb{C}$, $k \in \mathbb{Z}$, converges if $\lim_{K \rightarrow \infty} \prod_{|k| \leq K}(1 + a_k)$ exists. The limit is then also denoted by $\prod_{k \in \mathbb{Z}}(1 + a_k)$. The infinite product $\prod_{k \in \mathbb{Z}}(1 + a_k)$ converges absolutely if $\prod_{k \in \mathbb{Z}}(1 + |a_k|)$ converges.

Product representations: For any $\varphi \in L_c^2$, $\Delta^2(\lambda, \varphi) - 4$ and $\dot{\Delta}(\lambda, \varphi)$ admit product representations. For any given element in L_c^2 , choose N_0 and \mathcal{W} as in Proposition 2.1. According to Proposition 2.1, for any $\varphi \in \mathcal{W}$, $\dot{\Delta}(\lambda, \varphi)$ admits $2N_0 - 1$ zeros in the disk $D_0((N_0 - \frac{3}{4})\pi)$. For convenience list them in lexicographic order, $\dot{\lambda}_{-N_0+1}(\varphi) \preccurlyeq \dot{\lambda}_{-N_0+2}(\varphi) \preccurlyeq \dots \preccurlyeq \dot{\lambda}_{N_0-1}(\varphi)$. The remaining zeros are listed as in Proposition 2.1. The proof of the following statement can be found in [5, Lemma 6.5, Lemma 6.8].

Proposition 2.3. *For any $\varphi \in \mathcal{W}$ and $\lambda \in \mathbb{C}$*

$$\Delta^2(\lambda, \varphi) - 4 = -4 \prod_{k \in \mathbb{Z}} \frac{(\lambda_k^+(\varphi) - \lambda)(\lambda_k^-(\varphi) - \lambda)}{\pi_k^2},$$

and

$$\dot{\Delta}(\lambda, \varphi) = 2 \prod_{k \in \mathbb{Z}} \frac{\dot{\lambda}_k(\varphi) - \lambda}{\pi_k}.$$

Standard and canonical roots: Denote by $\sqrt[+]{z}$ the branch of the square root defined on $\mathbb{C} \setminus \{z \in \mathbb{R} : z \leq 0\}$ by $\sqrt[+]{1} = 1$. For any $a, b \in \mathbb{C}$, we define the standard root of $(a - \lambda)(b - \lambda)$ by the following relation:

$$(5) \quad \sqrt[s]{(a - \lambda)(b - \lambda)} = -\lambda \sqrt[+]{\left(1 - \frac{a}{\lambda}\right)\left(1 - \frac{b}{\lambda}\right)},$$

for all $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\left|\frac{a}{\lambda}\right|, \left|\frac{b}{\lambda}\right| \leq 1/2$. Let $G_{[a,b]}$ be a continuous simple curve connecting a and b . By analytic extension, (5) uniquely defines a holomorphic function on $\mathbb{C} \setminus G_{[a,b]}$ that we call the *standard root* of $(a - \lambda)(b - \lambda)$ on $\mathbb{C} \setminus G_{[a,b]}$. One has the asymptotic formula

$$\sqrt[s]{(a - \lambda)(b - \lambda)} \sim -\lambda \text{ as } |\lambda| \rightarrow \infty.$$

For any $\varphi \in \mathcal{W}$ and $\lambda \in \mathbb{C} \setminus (\sqcup_{k \in \mathbb{Z}} G_k)$ with \mathcal{W} and G_k , $k \in \mathbb{Z}$, given as above, we define the *canonical root* of $\Delta^2(\lambda, \varphi) - 4$ as

$$(6) \quad \sqrt[\varepsilon]{\Delta^2(\lambda, \varphi) - 4} := 2i \prod_{k \in \mathbb{Z}} \frac{\sqrt[s]{(\lambda_k^+(\varphi) - \lambda)(\lambda_k^-(\varphi) - \lambda)}}{\pi_k}.$$

To simplify notation, we occasionally will write $\mathcal{R}(\lambda, \varphi)$ for $\Delta^2(\lambda, \varphi) - 4$,

$$\mathcal{R}(\lambda, \varphi) := \Delta^2(\lambda, \varphi) - 4.$$

The proof of the following lemma is straightforward and hence omitted.

Lemma 2.1. *Let \mathcal{W} be given as above. For any $\varphi \in \mathcal{W}$, the canonical root (6) defines a holomorphic function on $\mathbb{C} \setminus (\sqcup_{k \in \mathbb{Z}} G_k)$.*

For any $\varphi \in \mathcal{W}$, define the *canonical sheet* (or *canonical branch*) of the open Riemann surface $\mathcal{C}_\varphi^\bullet$,

$$(7) \quad \mathcal{C}_\varphi^c := \{(\lambda, w) \in \mathbb{C}^2 \mid \lambda \in \mathbb{C} \setminus (\sqcup_{k \in \mathbb{Z}} G_k), w = \sqrt[c]{\Delta^2(\lambda, \varphi) - 4}\}.$$

As in the Introduction, denote by A_k , $k \in \mathbb{Z}$, the cycles on the canonical sheet \mathcal{C}_φ^c such that for any $k \in \mathbb{Z}$,

$$\pi(A_k) = \Gamma_k,$$

where $\pi : \mathcal{C}_\varphi \rightarrow \mathbb{C}$, $(\lambda, w) \mapsto \lambda$.

Finally, we need the following result on the A_k -periods of the holomorphic 1-form $\frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}} d\lambda$ on $\mathcal{C}_\varphi^\bullet$.

Lemma 2.2. *Let \mathcal{W} be given as above. Then, for any $\varphi \in \mathcal{W}$ and for any $k \in \mathbb{Z}$,*

$$\int_{\Gamma_k} \frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}} d\lambda \in 2\pi i\mathbb{Z}.$$

Furthermore, for any $\psi \in L_c^2$ there exist an open neighborhood \mathcal{U} of ψ in L_c^2 and an integer $N_0 \geq 1$ so that for any $\varphi \in \mathcal{U}$ statements (i)–(iii) of Proposition 2.1 hold and for any $|k| \geq N_0$,

$$\int_{\partial D_k(\pi/4)} \frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}} d\lambda = 0,$$

where $\partial D_k(\pi/4)$ is the counterclockwise oriented boundary of the disk $D_k(\pi/4)$.

Proof. Take $\varphi \in \mathcal{W}$ and consider the holomorphic function $f(\lambda) := \Delta(\lambda) + \sqrt[c]{\Delta(\lambda)^2 - 4}$ defined for $\lambda \in \mathbb{C} \setminus (\sqcup_{k \in \mathbb{Z}} G_k)$. Note that $f(\lambda)$ does not vanish and hence the logarithm $\log f(\lambda)$ is well defined as a multi-valued function on $\mathbb{C} \setminus (\sqcup_{k \in \mathbb{Z}} G_k)$. Along any simple closed C^1 -smooth curve $\gamma : [0, 1] \rightarrow \mathbb{C} \setminus (\sqcup_{k \in \mathbb{Z}} G_k)$ one can choose $g(t) \in \log f(\gamma(t))$ so that g is continuous on $0 \leq t \leq 1$. As $\gamma(0) = \gamma(1)$, one has $g(1) - g(0) \in 2\pi i\mathbb{Z}$. In view of the identity,

$$d(\log f(\lambda)) = \frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}} d\lambda,$$

the first statement of the Lemma then follows.

Let us prove the second statement of the Lemma. Take $\psi \in L_c^2$ and consider the set of potentials $I := \{t\psi \mid t \in [0, 1]\} \subseteq L_c^2$. In view

of Proposition 2.1 and the compactness of the set I in L_c^2 , there exist a connected open neighborhood \mathcal{U} of I in L_c^2 and an integer number $N_0 \geq 1$ such that for any $\varphi \in \mathcal{U}$ and for any $|k| \geq N_0$, the statements (i), (ii), and (iii) of Proposition 2.1 hold. Note that for any $|k| \geq N_0$, the map $\mathcal{U} \rightarrow \mathbb{R}, \varphi \mapsto \mathcal{P}_k(\varphi)$,

$$\mathcal{P}_k(\varphi) := \max_{\pm} \left\{ \frac{1}{2\pi i} \int_{\partial D_k(\pi/4)} \frac{\dot{\Delta}(\lambda, \varphi)}{\sqrt{\Delta(\lambda, \varphi)^2 - 4}} d\lambda \right\},$$

where the maximum is taken over the two different choices of the square root in the integrand, is continuous. As $\mathcal{P}_k(\varphi) \in \mathbb{Z}$ and as \mathcal{U} is connected and $0 \in \mathcal{U}$, it then follows that $\mathcal{P}_k(\varphi) = \mathcal{P}_k(0)$ for any $\varphi \in \mathcal{U}$ and for any $|k| \geq N_0$. Finally, as $\Delta(\lambda, 0) = 2 \cos(\lambda)$ we conclude that $\frac{\dot{\Delta}(\lambda, 0)}{\sqrt{\Delta^2(\lambda, 0) - 4}} d\lambda = \pm i d\lambda$, and hence $\mathcal{P}_k(0) = 0$. This completes the proof of the Lemma. q.e.d.

Denote

$$\Omega^* := \frac{\dot{\Delta}(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} d\lambda.$$

Using the second part of Lemma 2.2 we choose the neighborhood \mathcal{W} in L_\bullet^2 and the integer $N_0 \geq 1$ so that for any $\varphi \in \mathcal{W}$ and any $|k| \geq N_0$,

$$(8) \quad \int_{A_k} \Omega^* = 0.$$

In addition, for any $|k| < N_0$,

$$(9) \quad \int_{A_k} \Omega^* \in 2\pi i \mathbb{Z},$$

and does not depend on $\varphi \in \mathcal{W}$ as $\int_{A_k} \Omega^*$ takes discrete values and the neighborhood \mathcal{W} in L_\bullet^2 can be chosen connected.

3. Proof of Theorems 1.1 and 1.2

The aim of this section is to prove Theorem 1.1 and Theorem 1.2. Let $\mathcal{W} \subseteq L_\bullet^2$ be the neighborhood constructed in Section 2. Throughout this section we fix $\varphi \in \mathcal{W}$ and define the cycles $(A_m)_{m \in \mathbb{Z}}$ as in Section 2. Without further reference we will use the terminology introduced in Section 1 and Section 2.

Let $\zeta : \mathbb{C} \rightarrow \mathbb{C}$ be entire so that ζ vanishes on the set Z_φ of double eigenvalues of $L(\varphi)$. It then follows that the differential $\omega_\zeta = \frac{\zeta(\lambda)}{\sqrt{\Delta^2(\lambda) - 4}} d\lambda$ is locally L^2 -integrable. Using Stokes' theorem we will estimate $V(r) := \frac{i}{2} \int_{X_r} \omega_\zeta \wedge \overline{\omega_\zeta}$ where $X_r = \pi^{-1}(\{\lambda \in \mathbb{C} \mid |\lambda| < r\}) \cap \mathcal{C}_\varphi^\bullet$. In view of Proposition 2.1 and Proposition 2.2 one can choose $0 < \varepsilon_m < \pi/4, m > N_0$, with $\sum_{m > N_0} \varepsilon_m^2 < \infty$ so that $\lambda_m^\pm \in D_m(\varepsilon_m)$ and

$\lambda_{-m}^{\pm} \in D_{-m}(\varepsilon_m)$ for any $m > N_0$. Choose $r > 0$ so that for some $m > N_0$

$$(10) \quad m\pi + \varepsilon_m \leq r \leq (m+1)\pi - \varepsilon_{m+1}.$$

On the Riemann surface X_r consider the points $p^+ := \pi^{-1}(r) \cap \mathcal{C}_\varphi^c$ and $p^- := \pi^{-1}(r) \setminus \{p^+\}$ and choose a simple C^1 -smooth curve B_* on X_r (i.e., a C^1 -smooth map $[0, 1] \rightarrow X_r$ without self intersections) that connects p^+ with p^- and changes sheets when its projection $\pi(B_*)$ passes through the 0-th curve G_0 . If $\lambda_0^+ = \lambda_0^- = \tau_0$ and hence G_0 is the constant curve τ_0 we allow the curve B_* to pass through the point $(\tau_0, 0) \in \mathcal{C}_\varphi$ that is excluded from X_r . The inverse image $\pi^{-1}(\partial D_0(r))$ consists of two simple closed curves with images

$$(11) \quad C_r^+ := \pi^{-1}(\{\lambda \in \mathbb{C} \mid |\lambda| = r\}) \cap \mathcal{C}_\varphi^c,$$

and

$$C_r^- := \pi^{-1}(\{\lambda \in \mathbb{C} \mid |\lambda| = r\}) \setminus C_r^+,$$

that we orient so that their projections $\pi(C_r^\pm) \subseteq \mathbb{C}$ have counterclockwise orientation. Furthermore, for any $1 \leq |k| \leq m$, denote by B'_k a simple C^1 -smooth curve in X_r that starts and ends at p^+ and changes sheets twice – first when its projection $\pi(B'_k)$ passes through G_k and then through G_0 . If the image of G_k is a point we proceed as above and allow the curve B'_k to pass through the point $(\tau_k, 0) \in \mathcal{C}_\varphi$. Similarly, for any $1 \leq |k| \leq m$, denote by A'_k a simple C^1 -smooth curve in X_r that starts and ends at p^+ and that is homologous to A_k . The curves B_*, C_r^+, A'_k , and $B'_k, 1 \leq |k| \leq m$, considered above, are chosen so that they intersect each other only at p^+ . Denote by \tilde{X}_r the surface obtained from X_r by cutting it along the curves B_* and $A'_k, B'_k, 1 \leq |k| \leq m$. Then \tilde{X}_r is a disk and its boundary $\partial\tilde{X}_r$ can be represented as a composition of the following curves (composed in the order of their appearance): $C_r^+, B^*, C_r^-, (B^*)^{-1}, B'_1, A'_1, (B'_1)^{-1}, (A'_1)^{-1}, \dots, B'_m, A'_m, (B'_m)^{-1}, (A'_m)^{-1}$. Consider the function $F : \tilde{X}_r \rightarrow \mathbb{C}$ given by

$$(12) \quad F(p) := \int_{p^+}^p \omega_\zeta,$$

for $p \in \tilde{X}_r$. Note that the integral is independent of the choice of the path and hence F is well defined on \tilde{X}_r . Furthermore, introduce $a_k = \int_{A_k} \omega_\zeta$ ($0 \leq |k| \leq m$), $b_k := \int_{B'_k} \omega_\zeta$ ($1 \leq |k| \leq m$), $b_* = \int_{B_*} \omega_\zeta$, and $c_r^\pm = \int_{C_r^\pm} \omega_\zeta$. By Stokes' theorem

$$\begin{aligned} -2iV(r) &= \iint_{\tilde{X}_r} d(F\overline{\omega_\zeta}) = \int_{\partial\tilde{X}_r} F\overline{\omega_\zeta} \\ &= \int_{C_r^+} F\overline{\omega_\zeta} + \int_{C_r^-} F\overline{\omega_\zeta} - c_r^- \overline{b_*} - \sum_{1 \leq |k| \leq m} (a_k \overline{b_k} - \overline{a_k} b_k), \end{aligned}$$

where we used that $\int_{A'_k} \omega_\zeta = a_k$ for any $1 \leq |k| \leq m$ as A'_k and A_k are homologous. Note that for $z^- \in C_r^-$

$$F(z^-) = \int_{C_r^+} \omega_\zeta + \int_{B_*} \omega_\zeta + \int_{p^-}^{z^-} \omega_\zeta = c_r^+ + b_* + \int_{p^+}^{z^+} -\omega_\zeta,$$

where $z^+ \in C_r^+$ is determined by $\pi(z^+) = \pi(z^-)$ and the minus sign stems from passing to the canonical sheet. Hence

$$\begin{aligned} \int_{C_r^-} F(z^-) \overline{\omega_\zeta} &= - \int_{C_r^+} (c_r^+ + b_* - F(z^+)) \overline{\omega_\zeta} \\ &= -|c_r^+|^2 - b_* \overline{c_r^+} + \int_{C_r^+} F \overline{\omega_\zeta} \end{aligned}$$

yielding

$$-2iV(r) = 2 \int_{C_r^+} F \overline{\omega_\zeta} - |c_r^+|^2 - c_r^- \overline{b_*} - b_* \overline{c_r^+} - \sum_{1 \leq |k| \leq m} (a_k \overline{b_k} - \overline{a_k} b_k).$$

Now assume that $a_k = 0$ for $0 \leq |k| \leq m$. As $\sum_{0 \leq |k| \leq m} A_k$ is homologous to C_r^+ it then follows that $c_r^+ = 0$ and as $c_r^- = -c_r^+$ one also has $c_r^- = 0$. We thus have proved that for any $m\pi + \varepsilon_m \leq r \leq (m+1)\pi - \varepsilon_{m+1}$ with $m > N_0$

$$(13) \quad V(r) = i \int_{C_r^+} F \overline{\omega_\zeta}.$$

This identity will be used in the proof of the following lemma. We recall that $\mathcal{R}(\lambda) = \Delta^2(\lambda) - 4$.

Lemma 3.1. *Let $\zeta : \mathbb{C} \rightarrow \mathbb{C}$ be entire with $\zeta|_{Z_\varphi} = 0$ and $\int_{A_j} \omega_\zeta = 0 \forall j \in \mathbb{Z}$. Then for any $m\pi + \varepsilon_m \leq r \leq (m+1)\pi - \varepsilon_{m+1}$ with $m > N_0$*

$$(14) \quad V(r) \leq \frac{1}{2} \left(r \int_0^{2\pi} \left| \frac{\zeta(re^{i\theta})}{\mathcal{R}(re^{i\theta})} \right| d\theta \right)^2,$$

and

$$(15) \quad V'(r) \geq \frac{2}{r\pi} V(r).$$

Proof of Lemma 3.1. Using polar coordinates we get from (12)

$$\begin{aligned} \int_{C_r^+} F \overline{\omega_\zeta} &= \int_0^{2\pi} F(re^{i\theta}) \overline{\left(\frac{\zeta(re^{i\theta})}{\sqrt{\mathcal{R}(re^{i\theta})}} d(re^{i\theta}) \right)} \\ &= r^2 \int_0^{2\pi} \left(\int_0^\theta \frac{\zeta(re^{i\theta_1})}{\sqrt{\mathcal{R}(re^{i\theta_1})}} ie^{i\theta_1} d\theta_1 \right) \overline{\left(\frac{\zeta(re^{i\theta})}{\sqrt{\mathcal{R}(re^{i\theta})}} \right)} (-i)e^{-i\theta} d\theta, \end{aligned}$$

where $\sqrt[\varepsilon]{\mathcal{R}(\lambda)}$ denotes the canonical root (6). Hence

$$\begin{aligned} V(r) &= \left| \int_{C_r^+} F(p) \overline{\omega_\zeta} \right| \\ &\leq r^2 \int_0^{2\pi} \left(\int_0^\theta \left| \frac{\zeta(re^{i\theta_1})}{\sqrt[\varepsilon]{\mathcal{R}(re^{i\theta_1})}} \right| d\theta_1 \right) d \left(\int_0^\theta \left| \frac{\zeta(re^{i\theta_1})}{\sqrt[\varepsilon]{\mathcal{R}(re^{i\theta_1})}} \right| d\theta_1 \right) \\ &= \frac{r^2}{2} \left(\int_0^{2\pi} \left| \frac{\zeta(re^{i\theta_1})}{\sqrt[\varepsilon]{\mathcal{R}(re^{i\theta_1})}} \right| d\theta_1 \right)^2. \end{aligned}$$

This proves the estimate (14). To get (15) note that

$$\begin{aligned} \frac{r^2}{2} \left(\int_0^{2\pi} \left| \frac{\zeta(re^{i\theta})}{\sqrt[\varepsilon]{\mathcal{R}(re^{i\theta})}} \right| d\theta \right)^2 &\leq \pi r^2 \int_0^{2\pi} \left| \frac{\zeta(re^{i\theta})}{\sqrt[\varepsilon]{\mathcal{R}(re^{i\theta})}} \right|^2 d\theta \\ &= \pi r \frac{d}{dr} \left(\int_0^r \int_0^{2\pi} \left| \frac{\zeta(\rho e^{i\theta})}{\sqrt[\varepsilon]{\mathcal{R}(\rho e^{i\theta})}} \right|^2 \rho d\theta d\rho \right) = \pi r \frac{V'(r)}{2}. \end{aligned}$$

Hence $V(r) \leq \frac{r\pi}{2} V'(r)$ as claimed. q.e.d.

Estimate (15) is now used to prove Theorem 1.1.

Proof of Theorem 1.1. Assume that $\zeta \not\equiv 0$ is an entire function satisfying $\int_{A_m} \omega_\zeta = 0$ for any $m \in \mathbb{Z}$. Then $V(r) \neq 0 \forall r > 0$ and in view of (15), for any $m > N_0$ and for any $m\pi + \varepsilon_m \leq r \leq (m + 1)\pi - \varepsilon_{m+1}$,

$$(\log V(r))' \geq \frac{2}{\pi r}.$$

Integrating this inequality over the interval $[m\pi + \varepsilon_m, (m + 1)\pi - \varepsilon_{m+1}]$ we obtain that

$$\frac{V((m + 1)\pi - \varepsilon_{m+1})}{V(m\pi + \varepsilon_m)} \geq e^{\frac{2}{\pi} (\log((m+1)\pi - \varepsilon_{m+1}) - \log(m\pi + \varepsilon_m))}.$$

This implies that for any $m \geq m_0 > N_0$,

$$V((m + 1)\pi) \geq V(m_0\pi) e^{\frac{2}{\pi} S(m, m_0)},$$

where

$$\begin{aligned} S(m, m_0) &:= \sum_{j=m_0}^m \log((j + 1)\pi - \varepsilon_{j+1}) - \log(j\pi + \varepsilon_j) \\ &= \log \frac{m + 1}{m_0} + \sum_{j=m_0}^m \log \left(1 - \frac{\varepsilon_{j+1}}{(j + 1)\pi} \right) - \sum_{j=m_0}^m \log \left(1 + \frac{\varepsilon_j}{j\pi} \right) \\ &\geq \log \frac{m + 1}{m_0} - O \left(\sum_{j=m_0}^{m+1} \frac{\varepsilon_j}{j\pi} \right). \end{aligned}$$

As $\sum_{j=m_0}^m \frac{\varepsilon_j}{j\pi} = O\left(\sum_{j=m_0}^m \varepsilon_j^2\right)^{1/2}$ one then concludes that

$$V((m + 1)\pi) \geq C(m + 1)^{2/\pi},$$

where $C > 0$ depends on $m_0 > N_0$ but not on $m \geq m_0$. q.e.d.

Proof of Theorem 1.2. Theorem 1.2 is an immediate consequence of Theorem 1.1. q.e.d.

4. Outline of proof of Theorem 1.3

In this section, we describe the main steps in the proof of Theorem 1.3. For the remaining part of the paper \mathcal{W} will denote the neighborhood $\mathcal{W} \subseteq L_\bullet^2$ constructed in Section 2 with $N_0 \geq 1$ so that Proposition 2.1, Proposition 2.2, and the identities (8) and (9) hold. Let $(A_m)_{m \in \mathbb{Z}}$ be the cycles on the canonical branch of $\mathcal{C}_\varphi^\bullet$ introduced in Section 2. Without further explanations, for any given $\varphi \in \mathcal{W}$ and $n \in \mathbb{Z}$ consider the following ansatz for the holomorphic differentials of Theorem 1.3

$$(16) \quad \Omega_\beta^n := \Omega^n - \omega_\beta^n,$$

where the forms Ω^n and ω_β^n are defined as follows:

$$(17) \quad \Omega^n := \begin{cases} \frac{\dot{\Delta}(\lambda)}{\lambda - \lambda_n} \frac{d\lambda}{\sqrt{\mathcal{R}(\lambda)}}, & |n| > N_0, \\ \frac{\dot{\Delta}(\lambda)}{\lambda - \lambda_{N_0}} \frac{d\lambda}{\sqrt{\mathcal{R}(\lambda)}}, & |n| \leq N_0, \end{cases}$$

and for any given $\beta = (\beta_k)_{k \neq n} \in \ell_n^1 \equiv \ell_{n, \mathbb{C}}^1 := \ell^1(\mathbb{Z} \setminus \{n\}, \mathbb{C})$

$$(18) \quad \omega_\beta^n := \frac{\xi_\beta^n(\lambda)}{\sqrt{\mathcal{R}(\lambda)}} d\lambda,$$

where

$$(19) \quad \xi_\beta^n(\lambda) := \begin{cases} \left(\sum_{|j| > N_0, j \neq n} \frac{\beta_j}{\lambda - \lambda_j} + \frac{p_\beta^n(\lambda)}{\prod_{|k| \leq N_0} (\lambda - \lambda_k)} \right) \frac{\dot{\Delta}(\lambda)}{\lambda - \lambda_n}, & |n| > N_0, \\ \left(\sum_{|j| > N_0} \frac{\beta_j}{\lambda - \lambda_j} + \frac{p_\beta^n(\lambda)}{\prod_{\substack{|k| \leq N_0 \\ k \neq N_0}} (\lambda - \lambda_k)} \right) \frac{\dot{\Delta}(\lambda)}{\lambda - \lambda_{N_0}}, & |n| \leq N_0, \end{cases}$$

and

$$p_\beta^n(\lambda) := \begin{cases} \sum_{j=0}^{2N_0} \beta_{j-N_0} \lambda^j, & |n| > N_0, \\ \sum_{j=0}^{n+N_0-1} \beta_{j-N_0} \lambda^j + \sum_{j=n+N_0}^{2N_0-1} \beta_{j-N_0+1} \lambda^j, & |n| \leq N_0. \end{cases}$$

Note that $\xi_\beta^n(\lambda)$ is entire. In the case $|n| \leq N_0$, it is convenient to write the polynomial $p_\beta^n(\lambda)$ in the following alternative way $p_\beta^n(\lambda) =$

$(\sum_{|j| \leq N_0, j \neq n} \beta_j \lambda^{j-\varepsilon_j^n}) \lambda^{N_0}$ where

$$(20) \quad \varepsilon_j^n := \begin{cases} 1, & j \geq n, \\ 0, & j < n. \end{cases}$$

We want to find $\beta = (\beta_k)_{k \neq n} \in \ell_{\hat{n}}^1$ so that

$$(21) \quad \frac{1}{2\pi} \int_{A_m} \Omega_{\beta}^n = \delta_{nm} \quad \forall m \in \mathbb{Z}.$$

The following proposition is proved in Section 6.

Proposition 4.1. *For any $n \in \mathbb{Z}$, $\varphi \in \mathcal{W}$, and $\beta \in \ell_{\hat{n}}^1$*

$$(22) \quad \sum_{m \in \mathbb{Z}} \frac{1}{2\pi} \int_{A_m} \Omega_{\beta}^n = \lim_{K \rightarrow \infty} \sum_{|m| \leq K} \frac{1}{2\pi} \int_{A_m} \Omega_{\beta}^n = 1.$$

In particular, for $\beta = 0$, $\Omega_{\beta}^n \equiv \Omega^n$ satisfies

$$\sum_{m \in \mathbb{Z}} \frac{1}{2\pi} \int_{A_m} \Omega^n = 1.$$

In view of Proposition 4.1, the system of equations (21) is equivalent to

$$(23) \quad \int_{A_m} \omega_{\beta}^n = \int_{A_m} \Omega^n \quad \forall m \neq n.$$

By multiplying the right and left hand side of the above equation by $m\pi - n\pi$ (if $|n| > N_0$) or $m\pi - N_0\pi$ (if $|n| \leq N_0$), we arrive at the following linear system for β ,

$$(24) \quad T^n \beta = b^n, \quad T^n = (T_{mj}^n)_{m, j \neq n}, \quad b^n = (b_m^n)_{m \neq n},$$

where for any $m \neq n$, b_m^n is given by

$$b_m^n := \begin{cases} \frac{1}{2\pi} \int_{\Gamma_m} \frac{m\pi - n\pi}{\lambda - \lambda_n} \frac{\dot{\Delta}(\lambda)}{\varepsilon \sqrt{\mathcal{R}(\lambda)}} d\lambda, & |n| > N_0, \\ \frac{1}{2\pi} \int_{\Gamma_m} \frac{m\pi - N_0\pi}{\lambda - \lambda_{N_0}} \frac{\dot{\Delta}(\lambda)}{\varepsilon \sqrt{\mathcal{R}(\lambda)}} d\lambda, & |n| \leq N_0, \end{cases}$$

and for $|n| > N_0$, T_{mj}^n is given by

$$T_{mj}^n := \begin{cases} \frac{1}{2\pi} \int_{\Gamma_m} \frac{m\pi - n\pi}{\lambda - \lambda_n} \frac{1}{\lambda - \lambda_j} \frac{\dot{\Delta}(\lambda)}{\varepsilon \sqrt{\mathcal{R}(\lambda)}} d\lambda, & |j| > N_0, j \neq n, \\ \frac{1}{2\pi} \int_{\Gamma_m} \frac{m\pi - n\pi}{\lambda - \lambda_n} \frac{\dot{\Delta}(\lambda)}{\prod_{|k| \leq N_0} (\lambda - \lambda_k)} \frac{\lambda^{N_0+j}}{\varepsilon \sqrt{\mathcal{R}(\lambda)}} d\lambda, & |j| \leq N_0, \end{cases}$$

whereas for $|n| \leq N_0$ one has

$$T_{mj}^n := \begin{cases} \frac{1}{2\pi} \int_{\Gamma_m} \frac{m\pi - N_0\pi}{\lambda - \lambda_{N_0}} \frac{1}{\lambda - \lambda_j} \frac{\dot{\Delta}(\lambda)}{\varepsilon \sqrt{\mathcal{R}(\lambda)}} d\lambda, & |j| > N_0, \\ \frac{1}{2\pi} \int_{\Gamma_m} \frac{(m\pi - N_0\pi) \dot{\Delta}(\lambda)}{\prod_{|k| \leq N_0} (\lambda - \lambda_k)} \frac{\lambda^{N_0+j-\varepsilon_j^n}}{\varepsilon \sqrt{\mathcal{R}(\lambda)}} d\lambda, & |j| \leq N_0, j \neq n. \end{cases}$$

Using Proposition 6.1 – an application of Theorem 1.2 – we prove in Section 7 the following:

Proposition 4.2. *For any $n \in \mathbb{Z}$ and $\varphi \in \mathcal{W}$ we have:*

- (i) $b^n \in \ell_{\hat{n}}^1$;
- (ii) $T^n : \ell_{\hat{n}}^1 \rightarrow \ell_{\hat{n}}^1$ is a linear isomorphism.

Denote by $\beta^n \equiv \beta^n(\varphi) \in \ell_{\hat{n}}^1$ the unique solution of (24), guaranteed by Proposition 4.2 and define,

$$(25) \quad \zeta_n(\lambda, \varphi) := \begin{cases} \frac{\dot{\Delta}(\lambda)}{\lambda - \lambda_n} - \xi_{\beta^n}^n(\lambda), & |n| > N_0, \\ \frac{\dot{\Delta}(\lambda)}{\lambda - \lambda_{N_0}} - \xi_{\beta^n}^n(\lambda), & |n| \leq N_0, \end{cases}$$

where $\xi_{\beta^n}^n$ is given by (19) with β^n substituted for β . The following proposition is proved in Section 7.

Proposition 4.3. *For any $n \in \mathbb{Z}$, $\beta^n : \mathcal{W} \rightarrow \ell_{\hat{n}}^1$ and $\zeta_n : \mathbb{C} \times \mathcal{W} \rightarrow \mathbb{C}$ are analytic maps. Furthermore, for any $\varphi \in \mathcal{W}$ and $n \in \mathbb{Z}$,*

$$\frac{1}{2\pi} \int_{A_m} \frac{\zeta_n(\lambda, \varphi)}{\sqrt{\mathcal{R}(\lambda, \varphi)}} d\lambda = \delta_{nm} \quad \forall m \in \mathbb{Z}.$$

To obtain uniform in $n \in \mathbb{Z}$ and locally uniform in \mathcal{W} estimates of the zeros of ζ_n we consider the following “limiting” linear system for $\beta = (\beta_k)_{k \in \mathbb{Z}} \in \ell^1 \equiv \ell_{\mathbb{C}}^1$,

$$(26) \quad T^* \beta = b^*,$$

where $T^* = (T_{mj}^*)_{m,j \in \mathbb{Z}}$ is given by

$$T_{mj}^* := \begin{cases} \frac{1}{2\pi} \int_{\Gamma_m} \frac{1}{\lambda - \lambda_j} \frac{\dot{\Delta}(\lambda)}{\varepsilon \sqrt{\mathcal{R}(\lambda)}} d\lambda, & |j| > N_0, \\ \frac{1}{2\pi} \int_{\Gamma_m} \frac{\lambda^{N_0+j}}{\prod_{|k| \leq N_0} (\lambda - \lambda_k)} \frac{\dot{\Delta}(\lambda)}{\varepsilon \sqrt{\mathcal{R}(\lambda)}} d\lambda, & |j| \leq N_0, \end{cases}$$

and $b^* := (b_m^*)_{m \in \mathbb{Z}}$ is given by

$$b_m^* := \frac{1}{2\pi} \int_{\Gamma_m} \frac{\dot{\Delta}(\lambda)}{\varepsilon \sqrt{\mathcal{R}(\lambda)}} d\lambda.$$

This linear system is equivalent to the condition

$$(27) \quad \int_{A_m} \Omega_{\beta}^* = 0 \quad \forall m \in \mathbb{Z},$$

where $\beta \in \ell^1$ and the holomorphic 1-form Ω_{β}^* on C_{φ}^{\bullet} is given by

$$\Omega_{\beta}^* := \Omega^* - \omega_{\beta}^*,$$

where

$$\omega_{\beta}^* := \frac{\xi_{\beta}^*(\lambda)}{\sqrt{\mathcal{R}(\lambda)}} d\lambda,$$

with

$$(28) \quad \xi_\beta^*(\lambda) := \left(\sum_{|j| > N_0} \frac{\beta_j}{\lambda - \lambda_j} + \frac{p_\beta^*(\lambda)}{\prod_{|j| \leq N_0} (\lambda - \lambda_j)} \right) \dot{\Delta}(\lambda),$$

and

$$p_\beta^*(\lambda) := \sum_{j=0}^{2N_0} \beta_{-N_0+j} \lambda^j.$$

Note that ξ_β^* is an entire function of λ and $p_\beta^*(\lambda)$ is a polynomial of degree at most $2N_0$. We can rewrite (27) as

$$\int_{A_m} \omega_\beta^* = \int_{A_m} \Omega^* \quad \forall m \in \mathbb{Z},$$

that leads to the linear system (26) in view of the definitions of the forms ω_β^* and Ω^* . Using Proposition 5.1 – another application of Theorem 1.2 – we prove in Section 7 the following:

Proposition 4.4. *For any $\varphi \in \mathcal{W}$, $T^* : \ell^1 \rightarrow \ell^1$ is a linear isomorphism.*

Recall that in view of (8) and (9), $b^* \equiv b^*(\varphi)$ is in ℓ^1 and does not depend on $\varphi \in \mathcal{W}$. Denote by $\beta^* \equiv \beta^*(\varphi) \in \ell^1$ the unique solution of (26) guaranteed by Proposition 4.4.

Proposition 4.4 is used in Section 8 to prove the uniform estimates of the zeros of ζ_n , stated in Theorem 1.3 – see Proposition 8.1 and Lemma 8.3 – and the product representation of ζ_n – see Corollary 8.1. Finally, in Lemma 8.4, it is proved that $\zeta_n(\cdot, \varphi)$ vanishes on the set $Z_\varphi \setminus \{\lambda_n^\pm(\varphi)\}$. Combining the results described above, the proof of Theorem 1.3 is complete.

5. Vanishing lemma

Let us fix $\varphi \in \mathcal{W} \subseteq L^2_\bullet$ where as stated in the beginning of Section 4 \mathcal{W} denotes the neighborhood constructed in Section 2. In this section, we prove the following:

Proposition 5.1. *Let $\beta \in \ell^1$ be arbitrary. If $\int_{A_m} \omega_\beta^* = 0$ for any $m \in \mathbb{Z}$, then $\beta = 0$.*

We prove Proposition 5.1 with the help of Theorem 1.2. To this end we prove the following lemmas:

Lemma 5.1. *If $\int_{A_m} \omega_\beta^* = 0$ for any $m \in \mathbb{Z}$ then $\xi_\beta^*|_{Z_\varphi} = 0$.*

Proof of Lemma 5.1. Assume that for some $k \in \mathbb{Z}$, $\lambda_k^- = \lambda_k^+ = \tau_k$. Then in view of (6) for $(\lambda, w) \in \mathcal{C}_\varphi^c$ near $(\tau_k, 0) \in \mathcal{C}_\varphi$,

$$(29) \quad \omega_\beta^* = \xi_\beta^*(\lambda) \frac{h(\lambda)}{\lambda - \tau_k} d\lambda,$$

where $h(\lambda)$ is a holomorphic function that is defined in an open neighborhood of τ_k and satisfies $h(\tau_k) \neq 0$. As by assumption $\int_{A_k} \omega_\beta^* = 0$ we conclude from (29) that $\xi_\beta^*(\tau_k) = 0$. q.e.d.

Lemma 5.1 implies that for any $\beta \in \ell^1$ as in Proposition 5.1 and for any $r > 0$

$$V_\beta(r) := \frac{i}{2} \int_{X_r} \omega_\beta^* \wedge \overline{\omega_\beta^*} < \infty.$$

Lemma 5.2. *If $\int_{A_j} \omega_\beta^* = 0$ for any $j \in \mathbb{Z}$, then for any $\delta > 0$,*

$$(30) \quad V_\beta(m\pi) = O(m^\delta) \quad \text{as } m \rightarrow \infty.$$

Proof of Lemma 5.2. Assume that $\int_{A_j} \omega_\beta^* = 0$ for any $j \in \mathbb{Z}$. Then by (14) in Lemma 3.1, for $r_m = (m + \frac{1}{2})\pi$ with $m > N_0$,

$$V_\beta(m\pi) \leq \frac{r_m^2}{2} \left(\int_0^{2\pi} \left| \frac{\xi_\beta^*(r_m e^{i\theta})}{\sqrt{{}^c\mathcal{R}(r_m e^{i\theta})}} \right| d\theta \right)^2.$$

By [5, Lemma C.5] and Proposition 2.2,

$$\frac{\dot{\Delta}(r_m e^{i\theta})}{\sqrt{{}^c\mathcal{R}(r_m e^{i\theta})}} = O(1) \quad \text{as } m \rightarrow \infty,$$

uniformly in $0 \leq \theta < 2\pi$. This together with the definition of ξ_β^* implies

$$(31) \quad \left| \frac{\xi_\beta^*(r_m e^{i\theta})}{\sqrt{{}^c\mathcal{R}(r_m e^{i\theta})}} \right| \leq C \left(\sum_{|j| > N_0} \frac{|\beta_j|}{|r_m e^{i\theta} - \dot{\lambda}_j|} + \frac{|p_\beta^*(r_m e^{i\theta})|}{\prod_{|j| \leq N_0} |r_m e^{i\theta} - \dot{\lambda}_j|} \right),$$

with a constant $C > 0$ independent of $m > N_0$ and $0 \leq \theta < 2\pi$. As $p_\beta^*(\lambda)$ is a polynomial in λ of degree at most $2N_0$ it follows that

$$(32) \quad \int_0^{2\pi} \frac{|p_\beta^*(r_m e^{i\theta})|}{\prod_{|j| \leq N_0} |r_m e^{i\theta} - \dot{\lambda}_j|} d\theta = O\left(\frac{1}{m}\right) \quad \text{as } m \rightarrow \infty.$$

For any $m > 2N_0$, we split the sum $\sum_{|j| > N_0} = \sum_{j \in J_1(m)} + \sum_{j \in J_2(m)}$ where $J_1(m)$ is the set

$$\left\{ j > N_0 \mid \left| j - \left(m + \frac{1}{2}\right) \right| \leq \frac{m}{2} \right\} \cup \left\{ j < -N_0 \mid \left| j + \left(m + \frac{1}{2}\right) \right| \leq \frac{m}{2} \right\},$$

and $J_2(m) := \mathbb{Z} \setminus ([-N_0, N_0] \cup J_1(m))$. Then for any $j \in J_2(m)$ with $m > 2N_0$ and any $0 \leq \theta < 2\pi$

$$|r_m e^{i\theta} - \dot{\lambda}_j| \geq \left| \left(m + \frac{1}{2}\right)\pi - j\pi \right| - \frac{\pi}{4} \geq \frac{(m+1)\pi}{2} - \frac{\pi}{4} \geq \frac{m\pi}{2}$$

implying that

$$(33) \quad \int_0^{2\pi} \sum_{j \in J_2(m)} \frac{|\beta_j|}{|r_m e^{i\theta} - \dot{\lambda}_j|} d\theta = O\left(\frac{1}{m}\right).$$

To estimate $\int_0^{2\pi} \sum_{j \in J_1(m)} \frac{|\beta_j|}{|r_m e^{i\theta} - \dot{\lambda}_j|} d\theta$ the integral $\int_0^{2\pi}$ is split up as follows: For $0 < \alpha < 1$, one has uniformly in $j \in J_1(m)$,

$$\int_{-\frac{1}{m^\alpha}}^{\frac{1}{m^\alpha}} \frac{d\theta}{|r_m e^{i\theta} - \dot{\lambda}_j|}, \int_{\pi - \frac{1}{m^\alpha}}^{\pi + \frac{1}{m^\alpha}} \frac{d\theta}{|r_m e^{i\theta} - \dot{\lambda}_j|} = O\left(\frac{1}{m^\alpha}\right),$$

as $|r_m e^{i\theta} - \dot{\lambda}_j| \geq \pi/4$. By choosing $N_1 \geq 2N_0$ sufficiently large we can ensure that for any $m \geq N_1$, $\theta \in [\frac{1}{m^\alpha}, \pi - \frac{1}{m^\alpha}]$ and $j \in J_1(m)$

$$(34) \quad \begin{aligned} |r_m e^{i\theta} - \dot{\lambda}_j| &\geq |r_m \sin \theta - \text{Im } \dot{\lambda}_j| \\ &\geq r_m \sin \theta - \frac{\pi}{4} \geq \left(\left(m + \frac{1}{2}\right) \sin \frac{1}{m^\alpha} - \frac{1}{4} \right) \geq 1. \end{aligned}$$

Note that $r_m \sin \theta - \frac{\pi}{4}$ is the distance from $r_m e^{i\theta}$ to the horizontal line $\text{Im } z = \frac{\pi}{4}$. Using (34), $\sin \theta \geq \frac{2}{\pi} \theta$ for $0 \leq \theta \leq \pi/2$ and taking $N_1 \geq 2N_0$ larger if necessary we get

$$\begin{aligned} \int_{\frac{1}{m^\alpha}}^{\pi - \frac{1}{m^\alpha}} \frac{d\theta}{|r_m e^{i\theta} - \dot{\lambda}_j|} &\leq 2 \int_{\frac{1}{m^\alpha}}^{\frac{\pi}{2}} \frac{d\theta}{r_m \sin \theta - \frac{\pi}{4}} \\ &= \frac{\pi}{r_m} \int_{\frac{1}{m^\alpha}}^{\pi/2} \frac{d\theta}{\theta - \pi^2/8r_m} = O\left(\frac{\log m}{m}\right), \end{aligned}$$

and similarly

$$\int_{\pi + \frac{1}{m^\alpha}}^{2\pi - \frac{1}{m^\alpha}} \frac{d\theta}{|r_m e^{i\theta} - \dot{\lambda}_j|} = O\left(\frac{\log m}{m}\right),$$

uniformly in $j \in J_1(m)$. Hence

$$(35) \quad \int_0^{2\pi} \sum_{j \in J_1(m)} \frac{|\beta_j|}{|r_m e^{i\theta} - \dot{\lambda}_j|} d\theta = O\left(\frac{1}{m^\alpha}\right) \text{ as } m \rightarrow \infty.$$

Combining (31), (32), (33) and (35) yields

$$V_\beta(m\pi) = O(m^{2-2\alpha}),$$

for any $0 < \alpha < 1$. This completes the proof of the lemma. q.e.d.

Proof of Proposition 5.1. By assumption ω_β^* with $\beta \in \ell_{\mathbb{C}}^1$ satisfies $\int_{A_m} \omega_\beta^* = 0$ for any $m \in \mathbb{Z}$. Then Lemma 5.1 implies that $\xi_\beta^*|_{Z_\varphi} = 0$. Hence we can apply Theorem 1.2 and Lemma 5.2 to conclude that $\xi_\beta^* \equiv 0$. Evaluating ξ_β^* at $\lambda = \dot{\lambda}_k$ with $|k| > N_0$ we get from (28) and Proposition 2.3 that

$$0 = \xi_\beta^*(\dot{\lambda}_k) = 2\beta_k \prod_{m \neq k} \frac{\dot{\lambda}_k - \dot{\lambda}_m}{\pi_m}.$$

As $|k| > N_0$, $\dot{\lambda}_k$ is a simple zero of $\dot{\Delta}(\lambda)$ (cf. Proposition 2.1) and hence $\prod_{m \neq k} \frac{\dot{\lambda}_k - \dot{\lambda}_m}{\pi_m} \neq 0$. We, therefore, conclude that $\beta_k = 0$ for any $|k| > N_0$ and thus in view of (28),

$$\xi_\beta^*(\lambda) = p_\beta^*(\lambda) \frac{\prod_{|m| > N_0} \frac{\lambda - \dot{\lambda}_m}{\pi_m}}{\prod_{|m| \leq N_0} \pi_m}.$$

As $\xi_\beta^* \equiv 0$ it follows that $p_\beta^* \equiv 0$ implying that $\beta_k = 0$ for $|k| \leq N_0$. We thus have proved that $\beta = 0$ as claimed. q.e.d.

For any given $n \in \mathbb{Z}$ and $\beta = (\beta_k)_{k \neq n} \in \ell_n^1$ consider the holomorphic 1-form $\frac{\xi_\beta^n(\lambda)}{\sqrt{\mathcal{R}(\lambda)}} d\lambda$ defined in (18). Arguing in the same way as in the proof of Proposition 5.1 one obtains

Proposition 5.2. *Let $n \in \mathbb{Z}$ and $\beta = (\beta_k)_{k \neq n} \in \ell_n^1$ be arbitrary. If $\int_{A_m} \omega_\beta^n = 0$ for any $m \in \mathbb{Z}$, then $\beta = 0$.*

6. Proof of Proposition 4.1

The aim of this section is to prove Proposition 4.1 concerning the identity of the sum of all A -periods of a holomorphic differential of the form Ω_β^n .

Proof of Proposition 4.1. As the proof in the two cases $|n| \leq N_0$ and $|n| > N_0$ are similar, we consider the case $|n| > N_0$ only. Recall that for any $|n| > N_0$ and $\beta \in \ell_n^1$

$$\Omega_\beta^n = \frac{1 - \eta_\beta^n(\lambda)}{\lambda - \dot{\lambda}_n} \frac{\dot{\Delta}(\lambda)}{\sqrt{\mathcal{R}(\lambda)}} d\lambda,$$

where

$$(36) \quad \eta_\beta^n(\lambda) = \sum_{|j| > N_0, j \neq n} \frac{\beta_j}{\lambda - \dot{\lambda}_j} + \frac{p_\beta^n(\lambda)}{\prod_{|k| \leq N_0} (\lambda - \dot{\lambda}_k)}$$

is a meromorphic function that can have poles only at the points $\dot{\lambda}_j, j \in \mathbb{Z}$. Let $r_m := m\pi + \pi/2$ for $m > N_0$ with N_0 as in Proposition 2.1. As $\sum_{|k| \leq m} A_k$ is homologous to C_r^+ (cf. (11)) one has for any $m > N_0$

$$(37) \quad \sum_{|k| \leq m} \frac{1}{2\pi} \int_{A_k} \Omega_\beta^n = \frac{1}{2\pi} \int_{C_{r_m}^+} \Omega_\beta^n.$$

In view of [5, Lemma C.5], Proposition 2.1, Proposition 2.2, and the definition of the canonical root (6)

$$\frac{\dot{\Delta}(r_m e^{i\theta})}{\sqrt{\mathcal{R}(r_m e^{i\theta})}} = \frac{1}{i} (1 + o(1)) \quad \text{as } m \rightarrow \infty,$$

uniformly for $0 \leq \theta < 2\pi$. This together with (36) implies that uniformly for $\lambda \in C_{r_m}^+$

$$(38) \quad \Omega_\beta^n = \frac{1}{i} \left(\frac{1}{\lambda - \dot{\lambda}_n} + O\left(\frac{\eta_n^\beta(\lambda)}{m}\right) + o\left(\frac{1}{m}\right) \right) d\lambda \text{ as } m \rightarrow \infty,$$

with constants independent of $\lambda \in C_{r_m}^+$ and $m > N_0$. Combining (37) with (38) one gets for $m > \max\{n, N_0\}$

$$(39) \quad \sum_{|k| \leq m} \frac{1}{2\pi} \int_{A_k} \Omega_\beta^n = 1 + O\left(\max_{C_{r_m}^+} |\eta_n^\beta(\lambda)|\right) + o(1),$$

with constants uniform in $m > \max\{n, N_0\}$. Arguing in a similar way as in the proof of Lemma 5.2 one sees that

$$\max_{\lambda \in C_{r_m}^+} |\eta_n^\beta(\lambda)| \rightarrow 0 \text{ as } m \rightarrow \infty.$$

This combined with (39) yields Proposition 4.1. q.e.d.

As an immediate Corollary of Proposition 4.1 we get the following result for $\omega_\beta^n = \Omega^n - \Omega_\beta^n = \Omega_\gamma^n|_{\gamma=0} - \Omega_\beta^n$.

Corollary 6.1. *For any $n \in \mathbb{Z}$ and any $\beta \in \ell_{\hat{n}}^1$,*

$$\sum_{m \in \mathbb{Z}} \frac{1}{2\pi} \int_{A_m} \omega_\beta^n = \lim_{K \rightarrow \infty} \sum_{|m| \leq K} \frac{1}{2\pi} \int_{A_m} \omega_\beta^n = 0.$$

Corollary 6.1 can be combined with Proposition 5.2 yielding the following:

Proposition 6.1. *Let $n \in \mathbb{Z}$ and $\beta = (\beta_k)_{k \neq n} \in \ell_{\hat{n}}^1$ be arbitrary. If $\int_{A_m} \omega_\beta^n = 0$ for any $m \in \mathbb{Z} \setminus \{n\}$, then $\beta = 0$.*

7. Existence of normalized differentials

The aim of this section is to study the operators T^n and T^* , introduced in Section 4, and to prove Proposition 4.2, Proposition 4.3, and Proposition 4.4. We begin with the study of T^* .

Lemma 7.1. *Locally uniformly on \mathcal{W} , the coefficients T_{mj}^* of T^* satisfy the following estimates:*

$$T_{mj}^* = \begin{cases} O\left(\frac{|\dot{\lambda}_m - \tau_m| + |\gamma m|}{j - m}\right), & j \in \mathbb{Z}, |m| > N_0, m \neq j, \\ 1 + \ell^2(m), & |j| > N_0, m = j, \\ O\left(\frac{1}{1 + |j|}\right), & j \in \mathbb{Z}, |m| \leq N_0. \end{cases}$$

Proof. Case $|j| > N_0$ and $\lambda_m^+ = \lambda_m^-$: Recall that by Proposition 2.2 and the definition of the canonical root

$$\frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\mathcal{R}(\lambda)}} = \frac{1}{i} \prod_{k \in \mathbb{Z}} \frac{\dot{\lambda}_k - \lambda}{\sqrt[s]{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}},$$

and that for $|j| > N_0$, T_{mj}^* is given by

$$T_{mj}^* = \frac{1}{2\pi} \int_{\Gamma_m} \frac{1}{\lambda - \dot{\lambda}_j} \frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\mathcal{R}(\lambda)}} d\lambda.$$

As $\lambda_m^+ = \lambda_m^- = \tau_m = \dot{\lambda}_m$ one has by the definition of the standard root, $\sqrt[s]{(\lambda_m^+ - \lambda)(\lambda_m^- - \lambda)} = \tau_m - \lambda$, and hence, if in addition $m \neq j$, $\frac{1}{\lambda - \dot{\lambda}_j} \frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\mathcal{R}(\lambda)}}$ is holomorphic near τ_m and thus $T_{mj}^* = 0$. If $m = j$ (and hence $|m| > N_0$) one gets

$$(40) \quad T_{mm}^* = \frac{1}{2\pi} \int_{\Gamma_m} \frac{1}{\lambda - \dot{\lambda}_m} \frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\mathcal{R}(\lambda)}} d\lambda$$

$$(41) \quad = \frac{1}{2\pi i} \int_{\Gamma_m} \frac{1}{\lambda - \tau_m} \prod_{k \neq m} \frac{\dot{\lambda}_k - \lambda}{\sqrt[s]{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}} d\lambda.$$

Therefore, by the residue theorem and the product estimate in [5, Lemma C.3]

$$T_{mm}^* = \prod_{k \neq m} \frac{\dot{\lambda}_k - \tau_m}{\sqrt[s]{(\lambda_k^+ - \tau_m)(\lambda_k^- - \tau_m)}} = 1 + \ell^2(m),$$

locally uniformly in \mathcal{W} .

Case $|j| > N_0$ and $\lambda_m^+ \neq \lambda_m^-$: If $m = j$, one uses again [5, Lemma C.3] to see that

$$\begin{aligned} T_{mm}^* &= \frac{1}{2\pi i} \int_{\Gamma_m} \frac{-1}{\sqrt[s]{(\lambda_m^+ - \lambda)(\lambda_m^- - \lambda)}} \prod_{k \neq m} \frac{\dot{\lambda}_k - \lambda}{\sqrt[s]{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}} d\lambda \\ &= -\frac{1}{2\pi i} \int_{\Gamma_m} \frac{d\lambda}{\sqrt[s]{(\lambda_m^+ - \lambda)(\lambda_m^- - \lambda)}} + \ell^2(m). \end{aligned}$$

A direct calculation shows that $\frac{1}{2\pi i} \int_{\Gamma_m} \frac{d\lambda}{\sqrt[s]{(\lambda_m^+ - \lambda)(\lambda_m^- - \lambda)}} = -1$ yielding the claimed estimate $T_{mm}^* = 1 + \ell^2(m)$ in this case. If $|m| > N_0$ but $m \neq j$, then

$$(42) \quad T_{mj}^* = \frac{1}{2\pi i} \int_{\Gamma_m} \frac{B_{mj}(\lambda) d\lambda}{\sqrt[s]{(\lambda_m^+ - \lambda)(\lambda_m^- - \lambda)}},$$

where

$$(43) \quad B_{mj} := -\frac{\dot{\lambda}_m - \lambda}{\sqrt{s(\lambda_j^+ - \lambda)(\lambda_j^- - \lambda)}} \prod_{k \neq m, j} \frac{\dot{\lambda}_k - \lambda}{\sqrt{s(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}}.$$

By deforming the contour Γ_m to the straight interval G_m (taken twice) one then sees (cf. [5, Lemma 14.3]) that

$$|T_{mj}^*| \leq \max_{\lambda \in G_m} |B_{mj}(\lambda)|.$$

For $\lambda \in G_m$, $|\dot{\lambda}_m - \lambda| \leq |\dot{\lambda}_m - \tau_m| + |\gamma_m|$ and

$$\left(\sqrt{s(\lambda_j^+ - \lambda)(\lambda_j^- - \lambda)} \right)^{-1} = O\left(\frac{1}{j - m}\right),$$

whereas again with [5, Lemma C.3], $\prod_{k \neq m, j} \frac{\dot{\lambda}_k - \lambda}{\sqrt{s(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}} = O(1)$ uniformly on \mathcal{W} . Altogether we thus conclude that in the case considered $T_{mj}^* = O\left(\frac{|\dot{\lambda}_m - \tau_m| + |\gamma_m|}{j - m}\right)$. Finally, if $|m| \leq N_0$ and hence $m \neq j$ (as we assume $|j| > N_0$), one has that (42) and (43) hold. Hence,

$$|T_{mj}^*| \leq \max_{\lambda \in \Gamma_m} \left| \frac{B_{mj}(\lambda)}{\sqrt{s(\lambda_m^+ - \lambda)(\lambda_m^- - \lambda)}} \right| \text{length}(\Gamma_m)/2\pi,$$

where $\text{length}(\Gamma_m)$ is the Euclidean length of Γ_m . Using that λ_m^\pm are inside Γ'_m and hence uniformly in \mathcal{W} separated from Γ_m and as by definition, different contours are apart by a uniform constant one concludes from the estimate $\left(\sqrt{s(\lambda_j^+ - \lambda)(\lambda_j^- - \lambda)} \right)^{-1} = O\left(\frac{1}{j}\right)$ and [11, Lemma C.3] that $T_{mj}^* = O\left(\frac{1}{j}\right)$ uniformly on \mathcal{W} .

Case $|j| \leq N_0$: In this case the coefficient T_{mj}^* is given by the formula

$$\begin{aligned} T_{mj}^* &= \frac{1}{2\pi} \int_{\Gamma_m} \lambda^{N_0+j} \frac{\dot{\Delta}(\lambda)}{\prod_{|k| \leq N_0} (\lambda - \dot{\lambda}_k)} \frac{d\lambda}{\sqrt[2]{\mathcal{R}(\lambda)}} \\ &= \frac{1}{2\pi i} \int_{\Gamma_m} \frac{-\lambda^{N_0+j}}{\prod_{|k| \leq N_0} \sqrt{s(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}} \prod_{|k| > N_0} \frac{\dot{\lambda}_k - \lambda}{\sqrt{s(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}} d\lambda. \end{aligned}$$

If $|m| > N_0$ we apply [11, Lemma 14.3] to conclude that,

$$|T_{mj}^*| \leq \max_{\lambda \in G_m} \left| \frac{-\lambda^{N_0+j}(\dot{\lambda}_m - \lambda)}{\prod_{|k| \leq N_0} \sqrt{s(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}} \prod_{|k| > N_0, k \neq m} \frac{\dot{\lambda}_k - \lambda}{\sqrt{s(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}} \right|.$$

As for $|m| > N_0$, one has

$$\max_{\lambda \in G_m} \left| \prod_{|k| \leq N_0} \frac{\lambda^{N_0+j}}{\sqrt{s(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}} \right| = O\left(\frac{1}{m}\right),$$

it follows by the product estimate [5, Lemma C.3] that

$$T_{mj}^* = O\left(\frac{|\dot{\lambda}_m - \tau_m| + |\gamma_m|}{m}\right),$$

uniformly on \mathcal{W} . If $|m| \leq N_0$ we use again that λ_m^\pm are inside Γ'_m , hence uniformly in \mathcal{W} separated from Γ_m , and that different contours are apart by a uniform constant, to see that $T_{mj}^* = O(1)$. The claimed estimates for T_{mj}^* are proved. q.e.d.

From Lemma 7.1 it immediately follows that T^* defines a bounded linear operator, $T^* : \ell^1 \rightarrow \ell^1$.

Proof of Proposition 4.4. Take $\varphi \in \mathcal{W}$. By Proposition 5.1, T^* is injective. We claim that $T^* - Id$ is a compact operator on ℓ^1 . Therefore, T^* is Fredholm and thus T^* is a linear isomorphism. To see that $T^* - Id$ is compact introduce for any $N > N_0$ the operator $K_N : \ell^1 \rightarrow \ell^1$,

$$K_N := \Pi_N \circ (T^* - Id),$$

where $\Pi_N : \ell^1 \rightarrow \ell^1$ is the projection,

$$(\beta_k)_{k \in \mathbb{Z}} \mapsto (\dots, 0, \beta_{-N}, \dots, \beta_N, 0, \dots).$$

Note that K_N is an operator of finite rank and, therefore, compact. By Lemma 7.1 we have for any $N \geq N_0$,

$$(T^* - Id - K_N)_{mj} = \begin{cases} O\left(\frac{|\dot{\lambda}_m - \tau_m| + |\gamma_m|}{j-m}\right), & j \in \mathbb{Z} \setminus \{m\}, |m| > N, \\ \ell^2(m), & |m| > N, j = m, \\ 0, & j \in \mathbb{Z}, |m| \leq N. \end{cases}$$

Hence there exists a constant $C > 0$ independent of $N \geq N_0$ so that for any $N \geq N_0$ and any $\beta \in \ell^1$,

$$\begin{aligned} \|(T^* - Id - K_N)\beta\|_{\ell^1} &\leq C \sum_{|m| > N} \sum_{j \neq m} \frac{|\dot{\lambda}_m - \tau_m| + |\gamma_m|}{|j - m|} |\beta_j| \\ &\quad + \sum_{|m| > N} \ell^2(m) |\beta_m|. \end{aligned}$$

Clearly,

$$\sum_{|m| > N} \ell^2(m) |\beta_m| = \left(\sup_{|m| > N} |\ell^2(m)| \right) \|\beta\|_{\ell^1} \rightarrow 0,$$

as $N \rightarrow \infty$. By changing the order of summation we get from the Cauchy–Schwartz inequality

$$\sum_{|m|>N} \sum_{j \neq m} \frac{|\dot{\lambda}_m - \tau_m| + |\gamma_m|}{|j - m|} |\beta_j| \leq C \|\beta\|_{\ell^1} \left(\sum_{|m|>N} |\dot{\lambda}_m - \tau_m|^2 + |\gamma_m|^2 \right)^{1/2}.$$

Altogether it then follows that

$$\|T^* - Id - K_N\|_{\mathcal{L}(\ell^1)} \rightarrow 0 \text{ as } N \rightarrow \infty,$$

showing that $T^* - Id$ is compact.

q.e.d.

Next we want to prove Proposition 4.2. First we establish the following two lemmas:

Lemma 7.2. *For any $n \in \mathbb{Z}$ and for any $\varphi \in \mathcal{W}$ the coefficients b_m^n , $m \neq n$, of b^n satisfy*

$$b_m^n - b_m^* = \begin{cases} O\left(\frac{|\dot{\lambda}_m - \tau_m| + |\gamma_m|}{m-n}\right), & |m| > N_0, \quad m \neq n, \\ O\left(\frac{1}{n}\right), & |m| \leq N_0, \quad m \neq n, \end{cases}$$

uniformly in $n \in \mathbb{Z}$ and locally uniformly in \mathcal{W} . In particular, it follows that b^n is in ℓ_n^1 .

Proof. As the cases $|n| > N_0$ and $|n| \leq N_0$ can be treated in the same way, we only consider the case $|n| > N_0$.

Case $|m| > N_0$: Taking into account that $b_m^* = \frac{1}{2\pi} \int_{\Gamma_m} \frac{\dot{\Delta}(\lambda)}{\varepsilon/\mathcal{R}(\lambda)} d\lambda$ and $b_m^n = \frac{1}{2\pi} \int_{\Gamma_m} \frac{m\pi - n\pi}{\lambda - \lambda_n} \frac{\dot{\Delta}(\lambda)}{\varepsilon/\mathcal{R}(\lambda)} d\lambda$ we see that

$$b_m^n - b_m^* = \frac{1}{2\pi i} \int_{\Gamma_m} \left(\frac{m\pi - n\pi}{\lambda - \lambda_n} - 1 \right) \frac{\dot{\lambda}_m - \lambda}{\sqrt{s(\lambda_m^+ - \lambda)(\lambda_m^- - \lambda)}} \Pi_m(\lambda) d\lambda,$$

where $\Pi_m(\lambda) := \prod_{k \neq m} \frac{\dot{\lambda}_k - \lambda}{\sqrt{s(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}}$. If $\lambda_m^+ = \lambda_m^-$, one has $\dot{\lambda}_m = \lambda_m^+ = \lambda_m^-$ and $\sqrt{s(\lambda_m^+ - \lambda)(\lambda_m^- - \lambda)} = \dot{\lambda}_m - \lambda$. As $m \neq n$ one then concludes from the analyticity of the integrand in $D_m(\pi/4)$ that $b_m^n = 0$. If $\lambda_m^+ \neq \lambda_m^-$ we first note that

$$\frac{m\pi - n\pi}{\lambda - \dot{\lambda}_n} - 1 = \frac{(m\pi - \lambda) + (\dot{\lambda}_n - n\pi)}{\lambda - \dot{\lambda}_n} = O\left(\frac{1}{m-n}\right),$$

where we used that $\dot{\lambda}_n - n\pi = O(1)$ by Proposition 2.1. Furthermore, by [5, Lemma C.3], $\Pi_m(\lambda) = O(1)$. By deforming the contour Γ_m to the straight interval G_m (taken twice) and by using that for $\lambda \in G_m$, $|\dot{\lambda}_m - \lambda| \leq |\dot{\lambda}_m - \tau_m| + |\gamma_m|$, one sees from [11, Lemma 14.3] that $b_m^n - b_m^* = O\left(\frac{|\dot{\lambda}_m - \tau_m| + |\gamma_m|}{m-n}\right)$.

Case $|m| \leq N_0$: We can argue similarly as above to conclude that $b_m^n - b_m^* = O(\frac{1}{n})$ (cf. Lemma 7.1).

Going through the arguments of the proof one verifies that the estimates obtained are uniform in $n \in \mathbb{Z}$ and locally uniform on \mathcal{W} . q.e.d.

The coefficients T_{mj}^n can be estimated using Lemma 7.1 by writing $T_{mj}^n = T_{mj}^* + R_{mj}^n$ ($m, j \neq n$) where R_{mj}^n is defined for $|n| > N_0$ as follows

$$R_{mj}^n := \begin{cases} \frac{1}{2\pi} \int_{\Gamma_m} \left(\frac{m\pi - n\pi}{\lambda - \dot{\lambda}_n} - 1 \right) \frac{\lambda^{N_0+j}}{\prod_{|k| \leq N_0} (\lambda - \dot{\lambda}_k)} \frac{\dot{\Delta}(\lambda)}{\varepsilon \sqrt{\mathcal{R}(\lambda)}} d\lambda, & |j| \leq N_0, \\ \frac{1}{2\pi} \int_{\Gamma_m} \left(\frac{m\pi - n\pi}{\lambda - \dot{\lambda}_n} - 1 \right) \frac{\dot{\Delta}(\lambda)}{\lambda - \dot{\lambda}_j} \frac{d\lambda}{\varepsilon \sqrt{\mathcal{R}(\lambda)}}, & |j| > N_0, \end{cases}$$

whereas in the case $|n| \leq N_0$,²

$$R_{mj}^n := \begin{cases} \frac{1}{2\pi} \int_{\Gamma_m} \left(\frac{m\pi - N_0\pi}{\lambda - \dot{\lambda}_{N_0}} - 1 \right) \frac{\lambda^{N_0+j-\varepsilon_j^n}}{\prod_{|k| \leq N_0, k \neq N_0} (\lambda - \dot{\lambda}_k)} \frac{\dot{\Delta}(\lambda)}{\varepsilon \sqrt{\mathcal{R}(\lambda)}} d\lambda, & |j| \leq N_0, \\ \frac{1}{2\pi} \int_{\Gamma_m} \left(\frac{m\pi - N_0\pi}{\lambda - \dot{\lambda}_{N_0}} - 1 \right) \frac{\dot{\Delta}(\lambda)}{\lambda - \dot{\lambda}_j} \frac{d\lambda}{\varepsilon \sqrt{\mathcal{R}(\lambda)}}, & |j| > N_0. \end{cases}$$

Note that by Proposition 2.2, for $\lambda \in \Gamma_m$ and $|n| > N_0$,

$$\frac{m\pi - n\pi}{\lambda - \dot{\lambda}_n} - 1 = \frac{(m\pi - \lambda) + (\dot{\lambda}_n - n\pi)}{\lambda - \dot{\lambda}_n} = O\left(\frac{1}{m - n}\right).$$

It is convenient to rewrite R_{mj}^n in the case $|n| > N_0$ as follows

$$(44) \quad \begin{cases} \frac{1}{2\pi i} \int_{\Gamma_m} \frac{(m\pi - \lambda) + (\dot{\lambda}_n - n\pi)}{\sqrt{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}} \frac{\lambda^{N_0+j}}{\prod_{|k| \leq N_0} \sqrt{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}} \Pi_n(\lambda) d\lambda, & |j| \leq N_0, \\ \frac{1}{2\pi i} \int_{\Gamma_m} \frac{(m\pi - \lambda) + (\dot{\lambda}_n - n\pi)}{\sqrt{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}} \frac{1}{\sqrt{(\lambda_j^+ - \lambda)(\lambda_j^- - \lambda)}} \Pi_{nj}(\lambda) d\lambda, & |j| > N_0, \end{cases}$$

where

$$\Pi_n(\lambda) := \prod_{|k| > N_0, k \neq n} \frac{\dot{\lambda}_k - \lambda}{\sqrt{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}}$$

and

$$(45) \quad \Pi_{nj}(\lambda) := \prod_{k \neq n, j} \frac{\dot{\lambda}_k - \lambda}{\sqrt{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}}.$$

In the case $|n| \leq N_0$, similar identities hold.

²See (20) for the definition of ε_j^n .

Lemma 7.3. *For any $n \in \mathbb{Z}$ and for any $\varphi \in \mathcal{W}$, the coefficients $R_{m,j}^n$ satisfy the following estimates:*

$$R_{m,j}^n = \begin{cases} O\left(\frac{|\dot{\lambda}_m - \tau_m| + |\gamma_m|}{(m-n)(j-m)}\right), & j \in \mathbb{Z} \setminus \{n\}, |m| > N_0, m \neq j, n, \\ \frac{\ell^2(m) + \ell^2(n)}{m-n}, & |j| > N_0, m = j, m \neq n, \\ O\left(\frac{1}{(m-n)(1+|j|)}\right), & j \in \mathbb{Z} \setminus \{n\}, |m| \leq N_0, m \neq n, \end{cases}$$

uniformly in n and locally uniformly on \mathcal{W} .

Proof. As the cases $|n| > N_0$ and $|n| \leq N_0$ are proved in the same way we will only consider the case $|n| > N_0$.

Throughout the proof we assume that $m \neq n$ and $j \neq n$. The proof is very similar to the one of Lemma 7.1.

Case $|j| > N_0$ and $\lambda_m^+ = \lambda_m^- (= \tau_m)$: We argue as in the proof of Lemma 7.1 to see that for $m \neq j$, $R_{m,j}^n = 0$. If $m = j$ (and hence $|m| > N_0$) one gets from the residue theorem and [5, Lemma C.3] that

$$\begin{aligned} R_{mm}^n &= \frac{(m\pi - \tau_m) + (\dot{\lambda}_n - n\pi)}{\sqrt[s]{(\lambda_n^+ - \tau_m)(\lambda_n^- - \tau_m)}} \prod_{k \neq m, n} \frac{\dot{\lambda}_k - \tau_m}{\sqrt[s]{(\lambda_k^+ - \tau_m)(\lambda_k^- - \tau_m)}} \\ &= O\left(\frac{(m\pi - \tau_m) + (\dot{\lambda}_n - n\pi)}{m - n}\right). \end{aligned}$$

By Proposition 2.2, $m\pi - \tau_m = \ell^2(m)$ and $\dot{\lambda}_n - n\pi = \ell^2(n)$, hence $R_{mm}^n = \frac{\ell^2(m) + \ell^2(n)}{m-n}$.

Case $|j| > N_0$ and $\lambda_m^+ \neq \lambda_m^-$: If $j = m$ (and hence $|m| > N_0$) one deforms the contour Γ_m to the straight interval G_m (taken twice) and obtains from [5, Lemma 14.3] that

$$\begin{aligned} R_{mm}^n &= \frac{1}{2\pi i} \int_{\Gamma_m} \frac{(m\pi - \lambda) + (\dot{\lambda}_n - n\pi)}{\sqrt[s]{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}} \frac{1}{\sqrt[s]{(\lambda_m^+ - \lambda)(\lambda_m^- - \lambda)}} \Pi_{nm}(\lambda) d\lambda \\ &= \frac{\ell^2(m) + \ell^2(n)}{m - n}, \end{aligned}$$

where $\Pi_{mn}(\lambda)$ is defined in (45). If $|m| > N_0$, but $m \neq j$, one argues similarly: Deforming the contour Γ_m to G_m (taken twice) one sees that

$$|R_{m,j}^n| \leq \max_{\lambda \in G_m} \left| \frac{(m\pi - \lambda) + (\dot{\lambda}_n - n\pi)}{\sqrt[s]{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}} \frac{(\dot{\lambda}_m - \lambda)}{\sqrt[s]{(\lambda_j^+ - \lambda)(\lambda_j^- - \lambda)}} \Pi_{mjn}(\lambda) \right|,$$

where $\Pi_{mjn}(\lambda) := \prod_{k \neq m, j, n} \frac{\dot{\lambda}_k - \lambda}{\sqrt[s]{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}}$. As for $|m| > N_0$ and $\lambda \in G_m$,

$$|\dot{\lambda}_m - \lambda| \leq |\dot{\lambda}_m - \tau_m| + |\gamma_m| \text{ and } \left(\sqrt[s]{(\lambda_j^+ - \lambda)(\lambda_j^- - \lambda)}\right)^{-1} = O\left(\frac{1}{j-m}\right),$$

one concludes that $R_{m,j}^n = O\left(\frac{|\dot{\lambda}_m - \tau_m| + |\gamma_m|}{(m-n)(j-m)}\right)$. Finally, if $|m| \leq N_0$ and

hence $m \neq j$ (as we assume $|j| > N_0$), one argues as in the proof of Lemma 7.1 to conclude that

$$R_{mj}^n = O\left(\frac{1}{(m-n)(1+|j|)}\right).$$

Case $|j| \leq N_0$: In this case R_{mj}^n is given by the first equation in (44). If $|m| > N_0$ note that

$$\begin{aligned} \max_{\lambda \in G_m} \left| \frac{\lambda^{N_0+j}}{\prod_{|k| \leq N_0} \sqrt[3]{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}} \right| &= O\left(\frac{1}{m}\right), \\ \prod_{|k| > N_0, k \neq n, m} \frac{\dot{\lambda}_k - \lambda}{\sqrt[3]{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}} &= O(1), \end{aligned}$$

and $|\dot{\lambda}_m - \lambda| \leq |\dot{\lambda}_m - \tau_m| + |\gamma_m|$ for any $\lambda \in G_m$. This together with [5, Lemma 14.3] implies that,

$$R_{mj}^n = O\left(\frac{|\dot{\lambda}_m - \tau_m| + |\gamma_m|}{(m-n)m}\right) \left(= O\left(\frac{|\dot{\lambda}_m - \tau_m| + |\gamma_m|}{(m-n)(j-m)}\right) \right).$$

In the remaining case $|m| \leq N_0$ we argue again as in the proof of Lemma 7.1 to see that $R_{mj}^n = O(1)$. The claimed estimates for R_{mj}^n are thus proved. Going through the arguments of the proofs one sees that the derived estimates hold uniformly in n and locally uniformly on \mathcal{W} .
q.e.d.

From Lemma 7.1 and Lemma 7.3 it immediately follows that for each $n \in \mathbb{Z}$, T^n defines a bounded linear operator, $T^n : \ell_{\hat{n}}^1 \rightarrow \ell_{\hat{n}}^1$.

Proof of Proposition 4.2. By Proposition 6.1, T^n is injective. Arguing as in the proof of Proposition 4.4 one sees that $T^n - Id$ is a compact operator on $\ell_{\hat{n}}^1$. Therefore, T^n is Fredholm and thus T^n a linear isomorphism. Finally, by Lemma 7.2, $b^n \in \ell_{\hat{n}}^1$.
q.e.d.

Now let us turn to the proof of Proposition 4.3. Recall that for any $n \in \mathbb{Z}$ and for any given φ in \mathcal{W} we denote by $\beta^n \equiv \beta^n(\varphi)$ the unique solution of $T^n \beta = b^n$. In this way we obtain maps

$$\beta^n : \mathcal{W} \rightarrow \ell_{\hat{n}}^1,$$

and

$$\zeta_n : \mathbb{C} \times \mathcal{W} \rightarrow \mathbb{C}, \quad (\lambda, \varphi) \mapsto \zeta_{\beta^n}^n(\lambda).$$

Similarly, for any given $\varphi \in \mathcal{W}$ we denote by $\beta^* \equiv \beta^*(\varphi)$ the unique solution of the linear system $T^* \beta = b^*$ (see Proposition 4.4). Recall that by Lemma 2.2, b^* is in l^1 and independent of $\varphi \in \mathcal{W}$. In this way we obtain a map

$$\beta^* : \mathcal{W} \rightarrow \ell^1.$$

Lemma 7.4. *For any $n \in \mathbb{Z}$, (i) $b^n : \mathcal{W} \rightarrow \ell_n^1$ and (ii) $T^n : \mathcal{W} \rightarrow \mathcal{L}(\ell_n^1)$ are analytic maps. In addition, (iii) $T^* : \mathcal{W} \rightarrow \mathcal{L}(\ell^1)$ is an analytic map.*

Combining Lemma 7.4 with Proposition 4.2 and Proposition 4.4 one gets

Corollary 7.1. *For any $n \in \mathbb{Z}$ the maps $\beta^n : \mathcal{W} \rightarrow \ell_n^1$ and $\beta^* : \mathcal{W} \rightarrow \ell^1$ are analytic.*

Proof of Lemma 7.4. (i) Let us first consider the case $|n| > N_0$. According to the definition of $b^n = (b_m^n)_{m \neq n}$ in Section 4,

$$b_m^n = \frac{1}{2\pi} \int_{\Gamma_m} \frac{m\pi - n\pi}{\lambda - \dot{\lambda}_n} \frac{\dot{\Delta}(\lambda)}{\sqrt[\varepsilon]{\Delta^2(\lambda) - 4}} d\lambda.$$

By [5, Theorem A.3] it suffices to show that b^n is locally bounded and weakly analytic. By Lemma 7.2 and the Cauchy–Schwartz estimate, b^n is locally bounded. As the dual of ℓ_n^1 is $\ell_n^\infty \equiv \ell^\infty(\mathbb{Z} \setminus \{n\}, \mathbb{C})$, in view of Montel’s theorem, the weak analyticity of b^n then follows once we prove that each component b_m^n , $m \neq n$, of b^n is analytic on \mathcal{W} . To this end let us analyze the integrand in the definition of b_m^n . Recall that $\Delta, \dot{\Delta} : \mathbb{C} \times L_c^2 \rightarrow \mathbb{C}$ are analytic maps. As $|n| > N_0$ by assumption, $\dot{\lambda}_n$ is a simple zero of $\dot{\Delta}(\lambda)$ (cf. Proposition 2.1) and hence we obtain from the implicit function theorem that $\dot{\lambda}_n : \mathcal{W} \rightarrow \mathbb{C}$ is analytic. By construction, $\sqrt[\varepsilon]{\Delta^2(\lambda) - 4}$ is analytic on $G \setminus \sqcup_{k \in \mathbb{Z}} G_k$. In view of the definition of Γ_m and G_m , $m \in \mathbb{Z}$ (Section 2) there exists $\varepsilon > 0$, independent of m , so that $\sqrt[\varepsilon]{\Delta^2 - 4}$ is analytic on $U_\varepsilon(\Gamma_m) \times \mathcal{W}$, where $U_\varepsilon(\Gamma_m)$ is the ε -tubular neighborhood of Γ_m , $U_\varepsilon(\Gamma_m) := \{\lambda \in \mathbb{C} \mid \text{dist}(\lambda, \Gamma_m) < \varepsilon\}$. It then follows that for any $m \neq n$, $b_m^n : \mathcal{W} \rightarrow \mathbb{C}$ is analytic.

In the case $|n| \leq N_0$, b_m^n is defined for any $m \neq n$ by

$$b_m^n = \frac{1}{2\pi} \int_{\Gamma_m} \frac{m\pi - N_0\pi}{\lambda - \dot{\lambda}_{N_0}} \frac{\dot{\Delta}(\lambda)}{\sqrt[\varepsilon]{\Delta^2(\lambda) - 4}} d\lambda.$$

By the choice of N_0 in Proposition 2.1, for any $\varphi \in \mathcal{W}$, $\dot{\lambda}_{N_0}$ is the only root of $\dot{\Delta}(\cdot, \varphi)$ in $D_{N_0}(\pi/4)$. Hence arguing as above, $\dot{\lambda}_{N_0} : \mathcal{W} \rightarrow \mathbb{C}$ is analytic. Using the same arguments as in the case $|n| > N_0$ one sees that $b^n : \mathcal{W} \rightarrow \mathbb{C}$ is analytic also in this case.

(ii) As the proofs for $|n| > N_0$ and $|n| \leq N_0$ are similar, we consider the case $|n| > N_0$ only. In this case, the coefficients T_{mj}^n ($m, j \in \mathbb{Z} \setminus \{n\}$) of T^n are given by

$$T_{mj}^n = \begin{cases} \frac{1}{2\pi} \int_{\Gamma_m} \frac{m\pi - n\pi}{\lambda - \dot{\lambda}_n} \frac{1}{\lambda - \dot{\lambda}_j} \frac{\dot{\Delta}(\lambda)}{\sqrt[\varepsilon]{\Delta^2(\lambda) - 4}} d\lambda, & |j| > N_0, \\ \frac{1}{2\pi} \int_{\Gamma_m} \frac{m\pi - n\pi}{\lambda - \dot{\lambda}_n} \frac{\dot{\Delta}(\lambda)}{\prod_{|k| \leq N_0} (\lambda - \dot{\lambda}_k)} \frac{\lambda^{N_0+j}}{\sqrt[\varepsilon]{\Delta^2(\lambda) - 4}} d\lambda, & |j| \leq N_0. \end{cases}$$

Again by [5, Theorem A.3] it suffices to show that T^n is locally bounded and weakly analytic. By Lemma 7.1 and Lemma 7.3, T^n is locally bounded (see the arguments in the proof of Proposition 4.4). It remains to show that T^n is weakly analytic. First note that by arguing as in (i) one sees that for any $m, j \in \mathbb{Z} \setminus \{n\}$, $T_{m_j}^n : \mathcal{W} \rightarrow \mathbb{C}$ is analytic. Since $T^n : \mathcal{W} \rightarrow \mathcal{L}(\ell_n^1)$ is locally bounded so is for any $m \neq n$ the map

$$T_m^n : \mathcal{W} \rightarrow \ell_n^\infty, \varphi \mapsto (T_{m_j}^n(\varphi))_{j \neq m}.$$

Using that the components $T_{m_j}^n$ of T_m^n are analytic it then follows from [11, Theorem A.3] that $T_m^n : \mathcal{W} \rightarrow \ell_n^\infty$ is analytic. As a consequence, for each $N > |n|$, the map $\Pi_N \circ T^n : \mathcal{W} \rightarrow \mathcal{L}(\ell_n^1)$ is analytic, where $\Pi_N : \ell_n^1 \rightarrow \ell_n^1$ denotes the projection

$$(\beta_j)_{j \neq n} \mapsto (\dots, 0, \beta_{-N}, \dots, \beta_N, 0, \dots).$$

To show that $T^n : \mathcal{W} \rightarrow \mathcal{L}(\ell_n^1)$ is weakly analytic it suffices to show that on any disk $\mathcal{D}_{\varphi,h} := \{\varphi + zh \mid z \in \mathbb{C}, |z| < 1\}$ with closure $\overline{\mathcal{D}_{\varphi,h}} \subseteq \mathcal{W}$, $\Pi_N \circ T^n$ converges in $\mathcal{L}(\ell_n^1)$ to T^n locally uniformly in $\mathcal{D}_{\varphi,h}$ as $N \rightarrow \infty$. Indeed, if this is the case it follows from the Weierstrass theorem that $T^n|_{\mathcal{D}_{\varphi,h}} : \mathcal{D}_{\varphi,h} \rightarrow \mathcal{L}(\ell_n^1)$ is analytic, establishing in this way that T^n is weakly analytic. To see that $\Pi_N \circ T^n$ converges locally uniformly on $\mathcal{D}_{\varphi,h}$ to T^n as $N \rightarrow \infty$, observe that $\overline{\mathcal{D}_{\varphi,h}}$ is a compact subset of \mathcal{W} . The claimed convergence thus follows from the estimates of Lemma 7.1 and Lemma 7.3 (cf. the proof of Proposition 4.4).

(iii) Arguing in the same way as above one shows that $T^* : \mathcal{W} \rightarrow \mathcal{L}(\ell^1)$ is analytic. q.e.d.

Proof of Proposition 4.3. Take $n \in \mathbb{Z}$. Let us begin by showing that $\beta^n = (T^n)^{-1}b^n$, $\beta^n : \mathcal{W} \rightarrow \ell_n^1$, is analytic. As by Lemma 7.4 (ii) $T^n : \mathcal{W} \rightarrow \mathcal{L}(\ell_n^1)$ is analytic and by Proposition 4.2 (ii), $T^n(\varphi) \in \mathcal{L}(\ell_n^1)$ is a linear isomorphism for any $\varphi \in \mathcal{W}$, it follows that $(T^n)^{-1} : \mathcal{W} \mapsto \mathcal{L}(\ell_n^1), \varphi \mapsto (T^n(\varphi))^{-1}$ is analytic as well. This combined with the analyticity of b^n , established in Lemma 7.4 (i), implies that β^n is analytic.

Let us now turn towards ζ_n . As the cases $|n| > N_0$ and $|n| \leq N_0$ are proved in the same way we consider only $|n| > N_0$. Then ζ_n is given by

$$\zeta_n : \mathbb{C} \times \mathcal{W} \rightarrow \mathbb{C}, (\lambda, \varphi) \mapsto \frac{\dot{\Delta}(\lambda)}{\lambda - \dot{\lambda}_n} - \xi_{\beta^n}^n(\lambda).$$

As $|n| > N_0$, $\dot{\lambda}_n : \mathcal{W} \rightarrow \mathbb{C}$ is analytic and so is $\mathbb{C} \times \mathcal{W} \rightarrow \mathbb{C}, (\lambda, \varphi) \mapsto \frac{\dot{\Delta}(\lambda)}{\lambda - \dot{\lambda}_n}$, as $\lambda - \dot{\lambda}_n$ is a factor in the product representation for $\dot{\Delta}(\lambda)$ (Proposition 2.3). Recall that for any $\beta \in \ell_n^1$ and $|n| > N_0$,

$$\xi_{\beta}^n(\lambda) = \sum_{|j| > N_0} \beta_j \frac{\dot{\Delta}(\lambda)}{(\lambda - \dot{\lambda}_j)(\lambda - \dot{\lambda}_n)}$$

$$+ \left(\sum_{j=0}^{2N_0} \beta_{j-N_0} \lambda^j \right) \frac{\dot{\Delta}(\lambda)}{(\lambda - \dot{\lambda}_n) \prod_{|k| \leq N_0} (\lambda - \dot{\lambda}_k)}.$$

As above one argues that $\frac{\dot{\Delta}(\lambda)}{(\lambda - \dot{\lambda}_j)(\lambda - \dot{\lambda}_n)}$ and $\frac{\dot{\Delta}(\lambda)}{(\lambda - \dot{\lambda}_n) \prod_{|k| \leq N_0} (\lambda - \dot{\lambda}_k)}$ are analytic on $\mathbb{C} \times \mathcal{W}$. In addition, they are locally bounded uniformly in j . By [11, Theorem A.3] it then follows that the mapping $\mathbb{C} \times \mathcal{W} \rightarrow \ell_n^\infty$, that assigns to $(\lambda, w) \in \mathbb{C} \times \mathcal{W}$ the sequence

$$\left(\left(\frac{\lambda^{N_0+j} \dot{\Delta}(\lambda)}{\prod_{|k| \leq N_0} (\lambda - \dot{\lambda}_k) \lambda - \dot{\lambda}_n} \right)_{|j| \leq N_0}, \left(\frac{\dot{\Delta}(\lambda)}{(\lambda - \dot{\lambda}_n)(\lambda - \dot{\lambda}_j)} \right)_{|j| > N_0, j \neq n} \right) \in \ell_n^\infty$$

is analytic. This combined with the result above, saying that $\beta^n : \mathcal{W} \rightarrow \ell_n^1$ is analytic it follows that

$$\xi_{\beta^n}^n : \mathbb{C} \times \mathcal{W} \rightarrow \mathbb{C}, \quad (\lambda, \varphi) \mapsto \xi_{\beta^n(\varphi)}^n(\lambda, \varphi)$$

is analytic. Finally, by construction, the identity $\frac{1}{2\pi} \int_{A_m} \frac{\zeta_n(\lambda)}{\sqrt{\mathcal{R}(\lambda)}} d\lambda = \delta_{nm}$ holds for any $n, m \in \mathbb{Z}$. q.e.d.

To finish this section we prove results for the limiting behavior of b^n, T^n , and β^n as $|n| \rightarrow \infty$. To this end introduce the maps $\hat{b}^n : \mathcal{W} \rightarrow \ell^1$, $\hat{T}^n : \mathcal{W} \rightarrow \mathcal{L}(\ell^1)$, and $\hat{\beta}^n : \mathcal{W} \rightarrow \ell^1$,

$$\hat{b}_m^n := \begin{cases} b_m^n, & m \neq n, \\ 0, & m = n, \end{cases} \quad \hat{\beta}_j^n = \begin{cases} \beta_j^n, & j \neq n, \\ 0, & j = n, \end{cases}$$

$$\hat{T}_{mj}^n := \begin{cases} T_{mj}^n, & j, m \in \mathbb{Z} \setminus \{n\}, \\ 0, & (j, m) \in \left((\mathbb{Z} \setminus \{n\}) \times \{n\} \right) \cup \left(\{n\} \times (\mathbb{Z} \setminus \{n\}) \right), \\ 1, & (j, m) = (n, n). \end{cases}$$

Lemma 7.5. *For any given $\varphi \in \mathcal{W}$,*

- (i) $\lim_{|n| \rightarrow \infty} \|\hat{b}^n - b^*\|_{\ell^1} = 0$;
- (ii) $\lim_{|n| \rightarrow \infty} \|\hat{T}^n - T^*\|_{\mathcal{L}(\ell^1)} = 0$;
- (iii) $\lim_{|n| \rightarrow \infty} \|\hat{\beta}^n - \beta^*\|_{\ell^1} = 0$.

Proof. Take $\varphi \in \mathcal{W}$. For simplicity of the notation we drop the argument φ in the quantities below.

(i) As $b_m^* = 0$ for $|m| \geq N_0$ (see (8)) we see from the definition of \hat{b}_m^n that for $|n| \geq N_0$,

$$\hat{b}_n^n - b_n^* = 0.$$

Moreover, by Lemma 7.2,

$$|\hat{b}_m^n - b_m^*| \leq \begin{cases} C \frac{|\dot{\lambda}_m - \tau_m| + |\gamma_m|}{|m-n|}, & |m| > N_0, m \neq n, \\ C \frac{1}{|m-n|}, & |m| \leq N_0, m \neq n. \end{cases}$$

By Proposition 2.2, $|\dot{\lambda}_m - \tau_m| + |\gamma_m| = \ell^2(m)$. Therefore, for $|n| \geq N_0$, $|\hat{b}_m^n - b_m^*| \leq C \frac{a_m}{|m-n|}$ where $(a_m)_{m \in \mathbb{Z}} \in \ell^2$. Thus, for $|n| \geq N_0$,

$$\begin{aligned} \sum_{m \in \mathbb{Z}} |\hat{b}_m^n - b_m^*| &\leq C \sum_{|m-n| \leq \frac{n}{2}, m \neq n} \frac{a_m}{|m-n|} + C \sum_{|m-n| > \frac{n}{2}} \frac{a_m}{|m-n|} \\ &\leq C \left(\sum_{|m-n| \leq \frac{n}{2}} a_m^2 \right)^{1/2} \left(\sum_{k \neq 0} \frac{1}{k^2} \right)^{1/2} \\ (46) \quad &+ C \left(\sum_{m \in \mathbb{Z}} a_m^2 \right)^{1/2} \left(\sum_{k \neq 0} \frac{1}{k^{4/3}} \right)^{1/2} \left(\frac{2}{|n|} \right)^{1/3} \end{aligned}$$

implying that $\lim_{|n| \rightarrow \infty} \|\hat{b}^n - b^*\|_{\ell^1} = 0$.

(ii) Using the same arguments as in the prove of (i) one concludes from Lemma 7.3 that the claimed convergence holds.

(iii) It follows from Proposition 4.2 and Proposition 4.4 that \hat{T}^n and T^* are linear isomorphisms in ℓ^1 . Hence (ii) implies that for any $\varphi \in \mathcal{W}$, $(\hat{T}^n)^{-1} \rightarrow (T^*)^{-1}$ in $\mathcal{L}(\ell^1)$ as $|n| \rightarrow \infty$. This together with (i) and $\hat{\beta}^n = (\hat{T}^n)^{-1} \hat{b}^n$ imply that for any $\varphi \in \mathcal{W}$,

$$\hat{\beta}^n = (\hat{T}^n)^{-1} \hat{b}^n \rightarrow (T^*)^{-1} b^* = \beta^*,$$

in ℓ^1 as $|n| \rightarrow \infty$. q.e.d.

8. Estimates of the zeros

In this section, we prove that the zeros of the analytic function $\zeta_n : \mathbb{C} \times \mathcal{W} \rightarrow \mathbb{C}$, introduced in Section 4, satisfy the properties stated in Theorem 1.3. The ansatz we have chosen for the ζ_n is well suited to obtain these claimed estimates. Recall from (25) that for any $|n| > N_0$

$$\zeta_n(\lambda) \equiv \zeta_n(\lambda, \varphi) = (1 - \eta_n(\lambda)) \frac{\dot{\Delta}(\lambda)}{\lambda - \dot{\lambda}_n},$$

where

$$\eta_n(\lambda) := \sum_{|j| > N_0, j \neq n} \frac{\beta_j^n}{\lambda - \dot{\lambda}_j} + \frac{p_n(\lambda)}{\prod_{|j| \leq N_0} (\lambda - \dot{\lambda}_j)},$$

and $p_n(\lambda) := p_{\beta^n}^n(\lambda, \varphi)$ is the polynomial introduced in Section 4 with β given by $\beta^n \equiv \beta^n(\varphi)$ of Proposition 4.3. Similarly, for $|n| \leq N_0$ we define

$$\zeta_n(\lambda) \equiv \zeta_n(\lambda, \varphi) = (1 - \eta_n(\lambda)) \frac{\dot{\Delta}(\lambda)}{\lambda - \dot{\lambda}_{N_0}},$$

where

$$\eta_n(\lambda) := \sum_{|j| > N_0} \frac{\beta_j^n}{\lambda - \dot{\lambda}_j} + \frac{p_n(\lambda)}{\prod_{|j| \leq N_0, j \neq N_0} (\lambda - \dot{\lambda}_j)}.$$

First note that for any $n \in \mathbb{Z}$, $\frac{\dot{\Delta}(\lambda)}{\lambda - \dot{\lambda}_n}$ is an entire function and by the argument principle one has in view of the choice of \mathcal{W} and Proposition 2.1 that for any $\varphi \in \mathcal{W}$

$$(47) \quad \frac{1}{2\pi i} \int_{\Gamma_m} \partial_\lambda \log \left(\frac{\dot{\Delta}(\lambda)}{\lambda - \dot{\lambda}_n} \right) d\lambda = 1 - \delta_{nm} \quad \forall |m| > N_0,$$

and

$$(48) \quad \dot{\lambda}_m = \frac{1}{2\pi i} \int_{\Gamma_m} \lambda \partial_\lambda \log \left(\frac{\dot{\Delta}(\lambda)}{\lambda - \dot{\lambda}_n} \right) d\lambda \quad \forall |m| > N_0, m \neq n,$$

whereas for any $N \geq N_0$

$$(49) \quad \frac{1}{2\pi i} \int_{\partial D_0(N\pi + \frac{\pi}{4})} \partial_\lambda \log \left(\frac{\dot{\Delta}(\lambda)}{\lambda - \dot{\lambda}_n} \right) d\lambda = \begin{cases} 2N, & |n| \leq N, \\ 2N + 1, & |n| > N. \end{cases}$$

Viewing $\zeta_n(\lambda)$ as a perturbation of $\frac{\dot{\Delta}(\lambda)}{\lambda - \dot{\lambda}_n}$ we want to argue in a similar fashion for $\zeta_n(\lambda)$. First we need to establish some auxiliary estimates.

Lemma 8.1. *For any $\varphi \in \mathcal{W}$, $\beta \in \ell^1$, $N > N_0 \geq 1$, $n \in \mathbb{Z}$, and $|m| > 2N$, one has*

- (i) $\sup_{\lambda \in \Gamma_m} \left(\sum_{|j| > N_0} \frac{|\beta_j|}{|\lambda - \dot{\lambda}_j|} \right) \leq C \left(|\beta_m| + \sum_{|j| > N, j \neq m} \frac{|\beta_j|}{|j - m|} + \|\beta\|_{\ell^1} \frac{1}{m} \right);$
- (ii) $\sup_{\lambda \in \Gamma_m} \left(\frac{|p_\beta^n(\lambda)|}{\prod_{|j| \leq N_0} |\lambda - \dot{\lambda}_j|} \right) \leq C \|\beta\|_{\ell^1} \frac{1}{m}, \quad |n| > N_0;$
- (iii) $\sup_{\lambda \in \Gamma_m} \left(\frac{|p_\beta^n(\lambda)|}{\prod_{|j| \leq N_0, j \neq N_0} |\lambda - \dot{\lambda}_j|} \right) \leq C \|\beta\|_{\ell^1} \frac{1}{m}, \quad |n| \leq N_0,$

where the constant $C > 0$ can be chosen uniformly in $N > N_0$, $n \in \mathbb{Z}$, $|m| > 2N$, and $\varphi \in \mathcal{W}$.

Proof. For any $|m| > 2N$ and $\lambda \in \Gamma_m$ one has by the choice of \mathcal{W} and Proposition 2.1, $1/|\lambda - \dot{\lambda}_m| \leq \pi/12$ as $\text{dist}(\dot{\lambda}_m, \Gamma_m) \geq \frac{\pi}{4} - \frac{\pi}{6}$. In view of the choice of \mathcal{W} and Proposition 2.1 it then follows that for $|m| > 2N$ and $\lambda \in \Gamma_m$

$$\begin{aligned} \sum_{|j| > N_0} \frac{|\beta_j|}{|\lambda - \dot{\lambda}_j|} &\leq C \left(|\beta_m| + \sum_{|j| > N, j \neq m} \frac{|\beta_j|}{|j - m|} \right) + \sum_{N_0 < |j| \leq N} |\beta_j| \frac{1}{|\lambda - \dot{\lambda}_j|} \\ &\leq C \left(|\beta_m| + \sum_{|j| > N, j \neq m} \frac{|\beta_j|}{|j - m|} \right) + C \|\beta\|_{\ell^1} \frac{1}{m}, \end{aligned}$$

where $C > 0$ can be chosen uniformly in $N > N_0$, $|m| > 2N$, and $\varphi \in \mathcal{W}$. Towards (ii) note that as $|\lambda| \geq 1$ one has

$$\frac{|p_\beta^n(\lambda)|}{\prod_{|j| \leq N_0} |\lambda - \lambda_j|} \leq \frac{1}{|\lambda|} \frac{\sum_{|j| \leq N_0} |\beta_j|}{\prod_{|j| \leq N_0} \left|1 - \frac{\lambda_j}{\lambda}\right|},$$

for any $|n| > N_0$ and $\varphi \in \mathcal{W}$. In addition, $|\lambda_j| < (N_0 + \frac{1}{4})\pi$ for any $|j| \leq N_0$ and $|\lambda| \geq m\pi - \pi/4$. Hence, $\prod_{|j| \leq N_0} \left|1 - \frac{\lambda_j}{\lambda}\right| \geq C$ where the constant $C > 0$ can be chosen uniformly in $\varphi \in \mathcal{W}$. This implies that

$$\sup_{\lambda \in \Gamma_m} \left(\frac{|p_\beta^n(\lambda)|}{\prod_{|j| \leq N_0} |\lambda - \lambda_j|} \right) \leq C \|\beta\|_{\ell^1} \frac{1}{m},$$

uniformly in $|n| > N_0$, $N > N_0$, $|m| > 2N$, and $\varphi \in \mathcal{W}$.

Item (iii) is proved in a similar fashion. q.e.d.

Next we want to estimate $\eta_n(\lambda)$ on $\partial D_0(r_{2N})$ where for any $m \in \mathbb{Z}$, $r_m := m\pi + \pi/4$.

Lemma 8.2. *For any $\varphi \in \mathcal{W}$, $\beta \in \ell^1$, $N > N_0 \geq 1$, and $n \in \mathbb{Z}$, one has*

- (i) $\sup_{\lambda \in \partial D_0(r_{2N})} \left(\sum_{|j| > N_0} \frac{|\beta_j|}{|\lambda - \lambda_j|} \right) \leq C \frac{\|\beta\|_{\ell^1}}{N} + C \sum_{|j| > N} |\beta_j|;$
- (ii) $\sup_{\lambda \in \partial D_0(r_{2N})} \left(\frac{|p_\beta^n(\lambda)|}{\prod_{|j| \leq N_0} |\lambda - \lambda_j|} \right) \leq C \|\beta\|_{\ell^1} \frac{1}{N}, \quad |n| > N_0;$
- (iii) $\sup_{\lambda \in \partial D_0(r_{2N})} \left(\frac{|p_\beta^n(\lambda)|}{\prod_{\substack{|j| \leq N_0, \\ j \neq N_0}} |\lambda - \lambda_j|} \right) \leq C \|\beta\|_{\ell^1} \frac{1}{N}, \quad |n| \leq N_0,$

where $C > 0$ can be chosen uniformly in $n \in \mathbb{Z}$, $N > N_0$, and $\varphi \in \mathcal{W}$.

Proof. To prove item (i) we split the sum $\sum_{|j| > N_0}$ into two parts: $\sum_{N_0 < |j| \leq N}$ and $\sum_{|j| > N}$. Clearly, for any $\lambda \in \partial D_0(r_{2N})$,

$$\sum_{N_0 < |j| \leq N} \frac{|\beta_j|}{|\lambda - \lambda_j|} \leq C \|\beta\|_{\ell^1} / N,$$

and

$$\sum_{|j| > N} \frac{|\beta_j|}{|\lambda - \lambda_j|} \leq \left(\sum_{|j| > N} |\beta_j|^2 \right)^{1/2} \left(\sum_{|j| > N} \frac{1}{|\lambda - \lambda_j|^2} \right)^{1/2} \leq C \sum_{|j| > N} |\beta_j|,$$

where $C > 0$ can again be chosen independently of $n \in \mathbb{Z}$, $N > N_0$, and $\varphi \in \mathcal{W}$. The estimates (ii) and (iii) are proved in the same way as items (ii), respectively, (iii) of Lemma 8.1. q.e.d.

Lemma 8.1 and Lemma 8.2 can be used to localize the zeros of $\zeta_n(\cdot, \varphi)$ for any $n \in \mathbb{Z}$ and $\varphi \in \mathcal{W}$. In fact, let $\psi \in L^2_\bullet$ be the potential appearing in the construction of \mathcal{W} in Section 2. Using that $\beta^n(\psi) \rightarrow \beta^*(\psi)$ in ℓ^1 as $|n| \rightarrow \infty$ (Lemma 7.5) and the fact that $\beta^n : \mathcal{W} \rightarrow \ell^1_n$ and $\beta^* : \mathcal{W} \rightarrow \ell^1$

are analytic (Corollary 7.1), we conclude that for any $\varepsilon > 0$ there exists $N_1 \geq 1$ and an open neighborhood $\mathcal{W}_1 \subseteq \mathcal{W}$ of ψ in L^2_\bullet such that for any $n \in \mathbb{Z}$ and for any $\varphi \in \mathcal{W}_1$,

$$(50) \quad \sum_{|m| \geq N_1} |\hat{\beta}_m^n| < \varepsilon.$$

Combining this with Lemma 8.1 and Lemma 8.2, shrinking the neighborhood \mathcal{W} and taking $N_1 \geq N_0 \geq 1$ larger if necessary we see that for any $N \geq N_1$, $n \in \mathbb{Z}$, $|m| > 2N$, and $\varphi \in \mathcal{W}$,

$$(51) \quad \sup_{\lambda \in \Gamma_m} |\eta_m(\lambda)| \leq 1/2,$$

and

$$(52) \quad \sup_{\lambda \in \partial D_0(r_{2N})} |\eta_m(\lambda)| \leq 1/2.$$

It then follows that for any $N \geq N_1$, $\lambda \in \Gamma_m$, $|m| > 2N$, as well as for any $\lambda \in \partial D_0(r_{2N})$,

$$\begin{cases} \left| \frac{\dot{\Delta}(\lambda)}{\lambda - \lambda_n} - \zeta_n(\lambda) \right| \leq \left| \eta_n(\lambda) \frac{\dot{\Delta}(\lambda)}{\lambda - \lambda_n} \right| \leq \frac{1}{2} \left| \frac{\dot{\Delta}(\lambda)}{\lambda - \lambda_n} \right|, & |n| > N_0, \\ \left| \frac{\dot{\Delta}(\lambda)}{\lambda - \lambda_{N_0}} - \zeta_n(\lambda) \right| \leq \left| \eta_m(\lambda) \frac{\dot{\Delta}(\lambda)}{\lambda - \lambda_{N_0}} \right| \leq \frac{1}{2} \left| \frac{\dot{\Delta}(\lambda)}{\lambda - \lambda_{N_0}} \right|, & |n| \leq N_0. \end{cases}$$

Hence by Rouché’s theorem and formulas (47) and (49) one has for any $N > N_1$ and $|m| > 2N$,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\Gamma_m} \partial_\lambda (\log \zeta_n(\lambda)) d\lambda &= 1 - \delta_{nm}, \\ \frac{1}{2\pi i} \int_{\partial D_0(r_{2N})} \partial_\lambda (\log \zeta_n(\lambda)) d\lambda &= \begin{cases} 4N, & |n| \leq 2N, \\ 4N + 1, & |n| > 2N. \end{cases} \end{aligned}$$

Remark 8.1. As these relations hold for *any* $N > N_1$ we also see that for any $\varphi \in \mathcal{W}$ and for any $N > N_1$ there are no zeros of $\zeta_n(\cdot, \varphi)$ outside of the union of the sets $D_m(\pi/4)$, $|m| > 2N$, and $D_0(r_{2N})$.

For any $|m| > 2N$ we denote the zero of $\zeta_n(\lambda)$ inside Γ_m by σ_m^n . By the argument principle one has (cf. (48))

$$\begin{aligned} \sigma_m^n &= \frac{1}{2\pi i} \int_{\Gamma_m} \lambda \partial_\lambda (\log \zeta_n(\lambda)) d\lambda \\ &= \dot{\lambda}_m + \frac{1}{2\pi i} \int_{\Gamma_m} \lambda \partial_\lambda (\log(1 - \eta_m(\lambda))) d\lambda. \end{aligned}$$

Integration by parts leads to

$$(53) \quad \sigma_m^n = \dot{\lambda}_m - \frac{1}{2\pi i} \int_{\Gamma_m} \log(1 - \eta_m(\lambda)) d\lambda.$$

Using (51) and Lemma 8.1 one sees that for $|m| > 2N$,

$$\begin{aligned} \sup_{\lambda \in \Gamma_m} \left| \log (1 - \eta_n(\lambda)) \right| &\leq 2 \sup_{\lambda \in \Gamma_m} \left| \eta_n(\lambda) \right| \leq \\ &\leq C_1 \left(|\beta_m^n| + \sum_{|j| > N_1, j \neq m, n} \frac{|\beta_j^n|}{|j - m|} + \|\beta^n\|_{\ell_n^1} / m \right), \end{aligned}$$

where $C_1 > 0$ is independent of $n \in \mathbb{Z}$, $\varphi \in \mathcal{W}$, and $|m| > 2N$. By using the Cauchy–Schwartz inequality and then changing the order of summation in the double sum we get

$$\begin{aligned} \sum_{|m| > 2N} \left(\sum_{j \neq m, n} \frac{|\beta_j^n|}{|j - m|} \right)^2 &\leq \sum_{|m| > 2N} \|\beta^n\|_{\ell_n^1} \sum_{j \neq m, n} \frac{|\beta_j^n|}{|j - m|^2} = \\ &= \|\beta^n\|_{\ell_n^1} \sum_{j \neq n} |\beta_j^n| \sum_{|m| > 2N, m \neq j} \frac{1}{|j - m|^2} \leq C_2 \|\beta^n\|_{\ell_n^1}^2, \end{aligned}$$

where $C_2 = 2 \sum_{k \geq 1} \frac{1}{k^2}$. Hence, we get in view of (50) that there exists $C > 0$ such that for any $n \in \mathbb{Z}$ and for any $\varphi \in \mathcal{W}$

$$\sum_{|m| > 2N} |\sigma_m^n - \dot{\lambda}_m|^2 \leq C.$$

Note that as $\zeta_n : \mathbb{C} \times \mathcal{W} \rightarrow \mathbb{C}$ is analytic by Proposition 4.3, the identity (53) also shows that $\sigma_m^n : \mathcal{W} \rightarrow \mathbb{C}$ is analytic for any $|m| > 2N$, $m \neq n$. By denoting $2N$ again by N , we get

Proposition 8.1. *For any $\psi \in L^2_\bullet$ there exists an open neighborhood \mathcal{W} of ψ in L^2_\bullet obtained from Section 2 after shrinking if necessary and $N \geq N_0$ so that for any $n \in \mathbb{Z}$ and for any $\varphi \in \mathcal{W}$, the entire function $\zeta_n(\lambda)$ has precisely $2N + 1$ $[2N]$ zeros inside $D_0(r_N)$ if $|n| > N$ [$|n| \leq N$]. For any $|m| > N$, $m \neq n$, $\zeta_n(\lambda)$ has precisely one zero, denoted by $\sigma_m^n \equiv \sigma_m^n(\varphi)$, in $D_m(\pi/4)$. There are no other zeros of $\zeta_n(\lambda)$ in \mathbb{C} . Moreover, $\sigma_m^n = \dot{\lambda}_m + \ell^2(m)$, $|m| > N$, uniformly in $n \in \mathbb{Z}$ and uniformly in \mathcal{W} , and for any $|m| > N$, $\sigma_m^n : \mathcal{W} \rightarrow \mathbb{C}$ is analytic.*

Proposition 8.1 implies that $\zeta_n(\lambda) \equiv \zeta_n(\lambda, \varphi)$ has in fact a product representation. List the roots of $\zeta_n(\lambda, \varphi)$ inside $D_0(r_N)$ in lexicographic order and with their multiplicities, σ_m^n , $|m| \leq N$, $m \neq n$.

Corollary 8.1. *For any $n \in \mathbb{Z}$ and $\varphi \in \mathcal{W}$,*

$$\zeta_n(\lambda) = -\frac{2}{\pi_n} \prod_{j \neq n} \frac{\sigma_j^n - \lambda}{\pi_j}.$$

Proof. Take $\varphi \in \mathcal{W}$. As the cases $|n| > N_0$ and $|n| \leq N_0$ are proved in the same way let us consider the case $|n| > N_0$. By [5, Theorem 2.2],

$\dot{\Delta}(\lambda)$ is an entire function of order 1 and so is $\frac{\dot{\Delta}(\lambda)}{\lambda - \lambda_n}$. By Proposition 4.3, $\zeta_n(\lambda)$ is an entire function and by Lemma 8.2,

$$\sup_{\lambda \in \partial D_0(r_{2N})} |1 - \eta_n(\lambda)| = 1 + o(1) \text{ as } N \rightarrow \infty.$$

It then follows that $\zeta_n(\lambda) = (1 - \eta_n(\lambda)) \frac{\dot{\Delta}(\lambda)}{\lambda - \lambda_n}$ is an entire function of order 1. Moreover, by Proposition 2.2 and Proposition 8.1 the exponent of convergence of the zeros of $\zeta_n(\lambda)$ is equal to 1 and the series $\sum_{|j| > N_0} \frac{1}{|\sigma_j^n|}$ diverges. This implies that the genus of $\zeta_n(\lambda)$ is equal to 1. By Hadamard’s factorization theorem

$$(54) \quad \zeta_n(\lambda) = \lambda^{\nu_n} e^{\tilde{a}_n \lambda + \tilde{b}_n} \prod_{\sigma_k^n \neq 0} E\left(\frac{\lambda}{\sigma_k^n}, 1\right),$$

where ν_n is the order of vanishing of $\zeta_n(\lambda)$ at $\lambda = 0$, $\tilde{a}_n, \tilde{b}_n \in \mathbb{C}$ are constants independent of $\lambda \in \mathbb{C}$, and $E(z, 1)$ is the canonical factor $E(z, 1) := (1 - z)e^z$. For $|m| > n$ we pair the factors $E\left(\frac{\lambda}{\sigma_{-m}^n}, 1\right) \cdot E\left(\frac{\lambda}{\sigma_m^n}, 1\right)$ and conclude from (54), Proposition 2.2, and Proposition 8.1, that $\zeta_n(\lambda)$ has a product representation of the form

$$e^{a_n \lambda + b_n} \prod_{k \neq n} \frac{\sigma_k^n - \lambda}{\pi_k} = e^{a_n \lambda + b_n} \lim_{K \rightarrow \infty} \prod_{|k| \leq K, k \neq n} \frac{\sigma_k^n - \lambda}{\pi_k}.$$

On the other hand, by Proposition 2.3, $\frac{\dot{\Delta}(\lambda)}{\lambda - \lambda_n} = -\frac{2}{\pi_n} \prod_{k \neq n} \frac{\lambda_k - \lambda}{\pi_k}$ and by [5, Lemma C.5], on the circles $|\lambda| = r_{2N}$,

$$\frac{\prod_{k \neq n} \frac{\sigma_k^n - \lambda}{\pi_k}}{\prod_{k \neq n} \frac{\lambda_k - \lambda}{\pi_k}} = 1 + o(1) \quad \text{as } N \rightarrow \infty.$$

By Lemma 8.2 and Proposition 8.1, on the circle $|\lambda| = r_{2N}$,

$$\frac{e^{a_n \lambda + b_n} \prod_{k \neq n} \frac{\sigma_k^n - \lambda}{\pi_k}}{-\frac{2}{\pi_n} \prod_{k \neq n} \frac{\lambda_k - \lambda}{\pi_k}} = 1 + o(1) \quad \text{as } N \rightarrow \infty.$$

It then follows that $a_n = 0$ and $e^{b_n} = -\frac{2}{\pi_n}$, yielding the claimed formula $\zeta_n(\lambda) = -\frac{2}{\pi_n} \prod_{k \neq n} \frac{\sigma_k^n - \lambda}{\pi_k}$. q.e.d.

The refined asymptotics of the zeros $(\sigma_m^n)_{m \neq n}$ of ζ_n stated below are proved in the same way as in [5, Lemma 14.12] and hence we omit its proof.

Lemma 8.3. *There exist $N \geq N_0$ so that*

$$\sigma_m^n = \tau_m + \gamma_m^2 \ell_m^2 \quad \forall |m| > N,$$

uniformly in $n \in \mathbb{Z}$ and locally uniformly in \mathcal{W} .

Finally, we prove the following:

Lemma 8.4. *For any $n \in \mathbb{Z}$ and $\varphi \in \mathcal{W} \subseteq L_{\bullet}^2$, the entire function $\zeta_n(\lambda)$ vanishes at $\lambda \in Z_{\varphi} \setminus \{\lambda_n^{\pm}(\varphi)\}$. If $\lambda_n^{\pm} \in Z_{\varphi}$ then $\zeta_n(\lambda)$ does not vanish at λ_n^{\pm} .*

Proof. Take $\varphi \in \mathcal{W}$. To see that $\zeta_n(\lambda, \varphi)$ vanishes on $Z_{\varphi} \setminus \{\lambda_n^{\pm}(\varphi)\}$ one argues as in the proof of Lemma 5.1. If $\lambda_n^{\pm} \in Z_{\varphi}$, then λ_n^{\pm} is a zero of order two of $\mathcal{R}(\lambda)$ and hence $\frac{\zeta_n(\lambda)}{\sqrt[\varepsilon]{\mathcal{R}(\lambda)}} d\lambda$ has a pole of order ≤ 1 at λ_n^{\pm} .

As $\int_{\Gamma_n} \frac{\zeta_n(\lambda)}{\sqrt[\varepsilon]{\mathcal{R}(\lambda)}} d\lambda = 2\pi$ we conclude that $\zeta_n(\lambda_n^{\pm}) \neq 0$. q.e.d.

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