J. DIFFERENTIAL GEOMETRY
96 (2014) 457-505

TAUT SUBMANIFOLDS AND FOLIATIONS

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Abstract

We give an equivalent description of taut submanifolds of complete Riemannian manifolds as exactly those submanifolds whose normal exponential map has the property that every preimage of a point is a union of submanifolds. It turns out that every taut submanifold is also $\mathbb{Z}_2$-taut. We explicitly construct generalized Bott-Samelson cycles for the critical points of the energy functionals on the path spaces of a taut submanifold that, generically, represent a basis for the $\mathbb{Z}_2$-cohomology. We also consider singular Riemannian foliations all of whose leaves are taut. Using our characterization of taut submanifolds, we are able to show that tautness of a singular Riemannian foliation is actually a property of the quotient.

1. Introduction

The terminology of taut submanifolds was introduced by Carter and West in [**CW72**], where they call a submanifold L of a Euclidean space V *taut* if all the squared distance functions $d_q^2 : L \to \mathbb{R}$, $d_q^2(p) = \|p-q\|^2$, with respect to points $q \in V$ that are not focal points of L, are perfect Morse functions for some field $\mathbb{F}$; i.e., if the number of critical points of index k of d_q^2 coincides with the k-th Betti number of L with respect to the field $\mathbb{F}$ for all k. If L is taut and $\mathbb{F}$ is a field as in the defintion of tautness, then L is also called $\mathbb{F}$-*taut*. Thus, geometrically, taut submanifolds are as round as possible. If one tries to generalize this defintion to submanifolds of arbitrary Riemannian manifolds, the problem arises that the squared distance function is not a priori everywhere smooth anymore. Namely, it is not differentiable in the intersection of the cut locus of the respective point q with the submanifold.

Using different approaches, Grove and Halperin [**GH91**] and, independently, Terng and Thorbergsson [**TT97**] generalized this notion to submanifolds L of complete Riemannian manifolds M by saying that L is taut if there exists a field $\mathbb{F}$ such that every energy functional $E_q(c) = \int_{[0,1]} \|\dot{c}(t)\|^2 dt$ on the space $\mathcal{P}(M, L \times q)$ of H^1-paths $c : [0,1] \to M$ from L to a fixed point $q \in M$ is a perfect Morse function

Received 3/19/2012.

with respect to $\mathbb{F}$ if q is not a focal point of L. The critical points of E_q are exactly the geodesics that start orthogonally to L and end in q. In particular, in the case where $M = V$ is a Euclidean space, there is an obvious way to identify a submanifold L with the space of segments in $\mathcal{P}(V, L\times q)$, and under this identification the map d_q^2 corresponds to E_q. Further, it is not hard to see that in this case the path space $\mathcal{P}(V, L\times q)$ admits the subspace of segments from L to q as a strong deformation retract. So the definitions agree for submanifolds of a Euclidean space and it turns out that this is indeed the right way to generalize tautness.

It is shown in [**TT97**] that if $L \hookrightarrow M$ is an $\mathbb{F}$-taut submanifold, then the energy functionals E_q are Morse-Bott functions for all points $q \in M$. Our first main result now states that this property actually characterizes taut submanifolds.

Theorem A. *A closed submanifold L of a complete Riemannian manifold M is taut if and only if all energy functionals $E_q : \mathcal{P}(M, L\times q) \to \mathbb{R}$ are Morse-Bott functions.*

In fact, if all the energy functionals are Morse-Bott functions, then the field with respect to which L is taut is $\mathbb{Z}_2$. Thus, as a direct consequence, we obtain the following result, which was, just as Theorem A, so far not even known in the case of a Euclidean space.

Theorem B. *If a submanifold is $\mathbb{F}$-taut, then it is also $\mathbb{Z}_2$-taut.*

Based on this result it is reasonable to consider only $\mathbb{Z}_2$-taut submanifolds, so that we no longer distinguish between $\mathbb{Z}_2$-taut and taut.

As the definition shows, tautness is a very special property. In some sense, it is a kind of homogeneity requirement for the pair (M, L). So it is no surprise that so far not many examples of taut submanifolds are known. This makes it all the more remarkable that taut submanifolds, if at all, often occur in families which then decompose the ambient space; e.g., an orbit decomposition induced by the isotropy representation of a symmetric space. It is for this reason that we study such families as they usually appear; i.e., singular Riemannian foliations with only taut leaves. We call such families *taut foliations*. As a main result in this direction we observe that tautness of a foliation is indeed a property of the quotient of the foliation, so that it actually makes sense to talk about taut quotients as equivalence classes of quotients under isometries.

Theorem C. *Let $\mathcal{F}$ and $\mathcal{F}'$ be closed singular Riemannian foliations on complete Riemannian manifolds M and M' such that their quotients $M/\mathcal{F}$ and $M'/\mathcal{F}'$ are isometric. Then $\mathcal{F}$ is taut if and only if $\mathcal{F}'$ is taut.*

Due to this result one could think about taut foliations as foliations with *pointwise taut* quotients, where we follow [**Le06**] and call a manifold pointwise taut if all of its points are taut (submanifolds). Of course, in general a quotient of a singular Riemannian foliation is far from being a manifold, but as soon as it is a nice space in the sense that one could use differential geometric methods, it turns out that this picture is reasonable. Viewed in this light, the largest class of spaces for which one has the appropriate tools available is the class of Riemannian orbifolds; i.e., spaces locally modeled by quotients of Riemannian manifolds modulo the action of a finite group of isometries. Now, given a taut singular Riemannian foliation $\mathcal{F}$ on M such that the quotient $M/\mathcal{F}$ is an orbifold, it follows that $M/\mathcal{F}$ is already a good Riemannian orbifold; that is to say $M/\mathcal{F}$ is isometric to N/Γ, where N is a Riemannian manifold and $\Gamma \subset \mathrm{Iso}(N)$ is a discrete group of isometries. This observation together with the last theorem leads directly to our next result, which mainly motivates our picture of pointwise taut quotients.

Theorem D. *Let $\mathcal{F}$ be a closed singular Riemannian foliation on a complete Riemannian manifold M. Then $\mathcal{F}$ is taut and $M/\mathcal{F}$ is an orbifold if and only if $M/\mathcal{F}$ is isometric to N/Γ, where N is a pointwise taut Riemannian manifold and $\Gamma \subset \mathrm{Iso}(N)$ is a discrete group of isometries of N.*

In view of applications, the more interesting direction of this result is that the existence of a pointwise taut quotient covering implies tautness of the foliation. The known examples of pointwise taut Riemannian manifolds are mainly two classes of spaces, together with Riemannian products of elements of these classes and their subcoverings. The first one is the class of symmetric spaces, which are pointwise taut by the work of Bott and Samelson [**BS58**], and the second class consists of manifolds without conjugate points; e.g., manifolds with non-positive curvature. In fact, if there are no conjugate points along any geodesic in a Riemannian manifold, the index of every critical point of a given energy functional is zero, hence all points in such a manifold are taut. A conjecture in [**TT97**] states that a compact pointwise taut Riemannian manifold that has the homotopy type of a compact symmetric space is symmetric. We want to mention as an aside that it is shown in [**TT97**] that in the case of a compact rank-one symmetric space this conjecture is equivalent to the Blaschke conjecture, which is still not settled.

It is therefore not a surprise that in all known examples of taut foliations with orbifold quotients, these quotients are isometric to a space $(N \times P)/\Gamma$, where N is a symmetric space, P is a manifold without conjugate points, and Γ is a discrete subgroup of isometries. Consider, for instance, the parallel foliation $\mathcal{F}$ of a Euclidean space V that is induced by an isoparametric submanifold L of V. Such a foliation is a singular Riemannian foliation and is also called an *isoparametric foliation*. In this case, the quotient $V/\mathcal{F}$ is isometric to $(p+\nu_p(L))/\Gamma$, where $p \in L$ is some point and Γ is the finite Coxeter group generated by the reflections across the focal hyperplanes in $p + \nu_p(L) \subset V$. So $V/\mathcal{F}$ admits a flat orbifold covering that is a manifold, thus $\mathcal{F}$ is taut. In particular, our result implies that isoparametric submanifolds are taut, which is well known by [**HPT88**]. More generally, if a closed singular Riemannian foliation $\mathcal{F}$ on a Riemannian manifold M is polar (i.e., through every regular point in M there exists a complete submanifold meeting all the leaves and always perpendicularly) every section covers the orbit space $M/\mathcal{F}$ (as an orbifold). We therefore see again that the orbits of hyperpolar actions (i.e., when the sections are flat) are taut. Since sections are always totally geodesic and totally geodesic submanifolds of symmetric spaces are symmetric spaces, we also reobtain the result from [**BG07**] that a polar action of a compact Lie group on a compact rank-one symmetric space is taut. In [**GT03**], Gorodski and Thorbergsson classified all taut irreducible representations of compact Lie groups as either hyperpolar and hence equivalent to the isotropy representation of a symmetric space, or as one of the exceptional representations of cohomogeneity three. Let $\rho : G \to \mathbf{O}(V)$ be an exeptional representation; i.e., the induced action of G on V has cohomogeneity equal to three. Then, the restriction of this action on the unit sphere $S \subset V$ has cohomogeneity two, so that S/G is isometric to a quotient S^2/Γ of the round 2-sphere with a finite Coxeter group Γ. Since the orbits of the G-action on V are taut if and only if the orbits of the G-action on S are taut, it follows from Theorem D again that the exeptional representations are taut.

Practically, the only way to prove that a given submanifold $L \hookrightarrow M$ is taut is the explicit construction of so called *linking cycles* for the energy. Namely, one has to find cycles that complete the local unstable manifolds associated to some Morse chart around the critical points below the corresponding critical energy. This concept is explained in Section 2.1. For the proof of Theorem A in Section 2.3 (cf. Theorem 2.9) we therefore first make the easy observation that all the energy functionals $E_q : \mathcal{P}(M, L \times q) \to \mathbb{R}$ are Morse-Bott if and only if the normal exponential map $\exp^\perp : \nu(L) \to M$ has *integrable fibers*, by which we mean that $(\exp^\perp)^{-1}(\exp^\perp(v))$ is a union of submanifolds for all vectors $v \in \nu(L)$. If so, we explicitely construct linking cycles for non-degenerate critical points (i.e., a basis for the (co-)homology of $\mathcal{P}(M, L \times q)$ if $q \in M$ is not a focal point) proving that L is taut. For this purpose, for a normal vector $v \in \nu(L)$, we define Z_v to be the set of all piecewise continuous paths $c : [0, 1] \to \nu(L)$ obtained as follows. Follow the segment tv toward the zero section up to the first focal vector $t_1 v$, then take an arbitrary normal vector v_1 in the fiber of $\exp^\perp$ through $t_1 v$ and follow the segment tv_1 toward the zero section up to first focal vector $t_2 v_1$, then take an arbitrary normal vector in the fiber through $t_2 v_1$ and repeat this procedure. By construction, for every $c \in Z_v$, $\exp^\perp \circ c$ is a broken geodesic from $\exp^{\perp}(v)$ to L and we define the space $\Delta_v \subset \mathcal{P}(M, L \times \exp^{\perp}(v))$ to consist of all broken geodesics obtained in this manner, reparameterized on $[0,1]$ after reversing the orientation. If the occurring focal multiplicities are locally constant, then Δ_v is an iterated fiber bundle and thus a compact manifold of the right dimension. In this case, its fundamental class ensures that it indeed provides a linking cycle. In the general case, Δ_v is a "fiber bundle" with singularities. Using sheaf cohomology we prove that it still carries a "fundamental class" and therefore actually represents a linking cycle. As mentioned above, Theorem B is then a direct consequence of this result.

In the second section we recapitulate the notion of singular Riemannian foliations and make some preliminary observations about taut foliations. As the main property, a geodesic either meets the leaves of a singular Riemannian foliation orthogonally at all or at none of its points. If a geodesic intersects one and hence all leaves perpendicularly, it is called *horizontal*. Roughly speaking, the possibilities for varying a horizontal geodesic through horizontal geodesics consist of variations of the projection of the geodesic to the quotient and of variations through horizontal geodesics, all of which meet the same leaves simultaneously. This results in an index splitting for horizontal geodesics into a *horizontal* and *vertical* index that we discuss in Section 3.3; the latter one counting the intersections with the *singular leaves* (with their multiplicities). In 3.4 we then prove Theorem C using Theorem A and the fact that the horizontal index is an intrinsic notion of the quotient (cf. Theorem 3.19). Using the arguments from our proof of Theorem C, we are able to give a general construction to obtain lots of examples of taut submanifolds, including all the known examples that occur in families. Finally, Theorem D is then proved as a special case of Theorem C (cf. Theorem 3.25).

At the end of this section we recall some basic facts about *infinitesimally polar foliations* (i.e., those foliations whose quotients are orbifolds) from [**LT10**] and reformulate our Theorem D for those foliations. Infinitesimally polar foliations can also be described as those foliations admitting a *canonical geometric resolution* (cf. [**L10**]); that is to say, a canonically related regular foliation with an isometric quotient. As a consequence of our results we therefore observe that the canonical resolution of an infinitesimally polar foliation is taut if and only if the foliation is taut.

Our proof of Theorem A (as well as of Theorem C) may be viewed as a generalization of the construction of Bott and Samelson in [**BS58**], which proves that the orbit foliation of a variationally complete action (i.e., when the focal points of the orbits are only caused by singular orbits) is taut. Given an orbit of such an action, Bott and Samelson came up with concrete cycles associated to the critical points of the energy on the space of paths to a fixed point that, generically, represent a basis for the $\mathbb{Z}_2$-(co-)homology of the corresponding path space. For such a generic critical point c, their cycle can be described as a connected component of the set of broken horizontal geodesics (i.e., broken geodesics that intersect all the orbits orthogonally) that have the same projection to the orbit space as c, hence coincide with our space $\Delta(c)$ from Lemma 3.21. Thus we reobtain their result as a special case of Corollary 3.23.

Finally, we must warn the reader that the use of the term "taut foliation" could lead to confusion. In the theory of (regular) foliations there are other definitions of tautness, such as geometrically or topologically taut foliations. But in this work, by a taut foliation, we always mean a singular Riemannian foliation all of whose leaves are taut submanifolds as defined above.

Acknowledgments. This article contains the main results of the author's PhD thesis written at the Universität zu Köln, Germany. The author would like to thank his teacher and supervisor Gudlaugur Thorbergsson for his guidance.

The author would also like to express his gratitude to Alexander Lytchak for his suggestion of the topic of taut foliations and plenty of instructive discussions, as well as for his support.

The financial support of the graduate school 1269 "Global Structures in Geometry and Analysis" of the DFG at the University of Cologne is gratefully acknowledged.

2. Taut Submanifolds

2.1. Linking Cycles. The terminology of tautness for submanifolds of a Euclidean space was introduced by Carter and West in [**CW72**]. They call a submanifold L of a Euclidean space V *taut* if there exists a field $\mathbb{F}$ such that for generic points $q \in V$, the squared distance functions $d_q^2 : L \to \mathbb{R}$, given by $d_q^2(p) = \|p - q\|^2$, are perfect with respect to the field $\mathbb{F}$. A definition similar to this can be used for submanifolds of the round sphere $S^n \subset V$.

Reminder. A Morse function on a complete Hilbert manifold P is a smooth function $f : P \to \mathbb{R}$ that is bounded below, has a nondegenerate critical set $\mathrm{Crit}(f)$, and satisfies Condition (C); i.e., if (p_n) is a sequence of points in P with $\{f(p_n)\}$ bounded and $\|df_{p_n}\| \to 0$, then (p_n) has a convergent subsequence. Thus Condition (C) can be regarded as an analogue of a compactness claim in the infinite-dimensional setting.

If p is a critical point for the Morse function f, then the index $\mathrm{ind}(p)$ is defined to be the dimension of a maximal subspace of T_pP on which the Hessian is negative definite; i.e., the number of independent directions in which f decreases. As in the finite dimensional case, a Morse function gives rise to a cell complex with one cell of dimension k for each critical point with index k, which is homotopy equivalent to P. If we set $P^r = \{p \in P | f(p) \leq r\}$, then the weak Morse inequalities say that if $\nu_k(a,b)$ denotes the number of critical points of index k in $f^{-1}(a,b)$ for regular values $a < b$, then $b_k(P^b, P^a; \mathbb{F}) \leq \nu_k(a,b)$ for all k, where $b_k(P^b, P^a; \mathbb{F})$ is the k-th Betti number of (P^b, P^a) with respect to the field $\mathbb{F}$ and f is called *perfect* (*with respect to* $\mathbb{F}$) if the weak Morse inequalities are equalities for all k and all regular values $a < b$. For a detailed background we refer the reader to Part II of [**PT88**].

A Morse-Bott function $f : P \to \mathbb{R}$ on a complete Hilbert manifold P is a smooth function whose critical set is the union of closed submanifolds and whose Hessian is non-degenerate in the normal direction. That is to say, every critical point lies in a closed submanifold whose tangent space coincides with the kernel of the Hessian at each point. If so, the index of a critical point is defined to be the index of the restriction of the Hessian to the normal space of the corresponding critical manifold. Since the Hessian depends continuously on the points of the critical manifolds, the index is constant along the connected components of the critical set.

Using different approaches, Grove and Halperin [**GH91**] as well as Terng and Thorbergsson [**TT97**] defined a general notion of taut immersions into a complete Riemannian manifold. In [**TT97**] it has been proven that for submanifolds of a Euclidean space the generalized definition of tautness coincides with the one previously known. We are going to introduce this generalized notion using the exposition in [**TT97**].

Let (M, g) be a complete Riemannian manifold and let $H^1(I, M)$ be the complete Riemannian Hilbert manifold of H^1-paths $I = [0,1] \to M$ with its canonical differentiable and metric structure; i.e., $H^1(I, M)$ is locally modeled on $H^1(I, \mathbb{R}^n)$. Recall that a path is of class H^1 if and only if it is absolutely continuous with finite energy. The charts for $H^1(I, M)$ are given by the identification of an open neighborhood of the zero section in $H^1(I, c^*(TM))$, for $c \in H^1(I, M)$ piecewise smooth, with an open neighborhood of c in $H^1(I, M)$ by the exponential map of M. Then, for such c, one has $T_c H^1(I, M) \cong H^1(I, c^*(TM))$ and the expression

$$\langle X, Y \rangle = \int_I g(X(t), Y(t))dt + \int_I g(\nabla X(t), \nabla Y(t))dt,$$

which is a priori defined for X, Y piecewise smooth, can be extended to a complete Riemannian metric on $H^1(I, M)$ (cf. [**Kl82**]). Furnished with this differentiable structure, the map $e : H^1(I, M) \to M \times M$, given by $e(c) = (c(0), c(1))$, defines a submersion.

Now for a proper immersion $\phi : L \to M$ into a complete Riemannian manifold and a point $q \in M$, we define the path space $\mathcal{P}_{(\phi,q)}(M, L)$ to be the pullback of $H^1(I, M)$ along the map $p \mapsto (\phi(p), q)$ from L into

$M \times M$; i.e., $\mathcal{P}_{(\phi,q)}(M,L)$ consists of pairs $(p,c) \in L \times H^1(I,M)$ with $\phi(p) = c(0)$ and $c(1) = q$. In particular, $\mathcal{P}_{(\phi,q)}(M,L)$ inherits a smooth structure that turns it into a complete Hilbert manifold and one can show that the induced energy functional $E_{(\phi,q)} : \mathcal{P}_{(\phi,q)}(M,L) \to \mathbb{R}$, defined by

$$E_{(\phi,q)}((p,c)) = \int_I \|\dot{c}(t)\|^2 dt,$$

is a Morse function if and only if q is not a focal point of L (along any normal geodesic). Recall that for a normal vector $v \in \nu(L)$, the point $\exp^\perp(v) \in M$ is called a *multiplicity μ focal point* of L along the geodesic $\exp^\perp(tv)$ and v is called a *multiplicity μ focal vector* if and only if $\dim(\ker(d\exp^\perp_v)) = \mu > 0$, where $\exp^\perp : \nu(L) \to M$ denotes the normal exponential map. The critical points of $E_{(\phi,q)}$ are exactly the pairs (p,γ), where γ is a geodesic, which starts perpendicularly to L and ends in q. By the famous theorem of Morse, the index of a critical point (p,γ) is then given by the sum

$$\operatorname{ind}((p,\gamma)) = \sum_{t\in(0,1)} \mu(t)$$

over the multiplicities $\mu(t)$ of the points $\gamma(t)$ as focal points of L along γ.

A geodesic γ that starts perpendicularly to L (i.e., $\dot\gamma(0) \in \nu(L)$) is called an *L-geodesic*. In this case we frequently use the notation γ_v to denote the L-geodesic $t \mapsto \exp^\perp(tv)$ (or respective restrictions). A vector field J along an L-geodesic γ is called an *L-Jacobi field* along γ if and only if it is a variational vector field of a variation of γ through L-geodesics and the nullity of a critical point (p,γ_v) of $E_{(\phi,\gamma_v(1))}$ is given by $\mu(v) = \dim(\ker(d\exp^\perp_v))$ and equals the dimension of the vector space

$$\{J | J \text{ is an } L\text{-Jacobi field along } \gamma_v, J(1) = 0\}.$$

In our setting of path spaces, every energy sublevel contains finite dimensional submanifolds consisting of broken geodesics, each of them homotopy equivalent to this sublevel, such that the restriction of the energy to each of these submanifolds has the same relevant behavior; i.e., the critical points of the restriction are exactly the critical points of the energy functional and their indices and nullities coincide. In particular, the indices and nullities are finite.

For these facts and a detailed discussion on path spaces and the energy functional we refer the reader who is not familiar with these notions to [**Kl82**] and [**Sak96**].

Finally, if ϕ is a closed embedding identifying L with its image $\phi(L)$ in M, we will drop the reference to the map ϕ and simply write $\mathcal{P}(M, L\times q)$ instead of $\mathcal{P}_{(\phi,q)}(M,L)$ for the space of H^1-paths from L to q.

Suppose that $M = V$ is a Euclidean space and that, for simplicity, $L \subset V$ is a closed submanifold. Then, for every fixed point q

in V, the submanifold L is diffeomorphic to the space of segments $\mathcal{S} = \{s_p(t) = p + t(q-p), p \in L\}$ in $\mathcal{P}(V, L \times q)$ and with respect to this identification the energy is given by the squared distance; i.e., by $E_q(s_p) = d_q^2(p) = \|p-q\|^2$. Moreover, it is not hard to see that this identification respects the critical behavior of these functions. Since all the critical points of the energy E_q are contained in $\mathcal{S}$ and, by convexity, the space $\mathcal{P}(V, L \times q)$ contains $\mathcal{S}$ as a strong deformation retract, the squared distance function d_q^2 is a perfect Morse function on L if and only if E_q is a perfect Morse function on $\mathcal{P}(V, L \times q)$. This observation led Terng and Thorbergsson to a natural generalization of the notion of a taut immersion into any complete Riemannian manifold M.

Definition 2.1. A proper immersion $\phi : L \to M$ of a manifold L into a complete Riemannian manifold (M, g) is called *taut* if there exists a field $\mathbb{F}$ such that the energy functional $E_{(\phi,q)} : \mathcal{P}_{(\phi,q)}(N, L) \to \mathbb{R}$, given by $E_{(\phi,q)}(p, c) = \int_I \|\dot{c}(t)\|^2 dt$, is a perfect Morse function with respect to the field $\mathbb{F}$ for every point $q \in M$ that is not a focal point of L. In particular, a point $p \in M$ is called a *taut point* if $\{p\}$ is a taut submanifold of M; i.e., $E_q : \mathcal{P}(M, p \times q) \to \mathbb{R}$ is perfect with respect to some field for every $q \in M$ that is not conjugate to p along some geodesic. If a submanifold L is taut and $\mathbb{F}$ is a field as in the definition of tautness, then L is also called $\mathbb{F}$-*taut*.

In [**Le06**] Leitschkis called a manifold with only taut points *pointwise taut* and we will continue with this notion.

Note 2.2. In [**TT97**] it is shown that a properly immersed, taut submanifold of a simply connected, complete Riemannian manifold is actually embedded. Because we will see in Section 3.2 that one can always assume that the ambient space is simply connected, we will proceed assuming that all submanifolds are embedded and closed, but all of our results will also hold in the case of a proper immersion. For this reason, if not otherwise stated, by a submanifold L of M we always mean an embedded submanifold and consider all submanifolds as subsets of M. Finally, a manifold is always assumed to be connected.

The only way that is known to prove tautness in general (i.e., that a given Morse function is perfect) is the concept of *linking cycles*, which we are going to explain now. For this reason let $f : P \to \mathbb{R}$ be again a Morse function on a complete Hilbert manifold. Then, for every $r \in \mathbb{R}$, the sublevel P^r contains only a finite number of critical points of f and we can assume that these critical points have pairwise distinct critical values. That the latter assumption is not restrictive follows from the fact that one can lift a small neighborhood of a critical point a little without changing the relevant behavior of the function. Moreover, using the flow of the negative gradient, one sees that for small ε the sublevel sets $P^{r+\varepsilon}$ and $P^{r-\varepsilon}$ have the same homotopy type unless r is a critical value. If so, let p be the critical point of f with $f(p)=r$ and choose an ε such that $(r-\varepsilon, r+\varepsilon)$ contains no critical value except r. If we denote the index of p by i then $P^{r+\varepsilon}$ has the homotopy type of $P^{r-\varepsilon}$ with an i-cell e_i attached to $f^{-1}(r-\varepsilon)$. Consider the following part of the long exact cohomology sequence of the pair $(P^{r+\varepsilon}, P^{r-\varepsilon})$ with coefficients in a field $\mathbb{F}$:

$$\underbrace{H^{i-1}(P^{r+\varepsilon}, P^{r-\varepsilon})}_{=0} \to H^{i-1}(P^{r+\varepsilon}) \to H^{i-1}(P^{r-\varepsilon}) \xrightarrow{\partial^*} \underbrace{H^{i}(P^{r+\varepsilon}, P^{r-\varepsilon})}_{\cong\mathbb{F}}$$
$$\to H^{i}(P^{r+\varepsilon}) \to H^{i}(P^{r-\varepsilon}) \to \underbrace{H^{i+1}(P^{r+\varepsilon}, P^{r-\varepsilon})}_{=0}$$

Since we are using coefficients from a field, we can switch between the more common homological and the (for our approach) more suitable cohomological point of view by dualization; i.e., $H^*(P^r;\mathbb{F}) \cong \mathrm{Hom}_{\mathbb{F}}(H_*(P^r;\mathbb{F}),\mathbb{F})$. Anyway, we see that by passing from $P^{r-\varepsilon}$ to $P^{r+\varepsilon}$ the only possible changes in homology or cohomology occur in dimensions $i-1$ and i. To understand this geometrically, let us have a look at what happens in homology. In the first case, the boundary ∂e_i of the attaching cell is an $(i-1)$-sphere in $P^{r-\varepsilon}$ that does not bound a chain in $P^{r-\varepsilon}$; i.e., e_i has as boundary the nontrivial cycle ∂e_i and so $\partial_* \neq 0$. In the second case, ∂e_i does bound a chain in $P^{r-\varepsilon}$, which we can cap with e_i to create a new nontrivial homology class in $P^{r+\varepsilon}$; that is to say, $\partial_* = 0$ and $H_i(P^{r+\varepsilon}) \cong H_i(P^{r-\varepsilon}) \oplus \mathbb{F}$.

We see that the Morse inequalities are equalities if and only if

$$H^i(P^{r+\varepsilon}, P^{r-\varepsilon}) \to H^i(P^{r+\varepsilon}) \text{ is nontrivial; i.e., } \partial^* \equiv 0,$$

or, equivalently,

$$H_i(P^{r+\varepsilon}) \to H_i(P^{r+\varepsilon}, P^{r-\varepsilon}) \text{ is surjective; i.e., } \partial_* \equiv 0,$$

for all critical points p of f.

For a critical point $p \in P$, one can show that

$$H^*(P^{f(p)+\varepsilon}, P^{f(p)-\varepsilon}) \cong H^*(P^{f(p)}, P^{f(p)} \setminus \{p\}).$$

Thus suppose that we have a map $h_p : \Delta_p \to P^{f(p)}$ for every critical point p such that the composition

$$H^i(P^{f(p)}, P^{f(p)} \setminus \{p\}) \xrightarrow{h_p^*} H^i(\Delta_p, h_p^{-1}(P^{f(p)} \setminus \{p\})) \to H^i(\Delta_p)$$

is nontrivial. In this case, because the connecting homomorphism ∂^* is a natural transformation, we have that

$$h_p^* \circ \partial^* = \partial^* \circ h_p^*$$

and we conclude that the map $H^i(P^{f(p)}, P^{f(p)} \setminus \{p\}) \to H^i(P^{f(p)})$ cannot be zero, so that f is a perfect Morse function under this assumption. If so, we call the pair (Δ_p, h_p) a *linking cycle for* p, the critical point p *of* *linking type*, and we say that the function $f : P \to \mathbb{R}$ is of *linking type* if all the critical points are of linking type. Of course, if f is perfect, then the inclusions of the corresponding sublevels define linking cycles, so that f is of linking type. Thus we see that a Morse function is perfect if and only if it is of linking type.

Note 2.3. At the end of this section we will prove that an $\mathbb{F}$-taut submanifold is always $\mathbb{Z}_2$-taut. For this reason, and due to the fact that when dealing with (co-)homology there is just a little chance to get general results with other coefficients, we restrict our attention to the case $\mathbb{F} = \mathbb{Z}_2$. From now on, saying taut we always mean $\mathbb{Z}_2$-taut and we drop the reference to the field everywhere.

Remark 2.4. Using finite-dimensional approximations of the path space we see that, in the setting we are interested in, singular cohomology is isomorphic to Čech cohomology (cf. Section 2.2). Because the latter groups satisfy a continuity property and are more easy to handle, we focus on the Čech cohomology groups in the following.

2.2. The Main Tool. As already mentioned in the introduction, the problem when dealing with general cycle constructions is the behavior of the focal data. The construction of Bott and Samelson works well, also in the general case, if the focal points along a variation do not collapse; i.e., if the cardinality of the intersections of normal geodesics with the focal set is locally constant. Unfortunately, the occurrence of focal collapses along a variation of normal geodesics cannot be avoided in general, but since these collapses depend continuously on the initial directions of the geodesics, it turns out that this indeed constitutes no problem for our goal. In the following will work this out using the theory of sheaves and sheaf cohomology as it is presented in [**B67**] and chapter 5 of [**Wa83**]. We refer the reader to these textbooks for the facts that we presume.

To fix a notation we recall the following:

Definition 2.5. A *sheaf (of Abelian groups)* on X is a pair $(\mathcal{A}, \pi)$, where

1) $\mathcal{A}$ is a topological space;
2) $\pi : \mathcal{A} \to X$ is a local homeomorphism;
3) Each fiber $\mathcal{A}_x = \pi^{-1}(x)$ is an Abelian group and is called the *stalk* of $\mathcal{A}$ at the point x;
4) The group operations are continuous; i.e., the map
$$\begin{array}{ccc} \mathcal{A} \times_\pi \mathcal{A} & \to & \mathcal{A}, \\ (\alpha, \beta) & \mapsto & \alpha - \beta, \end{array}$$
with $\mathcal{A} \times_\pi \mathcal{A} = \{(\alpha, \beta) \in \mathcal{A} \times \mathcal{A} | \pi(\alpha) = \pi(\beta)\}$, is continuous.

If the context is clear we will drop the reference to the map π and talk about the sheaf $\mathcal{A}$. For an Abelian group G, we say that $\mathcal{A}$ is a G-sheaf on X if all the fibers $\pi^{-1}(x)$ are isomorphic to G.

Given a continuous map $f : X \to B$ from a compact Hausdorff topological space X onto some nice space B with a fundamental class in $\mathbb{Z}_2$-(co-)homology (e.g., a manifold) such that all the fibers are compact manifolds of constant dimension, one would expect that the union over the base B of all the fibers defines a non-trivial class in $\mathbb{Z}_2$-(co-)homology of dimension equal to the sum of the (co-)homological dimension of B and the fiber dimension. In order to prove our main theorem later on, we are dependent on a tool like this because we want to construct explicit (co-)cycles with specified cohomological behaviour. The easiest way, known to the author, to prove such a statement is by means of sheaf cohomology.

The first step in this direction is the following easy observation for which we have not found any reference in the literature.

Lemma 2.6. *Let X be a locally compact Hausdorff topological space. Then there are no nontrivial $\mathbb{Z}_2$-sheaves on X.*

Proof. Assume that $(\mathcal{A}, \pi)$ is a $\mathbb{Z}_2$-sheaf on X; i.e., $\pi^{-1}(x) \cong \mathbb{Z}_2$ for all $x \in X$. Then each fiber $\pi^{-1}(x)$ has exactly one element that is not zero and we denote this nontrivial element by 1_x. Because the zero section $0 : X \to \mathcal{A}$ is a global section, it remains to observe that the well-defined map $1 : X \to \mathcal{A}$, given by $1(x) = 1_x$, is continuous, which is straightforward. Thus the map $X \times \mathbb{Z}_2 \to \mathcal{A}$, given by $(x, \varepsilon) \mapsto \varepsilon_x$ for $\varepsilon \in \{0, 1\}$, defines an isomorphism of sheaves. q.e.d.

A *presheaf* on a topological space X is a contravariant functor from the category of open subsets of X, where the morphisms are just the inclusions, to the category of Abelian groups; i.e., a function that assigns to each open set U an Abelian group $A(U)$ and to each pair $U \subset V$ a homomorphism, called the *restriction*, $r_{U,V} : A(V) \to A(U)$ in such a way that $r_{U,U} = 1_{A(U)}$ and $r_{U,V} \circ r_{V,W} = r_{U,W}$ whenever $U \subset V \subset W$. The set of (local) sections of a sheaf $\mathcal{A}$ is a presheaf in the obvious way. Conversely, given a presheaf A, taking fiberwise the direct limit over all neighborhoods of a fixed point, one gets a sheaf $\mathcal{A}$ by the so-obtained set of germs topologized by the natural sections induced by A. In this case, one says that the sheaf $\mathcal{A}$ is *generated* by the presheaf A.

Examples.

- For any standard cohomology theory H^* on X, the assignment given by $U \mapsto H^r(U; G)$ defines a presheaf, where the coefficients are taken to be any Abelian group G.
- Let $f : Y \to X$ be a continuous map between topological spaces. Then for every $r \geq 0$ there is an associated presheaf on X given by the prescription $U \to \check{H}^r(f^{-1}(U); G)$, where we denote the Čech cohomology by $\check{H}^*$. The sheaf $\mathcal{H}^r(f; G)$ generated by this presheaf is called the *Leray sheaf* of f on X.

There is a way to define general cohomology theories with coefficients in a presheaf or in a sheaf. These concepts coincide for paracompact spaces. But a developement of this theory would go beyond the scope of our discussion, the more so as it is not really necessary for our goal. For this reason we have to refer the reader to the literature; e.g., [**B67**], [**Sp66**], or [**Wa83**]. Because it is all we need, we just want to mention that in the case of a constant sheaf $X \times G$, this cohomology is exactly the same as the usual Čech cohomology with coefficients in G. Moreover, it can be shown that if X is a topological manifold of dimension n, then all the cohomology groups with coefficients in any sheaf vanish in dimensions greater than n. This is an important feature of which we make essential use.

Remark 2.7. It is shown in [**Sp66**] that in the cases we are interested in, the cohomology groups $H^r(X;\mathcal{G})$ with coefficients in the constant sheaf $\mathcal{G} = X \times G$ are nothing else than the Alexander-Spanier cohomology groups with coefficients in G and that the latter coincide with the Čech cohomology groups $\check{H}^r(X;G)$. In particular, it is therefore clear that we have long exact sequences, excision, and that the homotopy axiom holds. Finally, because we are dealing only with nice spaces, please recall that for compact subsets (K,L) of a manifold, the Čech cohomolgy groups $\check{H}^r(K,L)$ are isomorphic to the direct limit

$$\varinjlim \{H^r(U,V) | (K,L) \subset (U,V)\},$$

where the limit is taken over open subsets $(U,V) \supset (K,L)$.

As mentioned above, the cohomology groups $H^r(U;\mathcal{G})$ with coefficients in the constant sheaf $\mathcal{G} = X \times G$ are isomorphic with the corresponding Čech groups. Thus the generated sheaves are also isomorphic. Due to this, our definition of the Leray sheaf in this case behaves well with respect to the general definition of the Leray sheaf in the context of sheaf cohomology, which is, given a sheaf $\mathcal{A}$ on Y, generated by the presheaf $U \to H^r(f^{-1}(U);\mathcal{A})$.

We now come to the heart of this section, presenting a very powerful tool for our use. We formulate the following lemma in an easy to handle version, adjusted accordingly to our purpose, but the reader who goes through the proof will notice that it also holds under weaker assumptions; e.g., if B as in the claim has the cohomological behavior of a manifold.

Lemma 2.8. *Let X be a connected and compact Hausdorff topological space and let $f : X \to B$ be a continuous map onto a manifold B of dimension k. Assume that every fiber $f^{-1}(b)$ is a connected manifold of (constant) dimension n or, more generally, that $\check{H}^n(f^{-1}(b);\mathbb{Z}_2) \cong \mathbb{Z}_2$*

and $\check{H}^l(f^{-1}(b);\mathbb{Z}_2) = 0$ *for all* $l > n$*, where* $\check{H}^*$ *denotes the Čech cohomology. Then* X *has cohomological dimension* $n+k$*; i.e.,* $\check{H}^{n+k}(X;\mathbb{Z}_2) \cong \mathbb{Z}_2$ *and* $\check{H}^l(X;\mathbb{Z}_2) = 0$ *for* $l > n+k$*.*

Proof. Since X is compact and connected and f is surjective, B is compact and connected, too. If we consider cohomology with coefficients in the constant sheaf $\mathcal{Z}_2 = X \times \mathbb{Z}_2$, then, due to Theorem 6.1 in [**B67**], there exists a spectral sequence $\{E_r, d_r\}$ with $d_r : E_r^{m,l} \to E_r^{m+r,l-r+1}$ converging to $H^*(X;\mathcal{Z}_2)$ with E_2 page

$$E_2^{m,l} = H^m(B;\mathcal{H}^l(f;\mathcal{Z}_2)),$$

where $\mathcal{H}^l(f;\mathcal{Z}_2)$ denotes the above-defined Leray sheaf on B, generated by the presheaf $U \mapsto H^l(f^{-1}(U);\mathcal{Z}_2)$. Further, it is also proven there that, under this assumptions, the stalks $\mathcal{H}^l(f;\mathcal{Z}_2)_p$ of the Leray sheaf are isomorphic to the cohomology groups of the corresponding fibers $H^l(f^{-1}(b);\mathcal{Z}_2) \cong \check{H}^l(f^{-1}(b);\mathbb{Z}_2)$. Then, by our assumptions and Lemma 2.6, the nth Leray sheaf is the constant sheaf on B; that is to say, $\mathcal{H}^n(f;\mathcal{Z}_2) \cong B \times \mathbb{Z}_2$. Therefore, the entry $E_2^{k,n}$ is just given by the kth Čech cohomology group $\check{H}^k(B;\mathbb{Z}_2) \cong \mathbb{Z}_2$. Because B is a manifold of dimension k, all the groups $H^m(B;\mathcal{H}^l(f;\mathcal{Z}_2))$ vanish for $m > k$, by dimensional reasons mentioned above. Also, all the entries $E_2^{m,l}$ with $l > n$ vanish, because $\mathcal{H}^l(f;\mathcal{Z}_2)$ is the 0-sheaf in this case. But this means that the entry $E_2^{k,n}$ survives in the spectral sequence since it is the top-right entry in the nontrivial rectangle on the E_2 page. The second statement of the claim follows from the fact that if $m+l > k+n$ then $m > k$ or $l > n$. q.e.d.

2.3. An Equivalent Description. Having achieved our key tool in the last section, we are now able to prove our main result.

Theorem 2.9. *Let* $L \subset M$ *be a closed submanifold of a complete Riemannian manifold* M*. Then the following statements are equivalent.*

1. L *is taut.*
2. *All energy functionals are Morse-Bott functions.*
3. *The fibers of the normal exponential map* $\exp^\perp : \nu(L) \to M$ *are integrable.*

Proof. The implication 1. $\Rightarrow$ 2. holds by Theorem 2.8 of [**TT97**], where Terng and Thorbergsson show that if L is a taut submanifold (with respect to any field), then all the energy functionals are Morse-Bott functions, adapting the idea of Ozawa [**Oz86**] for the Euclidean case.

The equivalence of 2. and 3. is more or less by definition. If $S \subset \mathcal{P}(M, L \times q)$ is a critical submanifold of E_q, then the image of the map $S \to \nu(L), s \mapsto \dot{s}(0)$ is an integral manifold of the kernel distribution. Conversely, if $S \subset \nu(M)$ is an integral manifold of the kernel distribution (i.e., an open and compact submanifold of some fiber $(\exp^{\perp})^{-1}(q)$) then the image of the map $S \to \mathcal{P}(M, L \times q), v \mapsto \exp^{\perp}(tv)|_{[0,1]}$ defines a non-degenerate connected component of the critical set of E_q.

Therefore, it remains to show 3. $\Rightarrow$ 1. We prove this by constructing explicit linking cycles for $\mathbb{Z}_2$-coefficients.

Let us first fix some notation before we discuss our construction. For convenience, let $\eta = \exp^{\perp} : \nu(L) \to M$ denote the normal exponential map. By the ray through $v \in \nu(L)$ we mean the map $r_v : \mathbb{R}_0^+ \to \nu(L)$, given by $r_v(t) = tv$, and by the segment to v we mean $s_v = r_v|_{[0,1]}$. Then for $\alpha \geq 1$, the point $q = \eta(v)$ is a focal point of L along the geodesic $\gamma_{\alpha v} = \eta \circ s_{\alpha v}$ if and only if $d\eta_v$ is singular and the multiplicity of q at such a point is just the dimension of the kernel of $d\eta_v$. Recall that in the case where $d\eta_v$ is not onto, we call v a focal vector of multiplicity $\mu(v) = \dim(\ker(d\eta_v))$. Let us denote by C the union of all points in $\nu(L)$, where $d\eta$ is singular, and call it the (tangent) focal locus. We call every number in $r_v^{-1}(C)$ a focal time along the ray r_v. It is a well known fact that the focal times are discrete along any ray and that they depend continuously on the rays. Namely, that there are continuous functions

$$\lambda_i : S(1) = \left\{v \in \nu(L) : \|v\|^2 = 1\right\} \to \mathbb{R}$$

with $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ and $r_v^{-1}(C) = \{\lambda_i(v)\}_{i \geq 1}$ (cf. [**IT01**]). This implies that every vector $v \in \nu(L)$ has an open neighborhood U such that every ray that intersects U contains $\mu(v)$ focal vectors in U counted with multiplicities; i.e., if $\operatorname{im}(r_w) \cap U \neq \emptyset$, then

$$\sum_{t \in r_w^{-1}(U)} \mu(tw) = \sum_{t \in r_w^{-1}(U \cap C)} \mu(tw) = \mu(v).$$

Finally, we call a focal vector $v \in C$ regular if there is an open neighborhood U of v such that all rays that intersect U intersect $U \cap C$ exactly once. Due to Warner [**Wa65**] and Hebda [**Heb81**], the set C_R of regular focal vectors is an open and dense subset of C that is a codimension-one submanifold of $\nu(L)$ such that $T_v\nu(L) \cong T_vC_R \oplus \mathbb{R}v$ for all $v \in C_R$. Let $\nu(L)^R$ denote the set of vectors $v \in \nu(L) \setminus C$ such that s_v intesects C only in C_R; i.e., such that $w \mapsto \#\{s_w^{-1}(C)\}$ is constant on a neighborhood of v. Then $\nu(L)^R$ is obviously open in $\nu(L)$ and it is also dense. To see this, consider the function $n : \nu(L) \setminus C \to \mathbb{Z}$, given by

$$n(v) = \#\left\{s_v^{-1}(C)\right\},$$

which is lower semi-continuous by our above observations. Then

$$\begin{aligned} \nu(L)^R &= \{v \in \nu(L) \setminus C : n \text{ is constant on a neighborhood of } v\} \\ &= \left\{v \in \nu(L) \setminus C : s_v^{-1}(C) = s_v^{-1}(C_R)\right\}. \end{aligned}$$

Since the set of regular vectors $\nu(L) \setminus C$ is open and dense in $\nu(L)$, it is enough to show that $\nu(L)^R$ is dense in $\nu(L) \setminus C$. Thus assume that $\nu(L)^R$ is not dense in $\nu(L) \setminus C$. Then the complement of $\nu(L)^R$ in

$\nu(L) \setminus C$ contains an open set U. The function n admits its maximum n_r on every intersection $U \cap B(r)$ of U with an open tube $B(r)$ of radius r around the zero section. Choose r so large that $U \cap B(r) \neq \emptyset$. Due to the semi-continuity of n, the set $n^{-1}(n_r) \cap U \cap B(r)$ defines an open subset of $\nu(L) \setminus C$ on which n is constant, which clearly contradicts our definition of $\nu(L)^R$.

Suppose now that the singular kernel distribution $\bigcup_{v \in \nu(L)} \ker(d\eta_v)$ is completely integrable; i.e., through every point $v \in \nu(L)$ there is a $\mu(v)$-dimensional compact connected submanifold C_v with $T_w C_v = \ker(d\eta_w)$ for all $w \in C_v$. Then we have that $C_v \subset S(\|v\|)$, where $S(\|v\|)$ denotes the sphere bundle over L of normal vectors of length $\|v\|$, and the index $i(v) = \sum_{t \in (0,1)} \mu(tv)$ is constant along C_v. As above, for a unit vector $v \in S(1)$, we denote by $0 < \lambda_1(v) \leq \lambda_2(v) \leq \cdots$ the (continuous) focal times along the ray r_v, counted with their multiplicities.

Let us now define a function $m : \nu(L) \to [0, 1)$ which assigns to a normal vector $v \in \nu(L) \setminus \{0\}$ the number

$$m(v) = \max \{t \in (0, 1) | \mu(tv) \neq 0\}$$

if $i(v) > 0$, and $m(v) = 0$ if $i(v) = 0$ or if v belongs to the zero section. In particular, by our above observations, the restriction of m to each submanifold C_v is continuous, because of $m(v) = \lambda_i(v/\|v\|)/\|v\|$ for some i and $\|v\| \neq 0$.

Denote by $\mathfrak{C} = \bigcup_{v \in \nu(L)} C_v$ the η-fiber decomposition of the normal bundle $\nu(L)$ and define $Q : \nu(L) \to \nu(L)/\mathfrak{C}$ to be the natural quotient map. Since the fibers of Q are compact submanifolds of $\nu(L)$, which are connected components of the fibers of the continuous map η from the complete space $\nu(L)$ to the manifold M, the quotient is a locally compact Hausdorff space and the restriction of the projection to $\nu(L) \setminus C$ is a homeomorphism onto an open subspace of $\nu(L)/\mathfrak{C}$. The fiber norms on $\nu(L)$ push down to the distance function from the image $Q(0)$ of the zero section, so that every compact subset in $\nu(L)/\mathfrak{C}$ has to be of bounded distance from $Q(0)$. In particular, the map Q is proper and therefore closed.

We now define a natural cycle candidate Δ_v for every geodesic $\gamma_v = \eta \circ s_v$ with $v \in \nu(L) \setminus C$. Because η factorizes over $\nu(L)/\mathfrak{C}$ by a map $\bar{\eta} : \nu(L)/\mathfrak{C} \to M$, we will work in the quotient space and consider the space $P(\nu(L)/\mathfrak{C}, Q(0) \times Q(v))$ of continuous paths $c : [0, 1] \to \nu(L)/\mathfrak{C}$ from $Q(0)$ to $Q(v)$ with the compact open topology. The constructed cycles will embed in an energy preserving and obvious way into $\mathcal{P}(M, L \times \eta(v))$ under the map on the path space level induced by $\bar{\eta}$, so that we renounce the reference to the latter space for the rest of the proof.

Because the η-kernel distribution on $\nu(L)$ is integrable, there is a natural cycle through $Q \circ s_v$ intuitively having the right dimension. Namely, take Z_v to be the set of all piecewise continuous maps from

$[0,1]$ to $\nu(L)$ obtained (with the reversed orientation) as follows: Follow the segment s_v toward the zero section up to the first focal vector $m(v)v$, take a vector $w_1 \in C_{m(v)v}$ and follow the straight line tw_1 toward the zero section up to the first focal vector $m(w_1)w_1$. Then take an arbitrary focal vector w_2 in the corresponding leaf and follow the line tw_2 toward the zero section up to the first focal vector, then take an arbitrary focal vector w_3 in the corresponding leaf $C_{m(w_2)w_2}$ and follow the line tw_3 up to the first focal vector, and so on. This process will end after a finite number of steps and we can push down these piecewise continuous maps via Q, obtaining broken geodesics $[0,1] \to \nu(L)/\mathfrak{C}$ starting in $Q(0)$ and ending in $Q(v)$. We define Δ_v to be the injective image of Z_v under this map.

To be more precise, let us say that a tuple $c = (c_r, \ldots, c_1)$ is an η-polygon on $[0,1]$ if there exists a partition $0 = t_r < t_{r-1} \cdots < t_1 < t_0 = 1$ of the interval $[0,1]$ such that $c_i : [t_i, t_{i-1}] \to \nu(L)$ is given by $c_i(t) = r_{w_i}(t) = tw_i$ for some vector $w_i \in \nu(L)$. Then for $v \in \nu(L) \setminus \{0\}$, let Z_v be the set consisting of all η-polygons c on $[0,1]$ inductively defined as follows. If $i(v) = \sum_{t\in(0,1)} \mu(tv) = 0$, set $Z_v = \{s_v\}$, and if $i(v) = 1$ with

$$t_1 = m(v) = s_v^{-1}(C) = \frac{\lambda_1(v/\|v\|)}{\|v\|},$$

define Z_v to consist of pairs $c = (c_2(w), c_1)$, where $w \in C_{m(v)v}$ and

$$c_2(w) : [0, m(v)] \to \nu(L) \text{ is given by } c_2(w)(t) = tw$$

and

$$c_1 : [m(v), 1] \to \nu(L) \text{ is given by } c_1(t) = tv.$$

Note that, because of $w \in C_{m(v)v}$, we have

$$\lambda_1(w/\|w\|) = \|w\| = m(v)\|v\| = \lambda_1(v/\|v\|).$$

Now assume that we have already defined Z_w if $i(w) \leq n$ in such a way that it consists of all η-polygons $c = (c_r, \ldots, c_1)$ with segments $c_i : [t_i, t_{i-1}] \to \nu(L)$ satisfying $c_{i+1}(t_i) \in C_{c_i(t_i)}$ for $i \geq 1$. Let v be a normal vector with $i(v) = n+1$. Then for every vector w in the fiber $C_{m(v)v}$ through $m(v)v$ we have $i(w) = i(m(v)v) < i(v)$. We define the space Z_v to consist of pairs $(d(w), c_1)$ for some $w \in C_{m(v)v}$, where $c_1 : [m(v), 1] \to \nu(L)$ is defined by $c_1(t) = tv$ and $d(w) : [0, m(v)] \to \nu(L)$ is a linear reparameterization on $[0, m(v)]$ of an element $\tilde{d} \in Z_w$. This means that there exists an η-polygon $\tilde{d} = (\tilde{d}_r, \ldots, \tilde{d}_1) \in Z_w$ with $\tilde{d}_i$ defined on an interval $[\tilde{t}_i, \tilde{t}_{i-1}]$, for a partition $0 = \tilde{t}_r < \tilde{t}_{r-1} < \cdots < \tilde{t}_1 < \tilde{t}_0 = 1$, such that $d(w) = (d_r, \ldots, d_1)$ with segments $d_i : [t_i, t_{i-1}] \to \nu(L)$, given by $d_i(t) = \tilde{d}_i\left(\frac{t}{m(v)}\right)$, where $t_i = m(v)\tilde{t}_i$. If we set $c_{i+1} = d_i$, we have that $c = (d(w), c_1) = (c_{r+1}, \ldots, c_2, c_1)$ with $c_1 = s_v|_{[m(v),1]}$, and for $i \geq 2$

and $w_1 = v$ we obtain the formula

$$c_i(t) = s_{\frac{\|v\|}{\|w_i\|}w_i}(t), t \in \left[m(w_i)\frac{\|w_i\|}{\|v\|}, m(w_{i-1})\frac{\|w_{i-1}\|}{\|v\|}\right]$$

with $w_i \in C_{m(w_{i-1})w_{i-1}}$.

Thus, in the above notation, $t_i = m(w_i)\frac{\|w_i\|}{\|v\|}$. Note that this is well defined because of $m(w_{i-1})\|w_{i-1}\| = \|w_i\|$ and $m(w) = 0$ if $i(w) = 0$. We can regard an η-polygon on $[0,1]$ as a piecewise continuous map $c : [0,1] \to \nu(L)$, defined by $c(0) = 0$ and $c|_{(t_i,t_{i-1}]} = c_i|_{(t_i,t_{i-1}]}$, where, of course, here $c(0) = 0$ means the origin of the normal space that is uniquely defined by $c(\varepsilon)$ for some small number $\varepsilon > 0$. Anyway, there is a well-defined injective map

$$\bar{Q} : Z_v \to P(\nu(L)/\mathfrak{C}, Q(0) \times Q(v)),$$

given by $\bar{Q}(c)|_{[t_i,t_{i-1}]} = Q \circ c_i$ with energy

$$\begin{aligned} E(\bar{Q}(c)) &= (1 - m(w_1))\|v\|^2 + \sum_{i\geq 2} E(c_i) \\ &= (1 - m(w_1))\|v\|^2 + \sum_{i\geq 2}(m(w_{i-1})\frac{\|w_{i-1}\|}{\|v\|} - m(w_i)\frac{\|w_i\|}{\|v\|})\|v\|^2 \\ &= \|v\|^2. \end{aligned}$$

We define the space Δ_v to be the image $\bar{Q}(Z_v) \subset P(\nu(L)/\mathfrak{C}, Q(0) \times Q(v))$ with the relative topology; i.e., induced by the compact open topology on the path space $P(\nu(L)/\mathfrak{C}, Q(0)\times Q(v))$. With this topology the space Δ_v is compact because Q is proper. The map $\bar{Q} : Z_v \to \Delta_v$ is a bijection and we topologize Z_v by the postulation that this map is a homeomorphism. As mentioned above, we can regard Z_v as a space of piecewise continuous maps.

We follow this direction and define $e : Z_v \times [0,1] \to \nu(L)$ by $e(c,0) = c(0)$ and $e(c,t) = \lim_{t' \nearrow t} c_i(t')$ if $t \in (t_i, t_{i-1}]$, so that $t \mapsto e_t(c)$ is the required map. Let us set $\bar{e} : \Delta_v \times [0,1] \to \nu(L)/\mathfrak{C}$ for the continuous evaluation map, given by the prescription $\bar{e}(\bar{Q}(c), t) = \bar{Q}(c)(t)$, and consider the commutative diagram

$$\begin{tikzcd} Z_v \times [0,1] \arrow[r, "e"] \arrow[d, "\bar{Q}\times \mathrm{id}"] & \nu(L) \arrow[d, "Q"] \\ \Delta_v \times [0,1] \arrow[r, "\bar{e}"] & \nu(L)/\mathfrak{C} \end{tikzcd}$$

from which it follows that e is continuous in (c,t) if $\bar{Q}(c)(t) \notin Q(C)$. If we define $e_t = e(\cdot, t)$, then $e_{m(v)} : Z_v \to \nu(L)$ is also continuous. To see this, we first observe that $e_{m(v)}$ is continuous iff it is continuous when considered as a map to the submanifold $C_{m(v)v} \subset S(m(v)\|v\|)$. Thus take an open subset $U \subset C_{m(v)v}$ and note that by definition we have

$e_{m(v)}(c) = w \in U$ iff the image of $c_2|_{[m(w),m(v)]}$ is contained in the set $[m(\cdot),1] \cdot U = \{r(w)w | r(w) \in [m(w),1], w \in U\}$. Because m is bounded away from 1 on $C_{m(v)}$, we can find an $\varepsilon > 0$ such that $(1-\varepsilon,1) \cdot C_{m(v)} \subset \nu(L) \setminus C$, and because Q is an embedding on $\nu(L) \setminus C$ we have

$$e_{m(v)}^{-1}(U) = e_{m(v)-\varepsilon/2}^{-1}((1-\varepsilon,1) \cdot U) = \bar{Q}^{-1}(\bar{e}_{m(v)-\varepsilon/2}^{-1}(Q((1-\varepsilon,1) \cdot U))),$$

which is therefore open.

The crucial point with regard to our goal is that therefore the map

$$\begin{aligned} pr_v = e_{m(v)} \circ \bar{Q}^{-1} : \Delta_v &\to C_{m(v)v} \\ \bar{Q}(c) &\mapsto c(m(v)) \end{aligned}$$

is continuous, so Δ_v is given as the (continuous) family of fibers

$$\Delta_v = \bigcup_{w \in C_{m(v)v}} pr_v^{-1}(w) \cong \bigcup_{w \in C_{m(v)v}} \Delta_w,$$

what enables us to use an inductive argument to verify the right cohomological behavior. For this reason, we identify $pr_v^{-1}(w) \cong \Delta_w$ by restriction and reparametrization; i.e., forgetting the last irrelevant segment we regard a broken geodesic $\bar{Q}(c)$ in Δ_v as a path from $Q(0)$ to the furthermost breaking point $Q(C_{m(v)v})$, namely as a path in the space $\Delta_{c(m(v))}$.

By definition, $\Delta_v = \{s_v\}$ if $i(v) = 0$. If $i(v) = 1$, then $C_{m(v)v} \cong S^1$ and we have $pr_v^{-1}(w) \cong \Delta_w \cong \{s_w\}$ for all $w \in C_{m(v)v}$. Of course, in this case we have $\check{H}^1(\Delta_v) \cong \mathbb{Z}_2$ and $\check{H}^k(\Delta_v) = 0$ if $k > 1$. In the general case, we note again that for all $w \in C_{m(v)v}$ we have

$$i(w) = i(m(v)v) = i(v) - \mu(m(v)v) = i(v) - \dim(C_{m(v)v}).$$

In particular, if $i(v) > 0$, then $i(w) < i(v)$ for all $w \in C_{m(v)v}$. Thus, because of the fact that $pr_v^{-1}(w) \cong \Delta_w$, we can assume by induction that we have $\check{H}^{i(w)}(pr_v^{-1}(w)) \cong \mathbb{Z}_2$ and $\check{H}^k(pr_v^{-1}(w)) = 0$ if $k > i(w)$ for all $w \in C_{m(v)v}$. Applying Lemma 2.8 to the map $pr_v : \Delta_v \to C_{m(v)}$ it then follows that $\check{H}^{i(v)}(\Delta_v) \cong \mathbb{Z}_2$ and $\check{H}^k(\Delta_v) = 0$ if $k > i(v)$.

Having come this far, it remains to prove that the spaces Δ_v indeed represent linking cycles for generic geodesics $Q \circ s_v$, because using continuity arguments as in Section 3.1 this would imply tautness. But this follows from the fact that, due to [**Wa67**] and [**Heb81**], $\mathfrak{C}$ defines a smooth distribution on every connected component of the set of regular focal vectors, so that it is more or less obvious by our construction that $Q \circ s_v$ admits a manifold neighborhood in Δ_v for all $v \in \nu(L)^R$. Further, this neighborhood can be deformed into the local unstable manifold in some Morse chart around $Q \circ s_v$ because of the following expression of the tangent space that is a direct consequence of our construction.

Namely,

$$T_{Q\circ s_v}\Delta_v = \bigoplus_{k=1}^{r} \mathcal{J}(t_k),$$

where $s_v^{-1}(C) = \{t_1, \ldots, t_r\}$ and $\mathcal{J}(t_k)$ equals the vector space of continuous vector fields J along $Q \circ s_v$ such that $J|_{[0,t_k]}$ is an L-Jacobi field along $Q \circ s_v$ and $J|_{[t_k,1]} \equiv 0$. A direct computation or a look at the proof of the Index Theorem of Morse (cf. [**Sak96**]) now shows that the projection of $T_{Q\circ s_v}\Delta_v$ onto the tangent space at $Q \circ s_v$ of the unstable manifold corresponding to some Morse chart is an isomorphism; that is to say, locally around the critical point, Δ_v can be deformed into the unstable manifold of some Morse chart. In this case, if we denote by $\mathcal{P}_{L,\eta(v)}$ the space $\mathcal{P}(M, L \times \eta(v))$, the following commutative diagramm

$$\begin{tikzcd}
\check{H}^{i(v)}(\mathcal{P}^{\|v\|^2}_{L,\eta(v)}) \arrow[r] & \check{H}^{i(v)}(\Delta_v) \\
\check{H}^{i(v)}(\mathcal{P}^{\|v\|^2}_{L,\eta(v)}, \mathcal{P}^{\|v\|^2}_{L,\eta(v)} \setminus \{\eta \circ s_v\}) \arrow[u] \arrow[r, "\cong"] & \check{H}^{i(v)}(\Delta_v, \Delta_v \setminus \{Q \circ s_v\}) \arrow[u, "\cong"']
\end{tikzcd}$$

yields the claim if $v \in \nu(L)^R$ is not a focal vector. Since for manifolds Čech cohomology is isomorphic to singular cohomology and $\nu(L)^R$ is dense in $\nu(L)$, we deduce, with the same arguments as in the proof of Proposition 2.7 in [**TT97**], that the energy $E_q : \mathcal{P}(M, L \times q) \to \mathbb{R}$ is $\mathbb{Z}_2$-perfect for all points q that are not focal points of L. q.e.d.

Remark 2.10. As we mentioned in the last section, for compact subsets (K, L) of a manifold P the Čech cohomolgy groups $\check{H}^j(K, L)$ are isomorphic to the direct limit

$$\varinjlim \left\{H^j(U, V) | (K, L) \subset (U, V)\right\}$$

where the limit is taken over open subsets $(U, V) \supset (K, L)$. Therefore, one could also show directly that

$$H^{i(v)}(\mathcal{P}^{\|v\|^2}_{L,\eta(v)}, \mathcal{P}^{\|v\|^2}_{L,\eta(v)} \setminus \{\eta \circ s_v\}) \to \check{H}^{i(v)}(\Delta_v, \Delta_v \setminus \{\eta \circ s_v\}) \to \check{H}^{i(v)}(\Delta_v)$$

is nontrivial for all the spaces Δ_v with $v \in \nu(L) \setminus C$, because using the deformation retraction of $P^{\|v\|^2+\varepsilon}_{L,\eta(v)}$ onto the Morse complex one can assume that a neighborhood base of $\eta \circ s_v$ in Δ_v is contained in some ball around the origin in $\mathbb{R}^{i(v)}$.

As a direct consequence of the proof of Theorem 2.9 and the above remark, we obtain the following fact, which was so far not known even in the case of a Euclidean space.

Theorem 2.11. *If a closed submanifold of a complete Riemannian manifold is taut with respect to some field, then it is also $\mathbb{Z}_2$-taut.*

It is worth mentioning that although tautness is defined by means of perfect Morse functions, an analogous statement for Morse functions is wrong. This indicates the high degree of geometry involved in this setting.

Example. Consider the unit 3-sphere $S^3 \subset \mathbb{C}^2$ with the $\mathbb{Z}_p$-action, generated by

$$1 \cdot (z, w) = (\mathrm{e}^{\frac{i2\pi}{p}} z, \mathrm{e}^{\frac{i2\pi q}{p}} w)$$

with p and q relatively prime. The quotient $S^3/\mathbb{Z}_p$ is known as the *lens space* $L(p,q)$ with fundamental group $\pi_1(L(p,q)) \cong \mathbb{Z}_p$ and homology $H_k(L(p,q);\mathbb{Z}_p) \cong \mathbb{Z}_p$ for $k = 0,1,2,3$. Now, the Morse-Bott function $|z|^2$ is invariant under the $\mathbb{Z}_p$-action so it descends to a Morse-Bott function on $L(p,q)$ with critical set corresponding to the two critical circles $z = 0$ and $w = 0$ of index 0 and index 2, respectively. One can perturb this Morse-Bott function in the neighborhoods of the critical submanifolds by adding a bump function depending on the distance to the respective submanifold times a (perfect) Morse function on the respective circle so that the indices add. This, therefore, results in a $\mathbb{Z}_p$-perfect Morse function on $L(p,q)$. Now, if p is odd, we have $H_2(L(p,q);\mathbb{Z}_2) = 0$, so that this $\mathbb{Z}_p$-perfect Morse function is not perfect with respect to $\mathbb{Z}_2$. In particular, there are no taut immersions of $L(2k+1,q)$ into a Euclidean space. Indeed, Thorbergsson actually proved in [**T88**] that there are no taut immersions of $L(p,q)$ into a Euclidean space, except in the case of the projective space $\mathbb{R}P^3 = L(2,1)$, by showing that the first nontrivial homology group can only have torsion elements of order two.

3. Taut Foliations

Even if there are not many examples of taut submanifolds, a remarkable observation is that they often occur, if at all, in families, which then decompose the ambient space. In this section we therefore focus on taut families as they usually occur, namely on singular Riemannian foliations all of whose leaves are taut. For this reason, we first recall some basic facts about singular Riemannian foliations and make some preliminary observations that we need to prove our second result in Subsection 3.4, which characterizes taut singular Riemannian foliations by means of their quotients.

3.1. Singular Riemannian Foliations and Orbifolds. For a more detailed discussion on singular Riemannian foliations and the proofs of the following statements we refer to [**Mol88**] and [**LT10**].

Definition 3.1. Let $\mathcal{F}$ be a partition of a manifold M^{n+k} into connected, injectively immersed submanifolds with maximal dimension n. For a point $p \in M$, let L_p denote the element of $\mathcal{F}$ which contains p.

Set

$$T\mathcal{F} = \bigcup_{p \in M} T_p L_p.$$

Then the partition $\mathcal{F}$ is called a *singular foliation of M of dimension n/codimension k* iff the $C^\infty(M)$-module $\Gamma(T\mathcal{F})$ of smooth vector fields X tangential to $\mathcal{F}$ (i.e., with $X_p \in T_p L_p$ for all $p \in M$) exhaust $T_p L_p$ for every $p \in M$. We call the elements of $\mathcal{F}$ *leaves*. A leaf is *regular* if it has dimension n, otherwise *singular*. A point belonging to a regular leaf is *regular*, otherwise *singular*. By M_0 we denote the set of regular points and call it the *regular stratum*. If (M, g) is a Riemannian manifold, a singular foliation is called a *singular Riemannian foliation* if every geodesic in M that intersects one leaf orthogonally intersects every leaf it meets orthogonally.

We sometimes also speak about a singular Riemannian foliation $(M, \mathcal{F})$, or also $(M, g, \mathcal{F})$, if we want to abbreviate that $\mathcal{F}$ is a singular Riemannian foliation on the Riemannian manifold M, or (M, g).

Example. The set of orbits of an isometric Lie group action on a Riemannian manifold M is a singular Riemannian foliation, closed if and only if the group considered as a subgroup of the isometry group is closed.

For $d \leq n$, denote by M_d the subset of all points $p \in M$ with fixed leaf dimension $\dim(L_p) = n - d$. Since the dimension of the leaves varies lower semi-continuously, the set $\bigcup_{d' \leq d} M_{n-d'} = \{p \in M \mid \dim(L_p) \leq d\}$ is closed. Further, M_d is an embedded submanifold of M and the restriction of $\mathcal{F}$ to M_d is a (regular) Riemannian foliation. The main stratum M_0 is open, dense, and connected if M is connected. All the other singular strata have codimension at least 2 in M.

Let p be a point in $(M, g, \mathcal{F})$ and let B be a small open ball in L_p. Then there is a number $\varepsilon > 0$ and a *distinguished tubular neighborhood* U at p so that the following holds true:

1) The foot point projection $\pi : U \to B$ is well defined;
2) U is the image of the ε-discs $\nu^\varepsilon(B)$ in the normal bundle $\nu(B)$ of B under the exponential map and the map $\exp : \nu^\varepsilon(B) \to U$ is a diffeomorphism;
3) $T_q = \pi^{-1}(q)$ is a global transversal of $(U, \mathcal{F}|_U)$ for all $q \in B$; i.e., T_q meets all the leaves of $\mathcal{F}|_U$ and always transversally;
4) For each real number $\lambda \in [-1, 1] \setminus \{0\}$ the map $h_\lambda : U \to U$, given by $h_\lambda(\exp(v)) = \exp(\lambda v)$, for all $v \in \nu^\varepsilon(B)$, preserves $\mathcal{F}$.

Indeed, the fact that the geodesics perpendicular to B remain perpendicular to the leaves implies that if $d(q, B) = \delta$, then the connected component P_q of q in the open subset $L_q \cap U$ of the leaf L_q is entirely contained in the tube S_δ^B of radius δ around B. The leaf of $\mathcal{F}_U$ through

q, which is exactly P_q, is called the *plaque of $\mathcal{F}$ passing through q in the neighborhood U*. In particular, we have $B = P_p$ by construction. Moreover, we see that for all $q \in U$, the distance from q to B remains constant as q moves along the plaque P_q, thus the distance between the neighboring leaves is locally constant.

Definition 3.2. We say that a singular Riemannian foliation $(M, \mathcal{F})$ *has the property P* if every leaf of $\mathcal{F}$ has the property P; e.g., $\mathcal{F}$ is closed if all the leaves are closed subspaces of M.

It is well kown that the leaves of a closed singular Riemannian foliation $\mathcal{F}$ on a complete Riemannian manifold M admit global ε-tubes, so that the distance between two leaves is globally constant. In this case, the quotient $M/\mathcal{F}$ is a complete metric space, where the distance between two points is just the distance between the corresponding leaves as submanifolds of M.

Lemma 3.3. *Let $(M, \mathcal{F})$ be a singular Riemannian foliation. Then a leaf $L \in \mathcal{F}$ is embedded if it is closed.*

Proof. Let r be the dimension of L. Then M_{n-r} is an embedded submanifold of M and $\mathcal{F}|_{M_{n-r}}$ is a regular Riemannian foliation. Due to Molino [**Mol88**, p.22]), the statement is true for $(M_{n-r}, \mathcal{F}|_{M_{n-r}})$, so it is true for $(M, \mathcal{F})$. q.e.d.

Again, let $p \in M$ be a point and let B be a small open neighborhood in the leaf L_p through p. Then there is an $\varepsilon > 0$ and a distinguished tubular neighborhood (U, B, π) around p such that there is an embedding ϕ of U into the tangent spaces T_pM with $d\phi_p = \mathrm{Id}$ and a singular Riemannian foliation $\mathfrak{F}_p$ on T_pM, called the *infinitesimal singular Riemannian foliation of $\mathcal{F}$ at the point p*, that coincides with $\phi_*\mathcal{F}$ on $\phi(U)$ and such that $\mathfrak{F}_p$ is invariant under all rescalings $r_\lambda : T_pM \to T_pM, r_\lambda(v) = \lambda v$, for all $\lambda \neq 0$. In particular, $\mathfrak{F}_p$ is closed if and only if $\mathcal{F}$ is locally closed at p. If $\mathcal{F}$ is locally closed at p, the quotient $T_pM/\mathfrak{F}_p$ is a non-negatively curved Alexandrov space and the local quotient $U/\mathcal{F}$ is a metric space of curvature bounded below in the sense of Alexandrov. Further, the space $T_pM/\mathfrak{F}_p$ is the tangent space to this Alexandrov space at the leaf $L \cap U \in U/\mathcal{F}$. The inclusion $U \to M$ induces a map between the quotients $U/\mathcal{F} \to M/\mathcal{F}$, which is an open finite-to-one map if $\mathcal{F}$ is closed. So, assume that $\mathcal{F}$ is closed and M is complete. Then the quotient $M/\mathcal{F}$ is a complete metric space with the metric induced by the distance of the leaves of $\mathcal{F}$ (as submanifolds). Let T be a global ε-tube around L with the same ε as in the definition of the distinguished neighborhood U, then U is saturated (i.e., it is a union of leaves) and $T/\mathcal{F}$ is a neighborhood of L in the global quotient $M/\mathcal{F}$. In this case, there is a finite group of isometries Γ acting on the local quotient $U/\mathcal{F}$ that fixes the plaque $L \cap U \in U/\mathcal{F}$ such that $U/\mathcal{F}$ is isometric to $T/\mathcal{F}$.

A concept that is closely related to that of (Riemannian) foliations is the notion of (Riemannian) orbifolds. We therefore summarize the basic facts and defintions as a reminder and discuss orbifold coverings thereafter in more detail, because we will need this later on.

Reminder. As a manifold is locally modeled on open sets U of $\mathbb{R}^n$, orbifolds are locally modeled on finite quotients U/G, where $G \subset \mathrm{Diff}(U)$ is a finite group. Thus an *orbifold* $(B, \mathfrak{U})$ is a Hausdorff topological space B together with a maximal *orbifold atlas* $\mathfrak{U}$ consisting of compatible orbifolds charts (U, G, φ); i.e., the mapping $\varphi : U \to B$ induces a homeomorphism $U/G \cong \varphi(U)$ with $\varphi(U)$ open. The compatibility condition just means that the transition maps $(U_1, G_1, \varphi_1) \to (U_2, G_2, \varphi_2)$ commute with the respective projections. In this case there is (up to isomorphism) a well defined notion of *isotropy group* for the points in B as the isotropy group of any preimage point in an arbitrary chart. The points in B with trivial isotropy group are called *regular* otherwise *singular*. If, in addition, B is a metric space and there is a Riemannian metric on the U_i such that $G_i \subset \mathrm{Iso}(U_i)$ and the homeomorphisms $U_i/G_i \to \varphi_i(U_i)$ are isometric, then B is called a *Riemannian orbifold*.

Let B be a Riemannian orbifold. Then B is locally isometric to a quotient N/Γ, where Γ is a finite group of isometries of a smooth Riemannian manifold N. The tangent bundle TB of B is therefore locally also given as a finite quotient TN/Γ. Since geodesics are invariant under isometries, there is a local geodesic flow on TB and the orbifold geodesics are (locally) the projections of geodesics in the covering manifolds under the quotient map. For $v \in TB$, we denote by η_v the *orbifold-geodesic* in direction v.

For each orbifold-geodesic η_v, the curvature endomorphism along η_v is well defined. Therefore, the notions of Jacobi fields and conjugate points are also well defined. Let us now assume that B is complete as a metric space. Then each orbifold-geodesic is defined on $\mathbb{R}$ and the local geodesic flow is a global flow. Denote by B_0 the regular stratum and note that B is stratified by Riemannian manifolds, where the unique maximal stratum B_0 is open and dense in B. Take a regular point $b \in B_0$ and consider the orbifold exponential map $\exp : T_bB \to B$, given by $\exp(tv) = \eta_v(t)$. This map (since defined in metric terms) factors over local branched covers of B; i.e., for each $w \in T_bB$ there is a finite quotient $N/\Gamma_w = O \subset B$ with $\exp(w) \in O$, such that exp lifts on a neighborhood of w to a smooth map to N. The vector $w = tv$ is a conjugate vector along the geodesic η_v if and only if this lift has a non-injective differential at w. For a detailed discussion about orbifolds see [**MM03**].

Definition 3.4. A *covering orbifold* of an orbifold $(B, \mathfrak{U})$ is an orbifold $(\widetilde{B}, \widetilde{\mathfrak{U}})$, with a map $P : \widetilde{B} \to B$ between the underlying spaces such that each point $b \in B$ has a neighborhood $V = U/G$ for which each component $\widetilde{V}_i$ of $P^{-1}(V)$ is isomorphic to U/G_i for some subgroup $G_i \subset G$, and such that the isomorphisms commute with the projection P.

It is well known that any orbifold admits a *universal orbifold covering*, that is to say, for a regular base point $b_0 \in B$ there exists a pointed connected covering orbifold $P : \widetilde{B} \to B$ with base point $\tilde{b}_0$ projecting to b_0 such that for any other covering orbifold $P' : B' \to B$ with base point b_0' and $P'(b_0') = b_0$, there is a lift $Q : \widetilde{B} \to B'$ of P along P' to an orbifold covering. In particular, a universal orbifold covering is regular in the sense that its group of deck transformations acts simply transitive on a generic fiber. This group is denoted by π_1^{orb} and is called the *orbifold fundamental group*.

Definition 3.5. An orbifold is called *good* if it is a global quotient or, equivalently, if the universal covering orbifold is a manifold; i.e., there are no singular points.

In the case of a Riemannian orbifold all the definitions are to be modified in the obvious manner, so that we can speak about *Riemannian orbifold coverings* and *good Riemannian orbifolds*. Of course, the statement about the universal orbifold covering also holds in the Riemannian category.

Example. Again let N be a Riemannian manifold and let Γ be a discrete group of isometries of N. If $\widetilde{N}$ denotes the universal covering of N, then $\widetilde{N}$ is the universal Riemannian covering orbifold of N/Γ. This can be seen, for instance, by observing that every covering orbifold of N/Γ has to be of the form $\widetilde{N}/\widetilde{\Gamma}'$, where $\widetilde{\Gamma}'$ is a subgroup of the group $\widetilde{\Gamma}$ of deck transformations of $\widetilde{N}$ over N/Γ. Hence the two definitions of a good orbifold are indeed equivalent.

Because we will use it in what follows, we will formulate the next observation as a lemma.

Lemma 3.6. *Let $\mathcal{F}$ be a closed (regular) Riemannian foliation on a complete Riemannian manifold M and let $\widetilde{\mathcal{F}}$ denote its lift to the universal Riemannian covering $\widetilde{M}$ of M. Then the quotient $\widetilde{M}/\widetilde{\mathcal{F}}$ is a complete Riemannian manifold (i.e., $\widetilde{\mathcal{F}}$ is simple) if $M/\mathcal{F}$ is a good Riemannian orbifold. In particular, the orbifold covering $\widetilde{M}/\widetilde{\mathcal{F}} \to M/\mathcal{F}$ coincides with the universal Riemannian orbifold covering.*

Proof. Since $\mathcal{F}$ is closed its lift $\widetilde{\mathcal{F}}$ is closed, too (cf. Lemma 3.8), and the leaves of $\widetilde{\mathcal{F}}$ admit global ε-tubes, because $\widetilde{M}$ is complete. Due to [**Hae88**] or [**Sal88**], there is a surjective group homomorphism $\pi_1(\widetilde{M}) \to \pi_1^{orb}(\widetilde{M}/\widetilde{\mathcal{F}})$, where the latter group is the group of deck transformations of the universal orbifold covering of $\widetilde{M}/\widetilde{\mathcal{F}}$. Now, if $M/\mathcal{F}$ is a good Riemannian orbifold, its branched cover $\widetilde{M}/\widetilde{\mathcal{F}}$ is a good Riemannian orbifold, too. But then $\pi_1^{orb}(\widetilde{M}/\widetilde{\mathcal{F}}) = 1$ implies that $\widetilde{M}/\widetilde{\mathcal{F}}$ already coincides with its universal covering orbifold and is therefore a manifold. q.e.d.

3.2. Simplifications and Preliminaries. In this section we give some preliminary results to simplify the discussion of taut foliations; e.g., we will see that one can always assume that the manifold is simply connected (cf. Lemma 3.11) and that a dense family of leaves forces a foliation to be taut (cf. Corollary 3.16).

Definition 3.7. Let $\mathcal{F}$ be singular Riemannian foliation on a complete Riemannian manifold M. If $\mathcal{F}$ is closed, we call $\mathcal{F}$ *taut* if every leaf of $\mathcal{F}$ is taut.

Note that if $\mathcal{F}$ is closed, then by Lemma 3.3 all the leaves are embedded submanifolds. Further, if $\mathcal{F}$ is the trivial foliation given by the points of M and $\mathcal{F}$ is taut, then, as already defined in Section 2.1, we call M *pointwise taut.*

In order to prove tautness of a foliation, one can always assume that M is simply connected. To see this, we first need the closeness property of lifts.

Lemma 3.8. *Let $\mathcal{F}$ be closed and let $\pi : N \to M$ be a covering map. Then the lift $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ given by the involutive singular distribution $T\widetilde{\mathcal{F}} = \pi^*(T\mathcal{F})$ is closed.*

Remark 3.9. Note that the converse is false, as one can see by the dense torus foliation induced by the submersion $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = y - \lambda x$, where λ is irrational.

Proof. For every leaf L of $\mathcal{F}$, the preimage $\pi^{-1}(L) = \bigcup_i \widetilde{L}_i$ is a union of leaves of $\widetilde{\mathcal{F}}$ and the restriction $\pi|_{\widetilde{L}_i} : \widetilde{L}_i \to L$ is a covering projection for each i. Thus each leaf $\widetilde{L} \in \widetilde{\mathcal{F}}$ is a connected component of the closed saturated set $\pi^{-1}(\pi(\widetilde{L}))$ and hence closed. q.e.d.

Assume that $f : N \to M$ is a Riemannian submersion between complete Riemannian manifolds and $L \subset M$ is a closed submanifold. Then, by [**He60**], the map $f : N \to M$ is a locally trivial fiber bundle and therefore, for any point $\bar{q} \in f^{-1}(q)$, the spaces $\mathcal{P}(N, f^{-1}(L) \times \bar{q})$ and $\mathcal{P}(M, L \times q)$ are homotopy equivalent. Since f yields a 1:1 correspondece between the critical points and preserves their indices [**HLO06**, Lemma 6.1]), we obtain

Lemma 3.10. *If $f : N \to M$ is a Riemannian submersion between complete Riemannian manifolds and $L \subset M$ is a closed submanifold, then L is taut if and only if $f^{-1}(L)$ is taut.*

Now, since the homology of a path connected component injects in the homology of the whole space, it is not hard to see that a union of connected, closed submanifolds is taut if and only if its components are taut. From that we deduce

Lemma 3.11. *Let $\pi : N \to M$ be a Riemannian covering and let M be complete. If $\mathcal{F}$ is closed, then $\mathcal{F}$ is $\mathbb{F}$-taut if and only if the lift $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ to N is $\mathbb{F}$-taut.*

Given a closed singular Riemannian foliation $\mathcal{F}$ on a complete Riemannian manifold M, every leaf posses a global ε-tube. For a regular leaf L with such a global tube, the restriction of the foot point projection on a nearby regular leaf induces a finite covering map onto L. We say that L has *trivial holonomy* if all these coverings are diffeomorphisms. It is well known that the set of regular points whose leaves have trivial holonomy is open and dense in M.

In particular, all regular leaves of $\mathcal{F}$ have trivial holonomy; that is to say, the quotient $M_0/\mathcal{F}$ is a Riemannian manifold if the foliation is taut and M is simply connected. To see this, for $p, q \in M$, let $\Omega_{p,q}(M)$ denote the space of all paths from p to q. Then $\Omega_{p,q}(M) \simeq \Omega_{q,q}(M)$ and the long exact sequence of the path space fibration yields isomorphisms $\pi_i(M) \cong \pi_{i-1}(\Omega_{q,q}(M))$, which implies that $\Omega_{p,q}(M)$ is connected.

The fibration $\mathcal{P}(M, L \times q) \to L$ given by $c \mapsto c(0)$ gives this part of the corresponding long exact homotopy sequence:

$$\pi_0(\Omega_{q,q}(M)) \to \pi_0(\mathcal{P}(M, L \times q)) \to \pi_0(L) \to 1.$$

Thus $\mathcal{P}(M, L \times q)$ is connected. Now a leaf with nontrivial holonomy would yield at least two local minima for the energy on the path space of a neighboring generic leaf; i.e., a leaf without holonomy. By tautness, all the maps in homology are injective, which clearly contradicts our connectedness observation.

We are now able to state a characterization of taut regular foliations, which indeed also follows from our second main result, Theorem 3.19 below. For the notion of Riemannian orbifolds see Section 3.2.

Lemma 3.12. *Let $\mathcal{F}$ be a closed (regular) Riemannian foliation on a complete Riemannian manifold M. Then $\mathcal{F}$ is $\mathbb{F}$-taut if and only if the quotient $M/\mathcal{F}$ is a good Riemannian orbifold with a pointwise $\mathbb{F}$-taut universal covering orbifold; i.e., $M/\mathcal{F}$ is isometric to N/Γ with a simply connected Riemannian manifold N, all of whose points are $\mathbb{F}$-taut and $\Gamma \subset \mathrm{Iso}(N)$ is a discrete subgroup of isometries.*

Proof. Let $\widetilde{\mathcal{F}}$ denote the lift of $\mathcal{F}$ to the universal cover $\widetilde{M}$ of M. Then, by Lemma 3.11, $\widetilde{\mathcal{F}}$ is taut if and only if $\mathcal{F}$ is taut. By the discussion above, if $\widetilde{\mathcal{F}}$ is taut, then it is simple; i.e., given by the fibers of a Riemannian submersion. So if $\mathcal{F}$ is taut, the quotient map $\widetilde{M} \to \widetilde{M}/\widetilde{\mathcal{F}}$ is a Riemannian submersion between complete Riemannian manifolds and $\widetilde{M}/\widetilde{\mathcal{F}}$ is 1-connected by the exact sequence for fibrations. In this case, the map $\widetilde{M}/\widetilde{\mathcal{F}} \to M/\mathcal{F}$ coincide with the universal orbifold covering. On the other hand, assume that $M/\mathcal{F}$ is a good Riemannian orbifold. Then, due to Lemma 3.6, $\widetilde{\mathcal{F}}$ is simple; that is to say, $\widetilde{M}/\widetilde{\mathcal{F}}$ is a Riemannian manifold and we can reduce the problem to Lemma 3.10. q.e.d.

Remark 3.13. At the end of Section 3.4 we prove the corresponding statement for the class of foliations whose quotients are orbifolds and coefficients in $\mathbb{Z}_2$.

We end this section with some genericity results.

Lemma 3.14. *Let $\mathcal{F}$ be closed. Then $L \in \mathcal{F}$ is $\mathbb{F}$-taut if and only if the energy functional $E_q : \mathcal{P}(M, L \times q) \to \mathbb{R}$ is an $\mathbb{F}$-perfect Morse function for all non L-focal regular points $q \in M$. Further, $\mathcal{F}$ is $\mathbb{F}$-taut if and only if all regular leaves are $\mathbb{F}$-taut.*

Proof. Since the set of non-focal points of L, as well as the set of regular points, is open and dense in M, every neighborhood of a given point q contains a regular point that is not a focal point. Therefore, the same argument as in the proof of Proposition 2.7 in [**TT97**] yields the first claim.

For the second claim, assume that all regular leaves are $\mathbb{F}$-taut. Let N be a singular leaf and let $q \in M$ be a non-focal point of N. By our above observations, we can assume that the point q is regular. Let γ be a critical point of E_q^N. Then $\dot\gamma(0)$ is a regular vector of $\mathfrak{F}_{\gamma(0)}$, hence there exists an $\varepsilon > 0$ such that $L = L_{\gamma(\varepsilon)}$ is a regular leaf contained in a global tube of N and the point q is not a focal point of L. Denote by $\bar\gamma$ the restriction $\gamma|_{[\varepsilon,1]}$ after linear reparameterization on $[0,1]$. Then the horizontal geodesic $\bar\gamma$ is a critical point of the perfect Morse function $E_q^L : \mathcal{P}(M, L \times q) \to \mathbb{R}$. Now let $i = \mathrm{ind}(\bar\gamma)$ be the index of the geodesic $\bar\gamma$ and denote by $\kappa = E_q^L(\bar\gamma)$ its energy. For notational reasons let us set $\mathcal{L}^c = (E_q^L)^{-1}([0,c])$, resp. $\mathcal{N}^c = (E_q^N)^{-1}([0,c])$. Denote by $\sigma \in H_i(\mathcal{L}^\kappa)$ the completion of the local unstable manifold representing a nontrivial cycle in $H_i(\mathcal{L}^\kappa, \mathcal{L}^\kappa \setminus \{\bar\gamma\})$ associated to $\bar\gamma$. We then have $\mathrm{ind}(\gamma) = \mathrm{ind}(\bar\gamma)$ for small numbers ε, for continuity reasons.

The restriction of the foot point projection $R : L \to N, (p, v) \mapsto p$, induces a map

$$\begin{array}{rccc} \bar R : & \mathcal{L}^\kappa & \to & \mathcal{N}^{E_q^N(\gamma)} \\ & c & \mapsto & \tilde c, \end{array}$$

where $\tilde c$ is the curve that one gets by concatenation of the unique horizontal geodesic from $R(c(0))$ to $c(0)$ with c, followed by reparameterization between 0 and 1. The map $\bar R$ maps level sets to level sets. Moreover, $\bar R$ is clearly an immersion. Therefore, $\bar R_*(\sigma)$ is a cycle in $\mathcal{N}^{E_q^N(\gamma)}$ that can be deformed within a morse chart around γ into a cycle z that agrees with the unstable manifold at γ above the E_q^N-level $E_q^N(\gamma)-\delta$ for small δ. It follows that the homology class of z, and thus the homology class $\bar{R}_*(\sigma)$, is mapped onto a generator of $H_i(\mathcal{N}^{E_q^N(\gamma)}, \mathcal{N}^{E_q^N(\gamma)} \setminus \{\gamma\})$. Since the critical point γ was chosen arbitrarily, every local unstable manifold can be completed to a cycle in $H_{i(\gamma)}(\mathcal{N}^{E_q^N(\gamma)+\delta})$; i.e., the map $H_n(\mathcal{N}^{\lambda+\delta}) \to H_n(\mathcal{N}^{\lambda+\delta}, \mathcal{N}^{\lambda-\delta})$ is surjective for all n and regular values $\lambda \pm \delta$. Hence N is $\mathbb{F}$-taut. q.e.d.

Our last observation in this section is that a foliation $\mathcal{F}$ is $\mathbb{F}$-taut if and only if a dense family of regular leaves is $\mathbb{F}$-taut, where we call a family of leaves *dense* if their union is a dense set. The next lemma shows that tautness is a closed property relative to non-collapsing convergence. It is then straightforward to see that tautness of a dense family of regular leaves forces a foliation to be taut.

Lemma 3.15. *Let $\mathcal{F}$ be a closed singular Riemannian foliation on a complete manifold M, and let $\{L_n\}$ be a sequence of $\mathbb{F}$-taut regular leaves converging to a regular leaf L without holonomy; i.e., for every tubular neighborhood T of L there is a number $n_0 \in \mathbb{N}$ such that $L_n \subset T$, and the canonical projection $\pi : T \to L$ restricted to L_n is a diffeomorphism for every $n \geq n_0$. Then L is $\mathbb{F}$-taut.*

Proof. Let T be a tubular neighborhood of L and let $\pi : T \to L$ be the canonical projection. Choose a number $n_0 \in \mathbb{N}$ so large that $L_n \subset T$ for all $n \geq n_0$. Now let $q \in M$ be a non-focal point of L. Then for large n, the point q is not a focal point of L_n as well. We denote by $f_n : \mathcal{P}(M, L \times q) \to \mathcal{P}(M, L_n \times q)$ and $g_n : \mathcal{P}(M, L_n \times q) \to \mathcal{P}(M, L \times q)$, respectively, the induced maps between the path spaces that one gets by assigning to a curve c the curve $\gamma_{c(0)} \cdot c$ and then reparameterizing it between 0 and 1, where $\gamma_{c(0)}$ is the unique shortest geodesic between L_n and L that intersects L in $c(0)$, resp. $\gamma_{c(0)}$ is the unique shortest geodesic between L and L_n that intersects L_n in $c(0)$. Then f_n is, in an obvious way, a homotopy equivalence with homotopy inverse g_n.

Let γ be a critical point of E_q^L with $\kappa = E_q^L(\gamma)$. We can choose n so large (i.e., a tube T so small) that there is an $\varepsilon > 0$ such that $(\kappa - 3\varepsilon, \kappa + 3\varepsilon) \setminus \{\kappa\}$ contains only regular values and $g \circ f(P^{\kappa - 2\varepsilon}) \subset P^{\kappa - \varepsilon}$ with $f = f_n, g = g_n$, and $P^r = \mathcal{P}(M, L \times q)^r$, and P_n^r defined analogously. Moreover, we can deform $g \circ f : P^{\kappa-2\varepsilon} \to P^{\kappa-\varepsilon}$ into the inclusion $j : P^{\kappa-2\varepsilon} \hookrightarrow P^{\kappa-\varepsilon}$ below the κ-level of E_q^L; i.e.,

$$(g \circ f)_* = j_* : \quad H_*(P^{\kappa-2\varepsilon}) \quad \to \quad H_*(P^{\kappa-\varepsilon}).$$

Since $P^{\kappa-2\varepsilon}$ is a strong deformation retract of $P^{\kappa-\delta}$ for all $2\varepsilon > \delta > 0$, the above map is an isomorphism. In particular, the induced map in homology

$$f_* : H_*(P^{\kappa-2\varepsilon}) \to H_*(P_n^{\alpha})$$

with $\alpha = \max\{E_q^{L_n}(f(c)) : c \in (E_q^L)^{-1}(\kappa - 2\varepsilon)\}$ injective.

Set $i = \operatorname{ind}(\gamma)$ and define $\tilde{\alpha} = \max\{E_q^{L_n}(f(c)) : c \in (E_q^L)^{-1}(\kappa)\}$. Now consider the commutative diagram that comes from the long exact sequence for pairs of spaces together with the natural behavior of the connecting homomorphism

$$\begin{tikzcd}
H_i(P^\kappa) \arrow[r, "f_*"] \arrow[d] & H_i(P_n^{\tilde{\alpha}}) \arrow[d] \\
H_i(P^\kappa, P^{\kappa-2\varepsilon}) \arrow[r, "f_*"] \arrow[d, "\partial_*"'] & H_i(P_n^{\tilde{\alpha}}, P_n^\alpha) \arrow[d, "\tilde{\partial}_*"] \\
H_{i-1}(P^{\kappa-2\varepsilon}) \arrow[r, "f_*"] & H_{i-1}(P_n^\alpha)
\end{tikzcd}$$

Since L_n is taut, we have $\tilde{\partial}_* = 0$. So

$$f_* \circ \partial_* = \tilde{\partial}_* \circ f_* = 0.$$

As we have seen, the map $f_* : H_*(P^{\kappa-2\varepsilon}) \to H_*(P_n^\alpha)$ is injective. But this means $\partial_* = 0$; i.e., L is taut. q.e.d.

Now assume under the assumptions of Lemma 3.15 that all L_n are regular leaves without holonomy and that L is an exceptional leaf; i.e., has nontrivial holonomy. Due to Lemma 3.11, we can assume that M is simply connected. Then for large n, the leaf L would provide at least two local minima for L_n. Again, by tautness of L_n, the path space corresponding to L_n would be disconnected. But this is clearly a contradiction, since M is simply connected. So, L necessarily has trivial holonomy. Thus combining Lemma 3.11, Lemma 3.14, and Lemma 3.15 together with the fact that the set of regular leaves without holonomy is open and dense in $M/\mathcal{F}$, we have

Corollary 3.16. *The closed singular Riemannian foliation $\mathcal{F}$ is $\mathbb{F}$-taut if and only if a dense family of leaves is $\mathbb{F}$-taut.*

3.3. Index Splitting for Horizontal Geodesics. In this section we summarize some general observations on the focal indices of geodesics with respect to Lagrangian subspaces of the space of normal Jacobi fields, which we then apply to the case of a horizontal geodesic of a singular Riemannian foliation $(M, \mathcal{F})$. We will see that the focal data of the space of normal $L_{\gamma(a)}$-Jacobi fields along a regular horizontal geodesic $\gamma : [a, b] \to M$ are of two types. Namely, for $t \in (a, b)$, there is a vertical multiplicity $\dim(\mathcal{F}) - \dim(L_{\gamma(t)})$ that counts the intersections with the singular stratum, and a horizontal multiplicity that is, roughly speaking, the multiplicity of $\gamma(t)$ as a conjugate point of $\gamma(a)$ in the quotient $M/\mathcal{F}$ along the projection of γ. Our discussion is based on [**L09**], [**LT10**], and [**Wil07**].

If $\mathcal{F}$ is a singular Riemannian foliation on a Riemannian manifold (M, g) we call a geodesic γ *horizontal* if it meets all leaves of $\mathcal{F}$ perpendicularly. We will call such a geodesic $\gamma : [a, b] \to M$ regular, if $\gamma(a)$ and $\gamma(b)$ are regular points of $\mathcal{F}$.

A regular horizontal geodesic intersects the singular strata of $\mathcal{F}$ in only finitely many points $a < t_1 < \cdots < t_r < b$ (see [**LT10**, Cor.4.6]). We set

$$c(\gamma) = \sum_{i=1}^{r} \dim(L_{\gamma(a)}) - \dim(L_{\gamma(t_i)})$$

and call this number the *crossing number* of γ.

Definition 3.17. Let $\gamma : [a, b] \to M$ be a horizontal geodesic. An *$\mathcal{F}$-Jacobi field* along γ is a variational field through horizontal geodesics starting on the leaf $L_{\gamma(a)}$. An *$\mathcal{F}$-vertical Jacobi field* along γ is an $\mathcal{F}$-Jacobi field J with $J(t) \in T_{\gamma(t)} L_{\gamma(t)}$ for all t.

Let us recall some required facts about the Jacobi equation, Jacobi fields, and focal points. We refer to [**L09**] and [**Wil07**] for the proofs and a more detailed discussion of the following facts. Our summary follows the exposition given in [**LT10**].

Let $\gamma : [a, b] \to M$ be a geodesic and let $\mathcal{N}$ be the normal bundle of γ. Let Jac denote the space of all normal Jacobi fields along γ; i.e., solutions of the equation

$$\nabla^2 J + R(J, \dot\gamma)\dot\gamma = 0,$$

where R denotes the curvator tensor. By ω we denote the canonical symplectic form on Jac, defined by $\omega(J_1, J_2) = \langle \nabla J_1, J_2 \rangle + \langle J_1, \nabla J_2 \rangle$. For subspaces W of Jac, we denote by $W^\perp$ the orthogonal complement with respect to ω. A subspace $W \subset$ Jac is called *isotropic*, resp. *Lagrangian*, if $W \subset W^\perp$, resp. $W = W^\perp$.

For an isotropic subspace W and $t \in [a, b]$, we define the *W-focal index* of t to be $f^W(t) = \dim(W) - \dim(W(t))$ with $W(t) = \{J(t) \mid J \in W\}$. Note that we have the equality $f^W(t) = \dim(W^t)$, where we have set $W^t = \{J \in W \mid J(t) = 0\}$. One can show that the set of points with non-zero focal index is discrete and such points are called *W-focal*. The *W-index* of γ is defined by $\mathrm{ind}_W(\gamma) = \sum_{t \in [a,b]} f^W(t)$.

Set $(M, \mathcal{F}, g)$ as usual and let $\gamma : [a, b] \to M$ be a horizontal geodesic. Then the space $\Lambda^{L_{\gamma(a)}}$ of all normal $\mathcal{F}$-Jacobi fields is Lagrangian but depends not only on the maximal geodesic containing γ but also on the starting point $\gamma(a)$. To arrange this problem, consider the space W^γ consisting of all Jacobi fields along γ with the property that these fields are variational fields through horizontal geodesics γ_s with $\gamma_s(t) \in L_{\gamma(t)}$ for all t. One can show [**LT10**, Sec 4.5] that $W^\gamma(t) = \{J(t) \mid J \in W^\gamma\}$ coincides with $T_{\gamma(t)} L_{\gamma(t)}$, for all t, and by definition we have $W^\gamma \subset \Lambda^{L_{\gamma(a)}}$. The space W^γ is just the space of all $\mathcal{F}$-vertical Jacobi fields along γ and does not depend on the starting point, in contrast to $\Lambda^{L_{\gamma(a)}}$. If $d(\gamma)$ denotes the maximal dimension of $L(\gamma(t))$, then we have $d(\gamma) = \dim(W^\gamma)$. Moreover, the W^γ-focal points along γ are precisely the points t_i with $\dim(L_{\gamma(t_i)}) < d(\gamma)$ and the W^γ-focal index is $d(\gamma) - \dim(L_{\gamma(t_i)})$. In particular, for a regular horizontal geodesic γ, its crossing number $c(\gamma)$ coincides with the vertical index $\mathrm{ind}_{W^\gamma}(\gamma)$. If, in addition, we call a Jacobi field *horizontal* if it is the variational field of a variation of γ through horizontal geodesics, one can describe the space $(W^\gamma)^\perp$ as the space consisting of normal horizontal Jacobi fields.

We now recall the construction of Wilking [**Wil07**] of the transversal Jacobi equation in our situation. Again let $\gamma : [a, b] \to M$ be a geodesic and consider the normal bundle $\mathcal{N}$ of γ with the connection induced by the pull back. Let $R : \mathcal{N} \to \mathcal{N}$ denote the curvature endomorphism defined by $R(X) = R(X, \dot\gamma)\dot\gamma$. Let Jac be as above and consider an isotropic subspace W of Jac. Set

$$\widetilde{W}(t) = W(t) \oplus \left\{\nabla J(t) \mid J \in W^t\right\}$$

and note that $\widetilde{W}(t) = W(t)$ for every non W-focal $t \in [a, b]$.

Then Wilking observed (in a more generall setting) that $\widetilde{W}$ defines a smooth subbundle of $\mathcal{N}$. If we denote by $\mathcal{H}$ the orthogonal complement of $\widetilde{W}$ and by $P : \mathcal{N} \to \mathcal{H}$ the orthogonal projection, then P defines an identification $\mathcal{H} \cong \mathcal{N}/\widetilde{W}$ and we can define a smooth endomorphism field $A : \widetilde{W} \to \mathcal{H}$, by $A(J(t)) = P(\nabla J(t))$ and $A(\nabla J(t)) = 0$ for all $J \in W^t$. Consider the field $R^{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ of symmetric endomorphisms, defined by

$$R^{\mathcal{H}}(Y) = P(R(Y)) + 3AA^*(Y),$$

and denote by $\nabla^{\mathcal{H}}$ the induced covariant derivative on $\mathcal{H}$; i.e.,

$$\nabla^{\mathcal{H}}(Y) = P(\nabla Y).$$

Wilking proved in [**Wil07**] that for each Jacobi field $J \in W^\perp$ the projection $Y = P(J)$ is an $R^{\mathcal{H}}$-Jacobi field; i.e., $(\nabla^{\mathcal{H}})^2 J + R^{\mathcal{H}}(J) = 0$. Moreover, two R-Jacobi fields $J_1, J_2 \in W^\perp$ have the same projection to $\mathcal{H}$ if and only if $J_1 - J_2 \in W$. Thus the induced map

$$I : W^\perp/W \to \mathrm{Jac}^{R^{\mathcal{H}}}$$

is injective and by dimensional reasons it is an isomorphism. Hence $R^{\mathcal{H}}$-Jacobi fields are precisely the projections of Jacobi fields in $W^\perp$; and Lagrangians in $\mathrm{Jac}^{R^{\mathcal{H}}}$ are projections of Lagrangians in Jac that contain W. As a consequence we obtain

Lemma 3.18. *For each Lagrangian $\Lambda \subset$ Jac that contains W, we have the equality* $\mathrm{ind}_W(\gamma) + \mathrm{ind}_{\Lambda/W}(\gamma) = \mathrm{ind}_\Lambda(\gamma)$.

Example. Let $f : M \to B$ be a Riemannian submersion and let $\mathcal{F}(f)$ denote the induced foliation on M. Let γ be a horizontal geodesic in M and denote by $\bar{\gamma} = f(\gamma)$ its image in B. Consider the space W^γ of $\mathcal{F}(f)$-vertical Jacobi fields along γ; i.e., variational fields of variations of horizontal lifts of $\bar{\gamma}$. Then W^γ is an isotropic subspace, since it is contained in the space Λ^N of normal N-Jacobi fields, where we set $N = f^{-1}(f(\gamma(a)))$. In this case, for each t, the space $W^\gamma(t)$ is the vertical space of f through $\gamma(t)$, $\mathcal{H}$ is canonically identified with the normal bundle of $\bar{\gamma}$ in B and the transversal operator $R^{\mathcal{H}}$ coincides with the curvature endomorphism in the base space; i.e., the term AA^* is just the O'Neill tensor. So, the *horizontal index* $\mathrm{ind}_{\Lambda^N/W^\gamma}(\gamma)$ describes the index of $\bar{\gamma}$. The *vertical index* $\mathrm{ind}_{W^\gamma}(\gamma)$ is zero in this case, but in the much more general situation of a singular Riemannian foliation, the vertical index counts the intersections of γ with singular leaves and coincides with the crossing number in the regular case.

Recall that two points $c < d$ in $[a, b]$ are called *conjugate* if there is a non-zero Jacobi field $J \in \mathrm{Jac}$ with $J(c) = 0 = J(d)$. Thus the statement that the point a does not have conjugate points on (a, b) for the transversal Jacobi equation on $\mathcal{H}$, where $\mathcal{H}$ is the W^γ-transversal bundle as defined above, is equivalent to the equality $\mathrm{ind}_{\Lambda^{L_{\gamma(a)}}}(\gamma_0) = \mathrm{ind}_{W^\gamma}(\gamma_0)$, where γ_0 denotes the subgeodesic $\gamma_0 : (a, b) \to M$ of γ. To see this, note that, by Lemma 3.18, $\mathrm{ind}_{\Lambda^{L_{\gamma(a)}}}(\gamma_0) = \mathrm{ind}_{W^\gamma}(\gamma_0)$ is equivalent to the absence of focal points of $\Lambda^{L_{\gamma(a)}}/W^\gamma$ on the open interval (a, b). But $\Lambda^{L_{\gamma(a)}}/W^\gamma$ is by definition the Lagrangian in $\mathrm{Jac}(\mathcal{H})$ of all Jacobi fields Y with $Y(a) = 0$. Thus the statement that $\mathrm{ind}_{\Lambda^{L_{\gamma(a)}}/W^\gamma}(\gamma_0) = 0$ is equivalent to the fact that a does not have conjugate points with respect to the transversal Jacobi equation.

3.4. A Property of the Quotient. Dealing with singular Riemannian foliations, one focuses mainly on the horizontal geometry of the foliation; that is to say, the geometry of the quotient. For this reason, one is often interested in geometric properties of the foliation that can be read off the quotient and to consider *equivalence classes* of foliations by means of isometric quotients. An example of such a quotient property is infinitesimal polarity (cf. [**LT10**]), which is equivalent to the property that the quotients are Riemannian orbifolds. Our second main result now states that tautness of a foliation is actually also a property of the quotient, so that one can speak about *equivalence classes* of taut foliations by means of their leaf spaces.

Theorem 3.19. *Let $\mathcal{F}$ and $\mathcal{F}'$ be closed singular Riemannian foliations on complete Riemannian manifolds M and M' with isometric quotients. Then $\mathcal{F}$ is taut if and only $\mathcal{F}'$ is taut. In particular, if one of them is $\mathbb{F}$-taut, then both are $\mathbb{Z}_2$-taut.*

Before we begin with the proof of the theorem let us discuss and apply this result in the context of the known examples.

If $M = S^k$ is the round sphere and $\mathcal{F}$ is the trivial foliation by points, there is a well known cycle construction for critical points of the energy functional [**Mi63**, pp.95–96]) that shows that S^k is pointwise taut. Terng and Thorbergsson proved in [**TT97**] that the standard metric on the sphere is the only one with respect to which the sphere is pointwise taut.

Now consider the more general case $M/\mathcal{F} = N/\Gamma$, where N is a symmetric space. In their study of Morse theory of symmetric spaces, Bott and Samelson came up with concrete cycles that represent a basis in $\mathbb{Z}_2$-homology of generic path spaces $\mathcal{P}(N, p \times q)$ and that are in fact compact connected manifolds (see [**BS58**]) and coincide with those cycles we constructed in Theorem 2.9. In particular, symmetric spaces are pointwise taut. Therefore, the foliation $\mathcal{F}$ on M has to be taut by Theorem 3.19. Of course, one could allow an additional constant direction of nonpositive curvature in the quotient, because under this assumption there are no focal points in this direction.

We want to emphasize that the following corollary covers all known examples.

Corollary 3.20. *If $\mathcal{F}$ is a closed singular Riemannian foliation on a complete Riemannian manifold M and $M/\mathcal{F} = (N \times P)/\Gamma$ is a good Riemannian orbifold, where N is a symmetric space and P is a nonpositively curved manifold, then $\mathcal{F}$ is taut.*

Another application of Theorem 3.19 are foliations admitting generalized sections.

Example. Let M be a complete Riemannian manifold with an isometric action of a compact Lie group G. In [**GOT04**] the authors developed the concept of a generalized section for such an action. They call a connected, complete submanifold Σ of M a *k-section* if the following hold:

- Σ is totally geodesic;
- Σ intersects all orbits;
- for every G-regular point $p \in \Sigma$, the tangent space $T_p\Sigma$ contains the normal space $\nu_p(G(p))$ as a subspace of codimension k;
- if $p \in \Sigma$ is a G-regular point with $g(p) \in \Sigma$ for some $g \in G$ then $g(\Sigma) = \Sigma$.

Generalized sections are also called *fat sections* and the copolarity of (G, M) is defined by

$$\operatorname{copol}(G, M) = \min \{k \in \mathbb{N} \mid \text{ there is a } k\text{-section } \Sigma \subset M\}$$

and measures, roughly speaking, how far the action is from being polar; i.e., admitting a 0-section. If Σ is a fat section, then it is shown in [**Ma08**] that there is the *fat Weyl group* $W(\Sigma) = N_G(\Sigma)/Z_G(\Sigma)$ that acts on Σ with $G(p) \cap \Sigma = W(\Sigma)(p)$ if $p \in \Sigma$, inducing an isometry between the quotients $\Sigma/W(\Sigma) = M/G$. We therefore deduce that $(\Sigma, \mathcal{F}^W)$ is taut if and only if $(M, \mathcal{F}^G)$ is taut.

Before we start with the proof of Theorem 3.19, we now state a preparing lemma that says that focal points caused by singular leaves do not provide any difficulties when dealing with tautness. This fact was already discussed in [**No08**].

Lemma 3.21. *Let $\mathcal{F}$ be a closed singular Riemannian foliation on a complete Riemannian manifold M and let $L \in \mathcal{F}$ be a regular leaf. For every broken horizontal geodesic $c : [0,1] \to M$ from L to a point $q \in M$ that intersects the singular stratum discretely, let $\Delta(c)$ denote the space of broken horizontal geodesics in the path space $\mathcal{P}(M, L \times q)$ that have the same projection to the quotient $M/\mathcal{F}$ as c. Then $\Delta(c)$ carries a smooth structure of a compact (possibly non-connected) manifold of dimension $\sum_{t\in[0,1]} \dim(\mathcal{F}) - \dim(L_{c(t)})$ such that the inclusion into the path space $\mathcal{P}(M, L \times q)$ becomes an embedding.*

Proof. Given a leaf $L \in \mathcal{F}$ let $\nu^\varepsilon(L)$ be a global ε-tube of L. Then the pull back of $\mathcal{F}$ by the normal exponential map is invariant under the homotheties $r_\lambda(v) = \lambda v$ for all $\lambda \in [-1,1]\setminus\{0\}$, so that there is a unique singular foliation $\mathcal{G}(L)$ that extends the pull back to $\nu(L)$ satisfying this property. The singular foliation $\mathcal{G}(L)$ is closed if $\mathcal{F}$ is closed, and it is shown in Section 4 of [**LT10**] that $v, w \in \nu(L)$ are in the same leaf of $\mathcal{G}(L)$ if and only if $\gamma_w(t) \in L_{\gamma_v(t)}$ for all t, where as usual γ_v is the unique geodesic with $\dot\gamma_v(0) = v$. Let V be a small open neighborhood of v in the leaf $\mathcal{L}_v$ of $\mathcal{G}(L)$ through v. Then the vector space of variational vector fields of variations through geodesics γ_w with $w \in V$ coincides with the space W^{γ_v} of $\mathcal{F}$-vertical Jacobi fields along γ_v, as defined in the last section. Due to [**LT10**], one has

$$W^{\gamma_v}(t) = \{J(t) \mid J \in W^{\gamma_v}\} = T_{\gamma_v(t)} L_{\gamma_v(t)},$$

and we deduce that the map $\eta_t : \mathcal{L}_v \to L_{\gamma_v(t)}$, given by $\eta_t(w) = \exp(tw) = \gamma_w(t)$, is a submersion for all t, which is surjective if $\mathcal{F}$ is closed. In this case, all the preimages $\eta_t^{-1}(p)$ are compact submanifolds of $\mathcal{L}_v$ of dimension $\dim(\mathcal{L}_v) - \dim(L_{\gamma_v(t)})$. In particular, if L is a regular leaf, the dimension of such a preimage equals the difference $\dim(\mathcal{F}) - \dim(L_{\gamma_v(t)})$.

We will now describe the compact set $\Delta(c)$ as the total space of an iterated bundle. Since the general case requires no new ideas but only some more notation, we will assume for the rest of the proof that c as in the claim is smooth. So let $L \in \mathcal{F}$ be a regular leaf and let $\gamma = \gamma_v$ be a horizontal geodesic from L to a point $q \in M$. Let $\gamma^{-1}(M \setminus M_0) = \{t_i\}_{i=1,\dots,r}$ with $0 < t_r < \cdots < t_1 \le 1$ denote the times where γ crosses the singular stratum and set $L_i = L_{\gamma(t_i)}$ and $v_i = \dim(\mathcal{F}) - \dim(L_i)$. Note that if q is a regular point, the vertical index of γ is given by

$v(\gamma) = \sum_{i=1}^{r} v_i$. With the notation from above, let $\eta_i : \mathcal{L}_v \to L_i$ be the surjective submersion defined by $\eta_i(w) = \exp(t_i w)$. Starting with the furthermost singular leaf, we now define $V_1 = \eta_1^{-1}(\gamma(t_1)) \subset \mathcal{L}_v$ and identify this space with the subspace

$$\Delta_1 = \left\{ c_w \in \Delta(\gamma) | c_w|_{[0,t_1]} = \gamma_w|_{[0,t_1]} \text{ for } w \in V_1 \text{ and } c_w|_{[t_1,1]} = \gamma|_{[t_1,1]} \right\}$$

of $\Delta(\gamma)$ of (at most) once broken geodesics in the obvious way; i.e., by $w \mapsto c_w$. With this identification, Δ_1 inherits a smooth structure that turns it into an embedded submanifold of $\mathcal{P}(M, L \times q)$ of dimension v_1.

At the second step we define V_2 to be the twisted product

$$\mathcal{L}_v \times_\eta V_1 = \{(w_2, w_1) \in \mathcal{L}_v \times V_1 \mid \eta_2(w_2) = \exp(t_2 w_1)\},$$

which can be identified with the subspace Δ_2 of $\Delta(\gamma)$ that consists of all (at most) twice broken horizontal geodesics $c_{(w_2,w_1)}$ with $c_{(w_2,w_1)}|_{[0,t_2]} = \gamma_{w_2}|_{[0,t_2]}$ for some element $w_2 \in \mathcal{L}_v$, and $c_{(w_2,w_1)}|_{[t_2,1]} = c_{w_1}|_{[t_2,1]}$ for some $w_1 \in V_1$. With the induced smooth structure, Δ_2 becomes a submanifold of $\mathcal{P}(M, L \times q)$ with

$$\dim(\Delta_2) = \dim(\mathcal{L}_v) + \dim(V_1) - \dim(L_2) = v_2 + v_1.$$

Note that all we need to ensure that Δ_2 is a submanifold is the fact that the map $\eta_2 : \mathcal{L}_v \to L_2$ is a submersion, so that $\mathcal{L}_v \times V_1 \to L_2 \times L_2$ is transversal to the diagonal in $L_2 \times L_2$.

Now assume that for some $r-1 \geq j \geq 1$ we have already defined V_j as a submanifold of dimension $\sum_{i=1}^{j} v_i$ of the j-fold product $\mathcal{L}_v^j$, together with an identification $V_j \cong \Delta_j$ given by $(w_j, \ldots, w_1) \mapsto c_{(w_j,\ldots,w_1)}$. Then we inductively define V_{j+1} and Δ_{j+1} as follows. Set $V_{j+1} = \mathcal{L}_v \times_\eta V_j$, where again the twisted product is defined by

$$\mathcal{L}_v \times_\eta V_j = \{(w_{j+1}, w_j, \ldots, w_1) \in \mathcal{L}_v \times V_j \mid \eta_{j+1}(w_{j+1}) = \exp(t_{j+1} w_j)\},$$

which is therefore a submanifold of $\mathcal{L}_v^{j+1}$ of dimension

$$\begin{aligned} \dim(V_{j+1}) &= \dim(\mathcal{L}_v) + \dim(V_j) - \dim(L_{j+1}) \\ &= v_{j+1} + \dim(V_j) \\ &= \sum_{i=1}^{j+1} v_i. \end{aligned}$$

Finally, define Δ_{j+1} to be the subspace of $\Delta(\gamma)$ consisting of all (at most) $(j+1)$-fold broken horizontal geodesics $c_{(w_{j+1},w_j,\ldots,w_1)}$ such that

$$\begin{aligned} c_{(w_{j+1},w_j,\ldots,w_1)}|_{[0,t_{j+1}]} &= \gamma_{w_{j+1}}|_{[0,t_{j+1}]} \text{ for some } w_{j+1} \in \mathcal{L}_v \text{ and} \\ c_{(w_{j+1},w_j,\ldots,w_1)}|_{[t_{j+1},1]} &= c_{(w_j,\ldots,w_1)}|_{[t_{j+1},1]} \text{ for some } (w_j, \ldots, w_1) \in V_j. \end{aligned}$$

By construction, it is clear that there is a 1:1 correspondence between $(j+1)$-tuples $(w_{j+1}, \ldots, w_1) \in V_{j+1}$ and paths $c_{(w_{j+1},w_j,\ldots,w_1)}$ in Δ_{j+1}. Moreover, the identification $\Delta_{j+1} \cong V_{j+1}$ (as manifolds) via

$(w_{j+1}, w_j, \ldots, w_1) \mapsto c_{(w_{j+1}, w_j, \ldots, w_1)}$ turns Δ_{j+1} into a compact submanifold of $\mathcal{P}(M, L \times q)$ of dimension $\sum_{i=1}^{j+1} v_i$. In particular, this defines a smooth structure for $\Delta_r = \Delta(\gamma)$ with the desired properties. q.e.d.

Remark 3.22. The assumptions in Lemma 3.21 are adapted to our setting, but the conclusion also holds if the foliation is not closed or the manifold is not complete. For this fact, because being a manifold is a local property, one only has to localize the arguments given in the proof of the lemma. Further, if $c = \gamma$ is smooth, let W^γ denotes the space of $\mathcal{F}$-vertical Jacobi fields along γ (see Section 3.3) and let $t_1 < \cdots < t_r$ be the W^γ-focal times along γ. Define W_i^γ to be the space of continuous vector fields J along γ such that $J|_{[0,t_i]} \in W^\gamma|_{[0,t_i]}$ and J vanishes on $[t_i, 1]$. Then by our description in the proof of the lemma, we conclude that the tangent space of $\Delta(\gamma)$ at γ is given by

$$T_\gamma \Delta(\gamma) = \bigoplus_{i=1}^{r} W_i^\gamma.$$

As a consequence of Lemma 3.21, using the same notation we reprove the mentioned special case.

Corollary 3.23. *If there are no horizontal conjugate points (i.e., conjugate points for the transversal Jacobi equation) along the horizontal geodesic γ and $\gamma(1)$ is not a focal point of $L_{\gamma(0)}$, then γ is of linking type (for $E_{\gamma(1)}$) with respect to $\mathbb{Z}_2$.*

Proof. By Lemma 3.21, there is a compact manifold $\Delta(\gamma)$ through every regular horizontal geodesic γ consisting of broken horizontal geodesics, all having the same length as γ, and $\dim(\Delta(\gamma))$ coincides with the vertical index $v(\gamma)$. But by assumption, the index of γ is just the vertical index. Moreover, the statement about $T_\gamma\Delta(\gamma)$ and the discussion at the end of the proof of Theorem 2.9 ensures that if we look at a finite dimensional approximation of the path space, $\Delta(\gamma)$ is transversal to the ascending cell in a Morse chart around γ, so that $\Delta(\gamma)$ can be deformed into the descending cell, and hence defines a linking cycle for γ. q.e.d.

Proof of Theorem 3.19. Let us briefly sketch the idea of the proof. For $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ as in the claim, let us identify $B = M/\mathcal{F} = M'/\mathcal{F}'$ via an isometry and consider the following diagram

$$\begin{tikzcd} M \arrow[dr, "Q"'] & & M' \arrow[dl, "Q'"] \\ & B & \end{tikzcd}$$

Now assume that $\mathcal{F}$ is taut. In order to prove that $\mathcal{F}'$ is taut it suffices to prove that the normal exponential map of a generic leaf of $\mathcal{F}'$ has integrable fibers, by Theorem 2.9 and our genericity results from Section

3.1. Let, therefore, $L' \in \mathcal{F}'$ be a regular leaf without holonomy. In this case, the leaf $L = Q^{-1}(Q'(L')) \in \mathcal{F}$ is a regular leaf without holonomy, too, and its normal exponential map has integrable fibers, by assumption. For $v \in \nu(L)$, let Δ_v denote the connected component of the fiber through v that contains v and identify it with the manifold of horizontal geodesics from L to $\exp(v)$ that have initial velocity in Δ_v. Now, given a vector $v' \in \nu(L')$ with the same projection to B as v, we push Δ_v down to B and lift it to M' along Q' to obtain a space $\Delta'_{v'}$ of horizontal geodesics that start in L' and end in $\exp(v')$. The observation that the map $\Delta'_{v'} \to \nu(L')$, which assigns to a horizontal geodesic its starting direction, provides an integral manifold of the kernel distribution of the normal exponential map of L' through v' then finishes the proof.

Having sketched the proof, let us now work out the details. Given a point $p \in M$, the infinitesimal foliation $\mathfrak{F}_p$ splits as a product foliation $\mathfrak{F}_p = T_pL_p \times \mathfrak{F}_p^1$ on the tangent space $T_pM = T_pL_p \times \nu_p(L_p)$ so that we have $T_pM/\mathfrak{F}_p = \nu_p(L_p)/\mathfrak{F}_p^1$, which is the tangent space to a local quotient $U/\mathcal{F}$ at $L \cap U$, where U is a distinguished neighborhood of p. The map $U/\mathcal{F} \to M/\mathcal{F}$, induced by the inclusion $U \to M$, is a finite-to-one open map, given by the quotient map of the action of a finite group Γ of isometries, onto a neighborhood $T^\varepsilon/\mathcal{F}$ of L_p, where T^ε is a global ε-tube around L_p with the same ε as in the definition of the distinguished neighborhood U (cf. Section 3.1). Identifying $\nu^\varepsilon(L_p) \cong T^\varepsilon$ via the normal exponential map, we see that we can identify the tangent space $T_{L_p}B$ of B at L_p with $(\nu_p(L_p)/\mathfrak{F}_p^1)/\Gamma$. We therefore define the differential $dQ_p : T_pM \to T_{Q(p)}B$ of the projection $Q : M \to B$ at the point p to be the composition of projections $T_pM \to \nu_p(L_p) \to (\nu_p(L_p)/\mathfrak{F}_p^1)/\Gamma$ and write $Q_* : TM \to TB$ for the induced map; i.e., $Q_*(v) = dQ_{P(v)}(v)$. We use the analogous notations for $Q' : M' \to B$.

Since orbifold geodesics coincide if they coincide initially, we deduce from the fact that the set of $\mathcal{F}$-horizontal vectors v in TM with the property that $Q \circ \gamma_v$ is completely contained in the open and dense orbifold part of B has full measure in the subset of all horizontal vectors (cf. [**LT10**]), that, given an $\mathcal{F}$-horizontal vector v and an $\mathcal{F}'$-horizontal vector v' with the same projection (i.e., $Q_*(v) = Q'_*(v')$) we have $Q \circ \gamma_v(t) = Q' \circ \gamma_{v'}(t)$ for all $t \in \mathbb{R}$.

Now take a regular leaf $L \in \mathcal{F}$ and recall that in this case the foliation $\mathcal{G}(L)$ on $\nu(L)$ (from the proof of Lemma 3.21) is a regular foliation with closed leaves such that the intersection of every leaf $\mathcal{L} \in \mathcal{G}(L)$ with any normal space $\nu_p(L)$ is finite. Further, as explained there, for every normal vector $v \in \nu(L)$ the restriction of the normal exponential map to the leaf $\mathcal{L}_v$ induces a submersion $\eta_v : \mathcal{L}_v \to L_{\gamma_v(1)}$, so that all the fibers $\eta_v^{-1}(q)$ for $q \in L_{\gamma_v(1)}$ are (unions of) compact submanifolds of dimension $\dim(\mathcal{F}) - \dim(L_{\gamma_v(1)})$, since the normal exponential map is proper. In this case, the smooth map $\eta_v^{-1}(q) \to \nu_q(L_{\gamma_v(1)})$, given by $w \mapsto \dot\gamma_w(1)$, defines a smooth identification of the connected component of $\eta_v^{-1}(q)$ that contains w with the regular leaf $\mathfrak{L}_{\dot\gamma_w(1)}$ of $\mathfrak{F}_q^1$ through $\dot\gamma_w(1) \in \nu_q(L_{\gamma_v(1)})$, where $w \in \eta_v^{-1}(q)$ is any preimage of q. Moreover, due to the fact that a regular horizontal geodesic intersects the singular stratum discretely, we see that, for $v \in \nu(L)$ and a horizontal geodesic $\gamma'_{v'} : [0,1] \to M'$ with $Q \circ \gamma_v = Q' \circ \gamma'_{v'}$, their horizontal indices coincide, because the corresponding transversal Jacobi equations coincide along the regular parts. That is to say, the kernel $\ker((d\exp_M^\perp)_v)$ of the differential of the normal exponential map in v contains the subspace $T_v\mathcal{L}_v \subset T_v\nu(L)$ and the dimension of $\ker((d\exp_M^\perp)_v)/T_v\mathcal{L}_v$ is independent of the foliation, or to be more precise, an intrinsic datum of the quotient.

We now finish the proof as follows. Assume that $\mathcal{F}$ is taut. Combining Lemma 3.16 and the proof of Theorem 2.9, it remains to prove that for generic leaves L' of $\mathcal{F}'$ the normal exponential map $\exp_{M'}^\perp : \nu(L') \to M'$ has integrable fibers. Thus we can restrict our attention to a regular leaf $L' \in \mathcal{F}'$ without holonomy; i.e., $Q'(L')$ is a manifold point of B and the restriction of Q' to a tubular neighborhood of L' defines a Riemannian submersion. In particular, in this case the leaf $L = Q^{-1}(Q'(L')) \in \mathcal{F}$ is a regular leaf without holonomy, too. Let $v' \in \nu(L')$ be a horizontal vector and let us set $q' = \gamma_{v'}(1)$. Now, choose an $\mathcal{F}$-horizontal vector $v \in \nu(L)$ with $Q_*(v) = Q'_*(v')$ and set $q = \gamma_v(1)$. Then, by construction, $Q \circ \gamma_v = Q' \circ \gamma_{v'}$ and

$$\begin{aligned}&\dim(\ker((d\exp_{M'}^\perp)_{v'})) - (\dim(\mathcal{F}') - \dim(L_{\exp_{M'}^\perp(v')}))\\ &= \dim(\ker((d\exp_M^\perp)_v)) - (\dim(\mathcal{F}) - \dim(L_{\exp_M^\perp(v)})).\end{aligned}$$

Since $\mathcal{F}$ is taut, the connected component Δ_v of $(\exp_M^\perp)^{-1}(q)$ containing v is a compact submanifold of $\nu(L)$ that is smoothly foliated by the $(\dim(\mathcal{F}) - \dim(L_q))$-dimensional regular foliation whose leaf through a horizontal vector $w \in \Delta_v$ is given by $\mathcal{N}_w = (\exp_M^\perp)^{-1}(q) \cap \mathcal{L}_w$. Again, we can regard Δ_v as a saturated subset of the regular part of the singular Riemannian foliation $\mathfrak{F}_q^1$ on $\nu_q(L_q)$ via $d_v : \Delta_v \to \nu_q(L_q)$, defined by the prescription $d_v(w) = d(\exp_M^\perp)_w(w)$. Moreover, by our choice of L', the image of the composition $dQ_q \circ d_v$ is completely contained in the manifold part of $T_qM/\mathfrak{F}_q = \nu_q(L_q)/\mathfrak{F}_q^1$ [**LT10**, Sec. 4]), so that every leaf of $\mathfrak{F}_{q'}$ through a vector $w' \in \nu_{q'}(L'_{q'})$ with $dQ'_{q'}(w') \in dQ_q(d_v(\Delta_v))$ is also regular without holonomy.

If we therefore define $K_{q'} \subset \mathfrak{F}_{q'}^1$ to be the preimage

$$K_{q'} = (dQ'_{q'})^{-1}((dQ_q \circ d_v)(\Delta_v)),$$

then $K_{q'}$ is obviously a union of regular leaves of $\mathfrak{F}_{q'}^1$ without holonomy, namely of leaves of dimension $\dim(\mathcal{F}') - \dim(L'_{q'})$, completely contained in a concentric sphere. Further, because $dQ_q(d_v(\Delta_v))$ carries a natural smooth structure that turns it into a $(\dim(\Delta_v) - (\dim(\mathcal{F}) - \dim(L_q)))$-dimensional manifold, and the restriction of $dQ'_{q'}$ to the set of points lying on regular leaves without holonomy is a submersion, the set $K_{q'}$ is a compact submanifold of $\nu_{q'}(L'_{q'})$ of dimension equal to $\dim(\ker((d\exp^{\perp}_{M'})_{v'}))$.

Recall that the infinitesimal foliations are invariant under all non-zero homotheties. Thus, if we define $\Delta'_{v'} = \{-\dot{\gamma}_{w'}(1) \in \nu(L') | -w' \in K_{v'}\}$, it easily follows from our above discussion that $\Delta'_{v'}$ is a compact submanifold of $(\exp^{\perp}_{M'})^{-1}(q')$ containing v' whose tangent space satisfies $T_{w'}\Delta'_{v'} = \ker((d\exp^{\perp}_{M'})_{w'})$ for all $w' \in \Delta'_{v'}$. This proves the claim. q.e.d.

As already mentioned before, tautness of a submanifold $L \subset M$ requires very special symmetry of the pair (M, L) around the submanifold L, which clarifies the fact that there are not many examples of taut submanifolds actually known. By this reason, it is worth mentioning that the ideas of the last proof can be used to construct lots of examples. For this purpose, consider a closed singular Riemannian foliation $\mathcal{F}$ on M such that the space of leaves $M/\mathcal{F}$ is a good Riemannian orbifold N/Γ. Assume that there is a submanifold $S \subset N$ completely contained in the interior of a fundamental domain of the Γ-action, which we identify with $M/\mathcal{F}$, and consider the saturated preimage $T = Q^{-1}(S)$ that is a union of regular leaves without holonomy. Now let $v \in \nu_p(T)$ be a normal vector to T. Then every $\mathcal{F}$-vertical Jacobi field along γ_v (cf. Section 3.3) is also a T-Jacobi field along γ_v (i.e., $W^{\gamma_v} \subset \Lambda^T$) and similar arguments as in the proof of Theorem 3.19 can be used to see that the multiplicity of $Q_*(v)$ as a focal vector of S in N is the same as the difference of the multiplicity of v as a focal vector of T in M and the number $\dim(\mathcal{F}) - \dim(L_{\gamma_v(1)})$. Thus the following lemma is obtained, along the same lines as the proof of Theorem 3.19.

Lemma 3.24. *Let $\mathcal{F}$ be a closed singular Riemannian foliation on a complete Riemannian manifold M, such that the space of leaves $M/\mathcal{F}$ is isometric to a quotient N/Γ, where N is a Riemannian manifold and $\Gamma \subset \mathrm{Iso}(N)$ is a discrete group of isometries. Let $N_0 \subset N$ denote a fundamental domain of the Γ-action and identify $N_0 \cong M/\mathcal{F}$. Now assume that $S \subset N$ is a taut submanifold that is completely contained in the interior of N_0. Then if $Q : M \to N_0$ denotes the projection, the submanifold $Q^{-1}(S)$ is taut, too.*

In the case where $M = \mathbb{R}^{n+k}$ is the standard Euclidean space and $\mathcal{F}$ is an n-dimensional isoparametric foliation (i.e., the parallel foliation induced by an isoparametric submanifold L of dimension n) identify a section Σ with the Euclidean space $\mathbb{R}^k$. Then take a small taut submanifold $S \subset \mathbb{R}^k$ completely contained in the interior of a Weyl chamber associated to the finite Coxeter group generated by the reflections across the L-focal hyperplanes in Σ and consider the $\mathcal{F}$-saturated set

$T = \{p \in \mathbb{R}^{n+k} \mid L_p \cap S \neq \emptyset\}$. Then, due to the last lemma, T is a taut submanifold of $\mathbb{R}^{n+k}$.

We finally come to the refined version of Theorem 3.19.

Theorem 3.25. *Let $\mathcal{F}$ be a closed singular Riemannian foliation on a complete Riemannian manifold M. Then $M/\mathcal{F}$ is an orbifold and $\mathcal{F}$ is taut if and only if the quotient $M/\mathcal{F}$ is a good Riemannian orbifold with a pointwise taut universal covering orbifold.*

Let us assume that $M/\mathcal{F}$ is a good Riemannian orbifold. Then, by Theorem 3.19 the foliation $\mathcal{F}$ is taut if and only if the universal covering orbifold, that is a manifold in this case, is pointwise taut. Thus we prove Theorem 3.25 by the observation that in the orbifold case the quotient of a taut foliation is developable.

Lemma 3.26. *If $\mathcal{F}$ is a closed and taut singular Riemannian foliation on a complete Riemannian manifold M that has an orbifold quotient, then $M/\mathcal{F}$ is a good Riemannian orbifold.*

Proof. It is a well known fact that every Riemannian orbifold is the quotient of a regular Riemannian foliation. For instance, one could take the foliation $\widehat{\mathcal{F}}$ on the manifold $\widehat{M}$ of orthonormal frames of $M/\mathcal{F}$ induced by the almost free action of $O(k)$, where k is the dimension of the orbifold (cf. [**Hae84**]). By Theorem 3.19, $\mathcal{F}$ is taut if and only if $\widehat{\mathcal{F}}$ is taut and the latter is taut if and only if its lift $\widetilde{\mathcal{F}}$ to the universal covering is taut. Hence $\widetilde{\mathcal{F}}$ is taut, too. Since $\widetilde{M}$ is simply connected, $\widetilde{\mathcal{F}}$ has trivial holonomy (cf. Section 3.2) and therefore, $\widetilde{M}/\widetilde{\mathcal{F}}$ is a complete Riemannian manifold, which is also simply connected by the exact homotopy sequence. In particular, $\widetilde{M}/\widetilde{\mathcal{F}} \to M/\mathcal{F}$ is the universal orbifold covering. q.e.d.

The property that the quotient of a singular Riemannian foliation is a Riemannian orbifold can be described by means of the infinitesimal foliations. Namely, this class of foliations coincides with the class of singular Riemannian foliations whose infinitesimal foliations have sections. Let us recall that a singular Riemannian foliation $(M, g, \mathcal{F})$ admits *sections* if there exists a complete, immersed submanifold Σ_p through every regular point $p \in M$ that meets every leaf and always orthogonally. It is not hard to see that a section is totally geodesic in M. As an example, the set of orbits of a polar action is a singular Riemannian foliation admitting sections. Motivated by this example, we also speak about *polar foliations*. We end this section with a short recollection of the basic notions for those foliations and reformulate our result about foliations whose quotients are orbifolds.

Singular Riemannian foliation with sections are well understood and were studied, for example by Alexandrino and Töben. One nice feature of this class is that one can canonically construct a blow up that has the same horizontal geometry (cf. [**T06**]). In [**L10**] it is shown that the existence of such a *geometric resolution* of a singular Riemannian foliation is equivalent to the fact that the foliation carries at the infinitesimal level the information of a singular Riemannian foliation with sections (see Theorem 3.30). Such foliations are called *infinitesimally polar* and were first defined and discussed by Lytchak and Thorbergsson in [**LT10**].

Let us recall once more the notion of the infinitesimal foliation (cf. Section 3.2). Let $p \in M$ be a point and let B be a small open neighborhood in the leaf L_p through p. Then there is an $\varepsilon > 0$ and a distinguished tubular neighborhood (U, B, π) around p such that there is an embedding ϕ of U into the tangent spaces T_pM with $D_p\phi = \mathrm{Id}$ and a singular Riemannian foliation $\mathfrak{F}_p$ on T_pM, which we called the infinitesimal singular Riemannian foliation of $\mathcal{F}$ at the point p that coincides with $\phi_*\mathcal{F}$ on $\phi(U)$ and such that $\mathfrak{F}_p$ is invariant under all rescalings $r_\lambda : T_pM \to T_pM, r_\lambda(v) = \lambda v$, for all $\lambda \neq 0$.

One can consider $\mathfrak{F}_p$ as the blow up of $\mathcal{F}$ in the following sense. Identify U with $\phi(U)$. Set $U^\lambda = r_\lambda(U)$ for $\lambda > 0$. So $\bigcup_{\lambda>0} U^\lambda = T_pM$. Define the Riemannian metric g^λ on U^λ as $g^\lambda = \lambda^2 (r_\lambda)_* g$. Then the blow up metrics g^λ smoothly converge to the flat metric g_p. By construction, the restriction of $\mathfrak{F}_p$ to U^λ is a singular Riemannian foliation with respect to g^λ. Moreover, if $\dim(L_p) = r$, then the infinitesimal singular foliation $\mathfrak{F}_p$ on $T_pM = T_pM_{n-r} \times (T_pM_{n-r})^\perp$ is a product $\mathfrak{F}_p^v \times \mathfrak{F}_p^h$, where $\mathfrak{F}_p^v$ is the trivial foliation given by parallels of T_pL_p and the main part $\mathfrak{F}_p^h$ on $(T_pM_{n-r})^\perp$ is a singular Riemannian foliation, invariant under rescalings and with the origin as the only 0-dimensional leaf. Thus $\mathfrak{F}_p^h$ is the cone over a foliation of dimension $n-r$ on the unit sphere of $T_pM_{n-r}^\perp$, which is induced by the intersections of the nearby higher dimensional leaves with a slice through p. In particular, the foliation $\mathfrak{F}_p$ is polar if and only if its factor $\mathfrak{F}_p^h$ is polar.

Example. Let (M^m, g) be a complete, simply connected Riemannian manifold with a closed singular Riemannian foliation $\mathcal{F}$ of codimension@2 and $M/\mathcal{F} = S^2/\Gamma$. Then $\mathcal{F}$ is infinitesimally polar and Γ is a finite Coxeter group. Further, $\mathcal{F}$ is taut and therefore has no exceptional leaves. Let $L \in M/\mathcal{F}$ be a point of codimension 2; i.e., a corner. Take a point $p \in L^{m-k}$ and consider the infinitesimal singular Riemannian foliation $\mathfrak{F}_p^h$ on $(\nu_p(L), g_p)$. Since L has codimension 2 in $M/\mathcal{F}$, the singular Riemannian foliation $\mathfrak{F}_p^h$ is the cone foliation over a singular Riemannian foliation of codimension 1 on the unit sphere S^{k-1} in $\nu_p(L)$. By a result of Münzner (see [**Mü80**], [**Mü81**]), one therefore has $S^{k-1}/\mathfrak{F}_p^h = I_d$ for an interval I_d with length $|I_d| = \pi/d$ and $d \in \{1, 2, 3, 4, 6\}$; i.e., $\nu_p(L)/\mathfrak{F}_p^h$ is an open cone over I_d with angle π/d. Note that to obtain a local isometry between $\nu_p(L)/\mathfrak{F}_p^h$ and $U/\mathcal{F}$ for a neighborhood U around p, indeed one has to change the metric on $\nu_p(L)$, but this has no influence on the possible values of the angle, because the metrics coincide in 0. Now the finite-to-one mapping $U/\mathcal{F} \to M/\mathcal{F}$ between the local and global quotient is given by the quotient $(U/\mathcal{F})/W$, where W is a group acting on U by isometries. But the absence of exceptional leaves implies that W acts trivially, so that a neighborhood of L in $M/\mathcal{F}$ is isometric to $U/\mathcal{F}$. It follows by the known classification of S^2/Γ that the quotient $M/\mathcal{F}$ is either the whole sphere S^2, the hemisphere $S^2/\mathbb{Z}_2$, a sickle S^2/D_i with $i \in \{2,3,4,6\}$, or a spherical triangle with angles $(\pi/n_1, \pi/n_2, \pi/n_3)$ and $(n_1,n_2,n_3) \in \{(2,2,2),(2,2,3),(2,2,4),(2,2,6),(2,3,3),(2,3,4)\}$

Obviously, if $\iota : \Sigma \to M$ is a section of $\mathcal{F}$ then $d\iota_p(T_p\Sigma)$ is a section of $\mathfrak{F}_{\iota(p)}$. Thus a singular Riemannian foliation with sections is infinitesimally polar. Conversely, in the general case a section Σ of $\mathfrak{F}_p$ cannot be realized as the tangent space of a local section, because this is equivalent to the fact that the horizontal distribution given by $\mathcal{H} = \bigcup_{p\in M_0}(T_pL_p)^{\perp_{g_p}}$ over the regular stratum is integrable, what, under the assumption of completeness of M, is equivalent to existence of sections. A well known example of an infinitesimally polar singular Riemannian foliation that is not polar is given by the fibers of the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2(\frac{1}{2})$.

In [**LT10**] Lytchak and Thorbergsson proved the following:

Theorem 3.27. *Let $\mathcal{F}$ be a singular Riemannian foliation on a Riemannian manifold M. Let $p \in M$ be a point and let $\mathfrak{F}_p$ be the infinitesimal singular Riemannian foliation induced by $\mathcal{F}$ on the tangent space T_pM. Then the following are equivalent:*

1) *The infinitesimal singular Riemannian foliation $\mathfrak{F}_p$ is polar;*
2) *$\mathcal{F}$ is locally closed at p and a local quotient $U/\mathcal{F}$ of a neighborhood U of p is a Riemannian orbifold.*

In fact, in [**LT10**] it is shown that the statements above are equivalent to the non-explosion of the curvature in the local quotients as one approaches a boundary point p of M_0.

Using the description of Lytchak and Thorbergsson and Theorem 3.25 we obtain

Corollary 3.28. *Let $\mathcal{F}$ be a closed singular Riemannian foliation on a complete Riemannian manifold M. Then $\mathcal{F}$ is infinitesimally polar and taut if and only if the quotient $M/\mathcal{F}$ is a good Riemannian orbifold with a pointwise taut universal covering orbifold.*

If the infinitesimal singular Riemannian foliation $\mathfrak{F}_p$ on (T_pM, g_p) is polar, $\mathfrak{F}_p$ is an isoparametric foliation given by the parallel foliation induced by a regular and hence isoparametric leaf. Since isoparametric foliations on a Euclidean space are taut (see [**HPT88**]), the infinitesimal foliation $\mathfrak{F}_p$ is taut in this case. The next lemma states that this is always true.

Lemma 3.29. *Let $\mathcal{F}$ be a closed singular Riemannian foliation on a complete Riemannian manifold (M, g). If $\mathcal{F}$ is taut, then for every $p \in M$, the infinitesimal foliation $\mathfrak{F}_p$ on (T_pM, g_p) is taut.*

Proof. We will use the notation from the beginning of this section. Take a point $p \in M$ with $d = \dim(L_p)$ and $k = \operatorname{codim}(M_{n-d})$. Then we have an orthogonal splitting $T_pM = T_pM_{n-d} \oplus \nu_p(M_{n-d})$ that induces a splitting of $\mathfrak{F}_p$ into a product foliation $\mathfrak{F}_p = \mathfrak{F}_p^v \times \mathfrak{F}_p^h$, where the first factor is the foliation given by parallels of T_pL_p and the main part $\mathfrak{F}_p^h$ is the cone-foliation (i.e., invariant under all homotheties $r_\lambda(v) = \lambda \cdot v$) of a singular Riemannian foliation with compact leaves of dimension at least one on the unit sphere in $\nu_p(M_{n-d})$ (if $\nu_p(M_{n-d}) \neq \{0\}$). Since the path spaces of these product submanifolds are the products of the corresponding path spaces in the factors, and the critical points are exactly the tuples of critical points in these factors, so that the critical data behave additive, we conclude that $\mathfrak{F}_p$ is taut if and only if $\mathfrak{F}_p^h$ is taut, because T_pL_p is contractible and $\mathfrak{F}_p^v$ is the trivial foliation by parallels of T_pL_p. Further, because $\mathfrak{F}_p$ as well as the set of straight lines in T_pM, is invariant under homotheties, it follows that $\mathfrak{F}_p$ is taut if and only if the restricted foliation $\mathfrak{F}_p^h|_D$ is taut, where D is a small ball in $\nu_p(M_{n-d})$ around the origin. Note that such a ball D is always saturated. Let U be a distinguished tubular neighborhood around p and set $V = \phi(U)$ and $h = \phi_*\, g$, where $\phi : U \to T_pM$ with $\phi_*(\mathcal{F}|_U) = \mathfrak{F}_p|_V$ is an embedding as in the definition of the infinitesimal foliation at p. Now with respect to the metric h and for a small ball D around the origin in $\nu_p(M_{n-d})$, the closed singular Riemannian foliation $\mathfrak{F}_p|_{(T_pM_{n-d} \times D) \cap \phi(U')}$ is taut (i.e., the saturation of $\mathcal{F}|_{\phi^{-1}(D)}$ in U' is taut) where $U' \subset U$ is a smaller distinguished tubular neighborhood at p that contains $\phi^{-1}(D)$. To see this, we can choose U' so small that we have to consider only critical points γ in U' with energy r such that the whole ball of radius r around $\gamma(1)$ is contained in U, so that we have

$$P(U, L_{\gamma(0)} \cap U \times \gamma(1))^r = P(M, L_{\gamma(0)} \times \gamma(1))^r$$

and conclude by tautness of $\mathcal{F}$ that all the local unstable manifolds can be completed in U below the energy r. Thus, since U and U' can be assumed to be diffeomorphic, the local unstable manifolds can also be completed in U', which implies our claim. If we now consider the blow up metrics h^λ on $V^\lambda = \{\lambda v | v \in V\}$ as defined above; i.e.,

$$h^\lambda = \lambda^2 \cdot (r_\lambda)_*\, h,$$

it follows that our restricted foliation is also taut with respect to the metrics h^λ. But the constant metric g_p is just the flat limit $\lim_{\lambda \to \infty} h^\lambda$ and we deduce that $\mathfrak{F}_p$ is taut with respect to g_p, because it is not hard to see that if a sequence of perfect Morse functions converge to a Morse function, this limit has to be perfect. This together with our genericity results from Section 3.2 finish the proof. q.e.d.

In the standard picture of an isometric action of a Lie group G on a Riemannian manifold M, one could ask if there exists another Riemannian manifold $(\widehat{M}, \hat{g})$, canonically related to M, on which G acts by isometries in such a way that all orbits of this action have the same dimension, because the singular leaves are the main source of difficulties if one tries to understand the geometric and topological properties of a foliation. If one additionally tries to resolve the action in a way that preserves the horizontal geometry, there was no such general construction known before the work of Lytchak [**L10**], who came up with a canonical resolution preserving the transverse geometry.

As the main result in [**L10**], Lytchak gave an equivalent characterization of the infinitesimally polar foliations as exactly those foliations that admit a *geometric resolution*, where he defined a geometric resolution of $(M, g, \mathcal{F})$ to be a smooth and surjective map $F : \widehat{M} \to M$ from a Riemannian manifold $(\widehat{M}, \hat{g})$ with a regular foliation $\widehat{\mathcal{F}}$ such that the following holds true: For all smooth curves c in $\widehat{M}$ the transverse lengths of c with respect to $\widehat{\mathcal{F}}$ and of $F(c)$ with respect to $\mathcal{F}$ coincide. The transverse length of a smooth curve $c : [a, b] \to M$ is defined as the length of the projection to local quotients

$$L_T(c) = \int_a^b \|P_{c(t)}(\dot{c}(t))\| dt,$$

where $P_q : T_qM \to (T_qL_q)^\perp =: H_q$ denotes the orthogonal projection. In particular, a map F, as in the definition of a geometric resolution, sends leaves of $\widehat{\mathcal{F}}$ to leaves of $\mathcal{F}$ and induces a length preserving map between the quotients. In general, a map between foliated manifolds that maps leaves to leaves is called foliated.

In [**L10**] Lytchak proved

Theorem 3.30. *Let (M, g) be a Riemannian manifold and let $\mathcal{F}$ be a singular Riemannian foliation on M. Then $(M, \mathcal{F})$ has a geometric resolution if and only if $\mathcal{F}$ is infintesimally polar. If $\mathcal{F}$ is infinitesimally polar, then there is a canonical resolution $F : \widehat{M} \to M$ with the following properties*

1) $\dim(\widehat{M}) = \dim(M)$*;*
2) F *induces an isometry between the spaces of leaves;*
3) $F|_{F^{-1}(M_0)} : F^{-1}(M_0) \to M_0$ *is a diffeomorphism;*
4) F *is proper and* 1*-Lipschitz.*

In particular, the resolution $\widehat{M}$ is compact or complete if M has the corresponding property. The isometry group Γ of $(M, \mathcal{F})$ acts by isometries on $(\widehat{M}, \widehat{\mathcal{F}})$ and the map $F : \widehat{M} \to M$ is Γ-equivariant. If $\mathcal{F}$ is given by *the orbits of a group G of isometries of M, then G acts by isometries on $\widehat{M}$, and $\widehat{\mathcal{F}}$ is given by the orbits of G. If M is complete, then the singular Riemannian foliation $\mathcal{F}$ has no horizontal conjugate points if and only if $\widehat{\mathcal{F}}$ has no horizontal conjugate points and $\mathcal{F}$ has sections if and only if $\widehat{\mathcal{F}}$ has sections.*

For the notion of horizontal conjugate points see Section 3.3.

Let us now say some words about his canonical resolution. Recall that the *Grassmannian bundle* $\mathfrak{G}_k(M)$ of a given manifold M^{n+k} consists fiberwise of the Grassmaniann manifolds

$$G_k(T_pM) = \{\sigma \subset T_pM \mid \sigma \text{ is a } k\text{-plane}\}$$

of the k-dimensional linear subspaces of the tangent space T_pM; that is to say, $\mathfrak{G}_k(M) = \bigcup_{p\in M} G_k(T_pM)$. For a detailed discussion of the Grassmanian bundle with its natural metric we refer the reader to [**Wie08**].

For a singular Riemannian foliation $\mathcal{F}$ of codimension k on M that has sections, Boulam defined in [**Bou95**] the set

$$\widehat{M}' = \{T_p\Sigma \mid \Sigma \text{ is a section through } p\}$$

of the Grassmannian bundle. Let $P : \widehat{M}' \to M$ denote the restriction of the canonical map $\mathfrak{G}_k(M) \to M$. Boualem constructed a differentiable structure on $\widehat{M}'$ and showed that there is some Riemannian metric on N such that the lifted partition $\widehat{\mathcal{F}}' = \left\{P^{-1}(L) \mid L \in \mathcal{F}\right\}$ becomes a regular Riemannian foliation on $\widehat{M}'$. The foliation $\widehat{\mathcal{F}}'$ is called the *blow up of* $\mathcal{F}$.

In [**T06**] Töben proved this result again with another technique and gives the following amplification: If we denote by h the natural Riemannian metric on $\mathfrak{G}_k(M)$ and by $\hat{g}' = \iota^* h$ the pull back on $\widehat{M}'$, then the pair $(\widehat{\mathcal{F}}', \widehat{\mathcal{F}}'^{\perp})$ is a bi-foliation on $\widehat{M}'$ with a Riemannian foliation $\widehat{\mathcal{F}}'$ and totally geodesic foliation $\widehat{\mathcal{F}}'^{\perp}$.

Therefore, in some sense, the sections of $\mathcal{F}$ play the role of a global benchmark and give rise to a resolution of the singularities. In [**L10**] Lytchak generalized this construction, replacing $\widehat{M}'$ by

$$\widehat{M} = \{\Sigma \subset T_pM \mid \Sigma \text{ is a section of } \mathfrak{F}_p \text{ through } 0\}$$

to infinitesimally polar singular Riemannian foliations. There is a unique Riemannian metric $\widehat{g}$ on $\widehat{M}$ such that its restriction to $T\widehat{\mathcal{F}} = T((P|_{\widehat{M}})^*\mathcal{F})$ coincides with the restriction of the canonical metric on the Grassmannian bundle and $\nu_\Sigma(\widehat{L}_\Sigma)$ is isometrically identified with Σ, where Σ is a section of $\mathfrak{F}_{P(\Sigma)}$. If we denote the restriction $P|_{\widehat{M}}$ by F, then $F : (\widehat{M}, \widehat{F}, \widehat{g}) \to (M, \mathcal{F}, g)$ is the canonical geometric resolution. Thus what is really needed for such a resolution is just the infinitesimal geometric information of a singular Riemannian foliation with sections, but not the actual existence of sections.

Using Theorem 3.30 and Theorem 3.19 we therefore obtain

Corollary 3.31. *Let $\mathcal{F}$ be a closed and infinitesimally polar singular Riemannian foliation on a complete Riemannian manifold M and let $(\widehat{M}, \widehat{\mathcal{F}})$ denote the canonical geometric resolution. Then $\widehat{\mathcal{F}}$ is taut if and only if $\mathcal{F}$ is taut.*

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