

EINSTEIN TYPE METRICS AND STABILITY ON VECTOR BUNDLES

NAICHUNG CONAN LEUNG

Abstract

In this paper we show that stability for holomorphic vector bundles are equivalent to the existence of solutions to certain system of Monge Ampère equations parametrized by a parameter k . We solve this fully nonlinear elliptic system by singular perturbation technique and show that the vanishing of obstructions for the perturbation is given precisely by the stability condition. This can be interpreted as an infinite dimensional analog of the equivalency between Geometric Invariant Theory and Symplectic Reduction for moduli space of vector bundles.

1. Introduction

This paper is largely grown out from the thesis [7] of the author under the direction of Professor S.T.Yau. However, a more concrete picture in terms of the infinite dimensional Geometric Invariant Theory (GIT) and the symplectic reduction will also be presented here.

We shall demonstrate that when we use GIT to study the moduli problem of vector bundles (following Gieseker), it is equivalent to finding certain canonical Einstein type metrics on the bundle E . The curvature of such metrics satisfies a fully nonlinear elliptic system of equations arised as moment map equations (the almost Hermitian Einstein equations):

$$[e^{\frac{i}{2\pi}R_A+k\omega}I_E Td(X)]^{(2n)} = \frac{1}{rk(E)}\chi(X, E \otimes L^k)\frac{\omega^n}{n!}I_E.$$

Received March 16, 1995.

Theorem 1 (Main Theorem). *Let E be an irreducible sufficiently smooth holomorphic vector bundle over a compact Kähler manifold X . Then E is Gieseker stable if and only if there exists an almost Hermitian Einstein metric on E .*

The proof of this theorem will be treated in the last section using a singular perturbation arguments. There, we start with a singular solution (corresponding to $k = \infty$) and attempt to perturb it and solve the equation for large k . We find that the obstruction for such a perturbation is precisely captured by the (Gieseker) stability behaviors of the bundle E .

In section two and three, we shall explain the source which makes these equations appear naturally. In finite dimension, GIT and symplectic quotient are closely related by a theorem of Kempf and Ness [5]. However, when we try to apply GIT to study moduli problem, we have to face the difficulty that the dimension of the space in question need not be bounded.

In our present case, we are interested in the moduli problem of holomorphic vector bundles and the GIT is worked out by Gieseker [4] for algebraic surfaces. We need to characterize the bundle E using its global holomorphic sections. To generate enough global holomorphic sections, we tensor E with high power k of a fix polarization L over X . Intuitively, the corresponding symplectic reduction theory would be on the space of holomorphic maps from X to a Grassmannian $Gr(r, N(k))$ where $N(k)$ goes to infinity together with k . On the space of all such maps, there is a canonical symplectic structure.

Instead of looking at $Map(X, Gr(r, N))$, we shall look at the space of connections on E , \mathcal{A} . These two spaces are closely related in the sense that any map will induce a connection on E by pulling back the Universal connection on the Universal bundle \mathbb{E} over $Gr(r, N(k))$. This map induces a homotopic equivalency between these two spaces in the limit as k goes to infinity. We will compare these spaces in more details in future. This map also induces a symplectic structure on \mathcal{A} as follows:

$$\Omega_k(D_A)(B, C) = \int_X Tr_E [B \wedge e^{(k\omega I_E + \frac{i}{2\pi} R_A)} \wedge C]_{sym} Td(X).$$

When we let k goes to infinity, this symplectic structure will be simplified to

$$\Omega(D_A)(B, C) = \int_X Tr BC \wedge \frac{\omega^{n-1}}{(n-1)!}.$$

More details and higher degree generalizations of such symplectic structures can be found in [8].

In section three, we then discuss the corresponding moment maps (the almost Hermitian Einstein equations) and symplectic quotients. By letting k goes to infinity, the moment map equations will be linearized and we recover the Hermitian Einstein equation:

$$\bigwedge R_A = \mu_E I_E.$$

The linearized theory is well understood by the work of Narasihan and Seshadri [12] in dimension one, Donaldson [3] in dimension two, and Uhlenbeck and Yau [14] in general. They showed that irreducible Hermitian Einstein bundles are equivalent to Mumford stable bundles. In our language, their result is interpreted as saying that in the limit where we let k equals infinity, the symplectic reduction theory does correspond to the 'limit' of the GIT for vector bundles over Kähler manifolds. The result in Uhlenbeck and Yau's paper is very important for us in constructing the singular solution for the perturbation arguments and also important in proving the existence of perturbation. In the last part of section three, we prove the (Gieseker) stability property for almost Hermitian Einstein bundles.

In the last section, we will solve the almost Hermitian Einstein equations on (Gieseker) stable bundles and prove the main theorem. The idea of how to solve it will be explained in the beginning of that section and we are not going to repeat them here. However, this gives us the infinite dimensional version of the equivalency between GIT for moduli of holomorphic vector bundles and symplectic reduction in Gauge theory. To put it another way, we find canonical Einstein type metrics on (Gieseker) stable bundles.

2. Infinite dimension GIT and symplectic reduction

In this chapter, we shall explain the relationship between geometric invariant theory and symplectic reduction. Our prime interest is the moduli space of holomorphic vector bundles. In our setting, both our space and the group which it acts on are of infinite dimensional. However, it has a finite dimensional approximation. We shall interpret the notion of Gieseker stability as an asymptotics stability of this approximation and the notion of Mumford stability as 'limiting' stability. We shall justify this picture which we portray here by showing the existence

of certain Einstein type metrics on (Gieseker) stable bundles.

We first recall basic results concerning the relationship between the Geometric Invariant Theory and the symplectic reduction. Readers can find more details from ([11]). Let X be a symplectic manifold of dimension $2n$ with symplectic form ω , and G be a compact Lie group which acts on X symplectically. Then there is a well known procedure to produce a quotient space within the symplectic category called the symplectic quotient provided that the G action is a Hamiltonian action. By an action to be Hamiltonian we mean that a moment map exists. That is,

$$\mu : X \rightarrow \mathfrak{g}^*,$$

where μ is G -equivariant with respect to the G action on X and the coadjoint action of G on the dual \mathfrak{g}^* of the Lie algebra of G , and μ satisfies

$$\omega(v, X_\phi) = d\mu(v)(\phi),$$

where v is a tangent vector on X , ϕ is an element of \mathfrak{g} and X_ϕ is the vector field generated by ϕ via the group action.

The construction of the symplectic quotient of X by G is the usual quotient space of G acting on $\mu^{-1}(0)$, which we denote by $X//G = \mu^{-1}(0)/G$. On the symplectic quotient space, there is a natural symplectic structure induced from ω . Notice that the dimension of the symplectic quotient is $\dim X - 2\dim G$ provided that 0 is a regular value of the moment map. In general, we can replace zero by other coadjoint orbits in the construction of symplectic quotient.

Now, suppose that X is a projective manifold inside a complex projective space $\mathbb{C}P^N$ and the G action on X factor through projective transformations of $\mathbb{C}P^N$. That is $G \rightarrow \mathbb{P}GL(N + 1, \mathbb{C})$. We can assume the complexification $G^{\mathbb{C}}$ of G acts on X via holomorphic transformations. Then one can identify the geometric invariant theory (GIT) quotient and the symplectic quotient. The GIT quotient space is certain equivalent classes of semi-stable points in X with respect to the $G^{\mathbb{C}}$ action. A theorem of Kempf and Ness states that the two quotients are the same.

Theorem 2. *A point $x \in X$ is semistable if and only if*

$$\overline{O_G(x)} \cap \mu^{-1}(0) \neq \emptyset.$$

Moreover, the inclusion from $\mu^{-1}(0)$ to the set of semi-stable points X^{ss} induces a homeomorphism from the symplectic quotient $X//G$ to the GIT quotient.

STABLE BUNDLES

We want to apply the Geometric Invariant Theory to construct a moduli space of holomorphic vector bundles over X . First of all, we know that for any holomorphic bundle E , there exists a constant k_0 (depending on E), such that, for all $k > k_0$, we have

- (i) $H^i(X, E \otimes L^k) = 0$, for all $i > 0$,
- (ii) $E \otimes L^k$ is generated by global sections.

Suppose that we have a family \mathcal{F} of holomorphic vector bundles (with fix Hilbert polynomial, $\chi(X, E \otimes L^k)$) such that one can find a uniform k_0 , so that for any $k > k_0$, (i) and (ii) work for each member $E \in \mathcal{F}$. Then, after tensoring with high power of L , each E can be determined by the behavior of its global sections. In other words, we hope to get a 'holomorphic' embedding of \mathcal{F} to the Grassmannian. We denote r to be the rank of these bundles and consider the following natural morphism:

$$\bigwedge^r H^0(X, E \otimes L^k) \rightarrow H^0(X, \det E \otimes L^{rk}).$$

For simplicity, we consider those bundles with a fix determinant line bundle M , $\bigwedge^r E = \det E = M$ for some fix line bundle M on X . Then $H^0(X, \det E \otimes L^{rk})$ becomes a fix vector space; we call it W . By (i) and the Riemann Roch theorem, the dimension of $H^0(X, E \otimes L^k)$ is also constant. If we 'pick' an identification of $H^0(X, E \otimes L^k)$ to some fix vector space V of that dimension, then we have a homomorphism corresponding to each E :

$$\bigwedge^r V \rightarrow W.$$

The upshot is that this map determines the holomorphic bundle E up to a choice of the base for V . So $G^{\mathbb{C}} = SL(V)$ acts on $Hom(\bigwedge^r V, W)$ and the quotient, if exists, would be the moduli space for the family \mathcal{F} .

In order to interpret stable point in $Hom(\bigwedge^r V, W)$ in terms of the bundle informations, Gieseker ([4]) introduced his definition of stability action and proved that a holomorphic bundle being Gieseker stable is the same as the corresponding homomorphism $\bigwedge^r V \rightarrow W$ being a stable point under $G^{\mathbb{C}} = SL(V)$. He also proved that in the case of a projective surface, such bundles form a bounded family which implies the existence

of a uniform k_0 that makes (i) and (ii) work for all Gieseker semi-stable bundles (with a fix Hilbert polynomial).

Definition 1 (Gieseker Stability) ([4]). Let E be a rank r holomorphic vector bundle (or coherent torsion-free sheaf in general) over a projective variety X with ample line bundle L . E is called Gieseker stable if for any nontrivial coherent subsheaf S of E , we have

$$\frac{\chi(X, S \otimes L^k)}{\text{rank}S} < \frac{\chi(X, E \otimes L^k)}{\text{rank}E}$$

for large enough k .

E is called Gieseker semi-stable if

$$\frac{\chi(X, S \otimes L^k)}{\text{rank}S} \leq \frac{\chi(X, E \otimes L^k)}{\text{rank}E}$$

for large enough k .

Notice that a nontrivial subsheaf S here is assumed to have rank strictly between 0 and r .

Theorem 2 (Gieseker) ([4]). *Let X be a projective surface. Then the moduli space of Gieseker stable torsion-free coherent sheaves with a fix Hilbert polynomial exists as a quasi-projective variety.*

Moreover, the moduli space of the equivalent classes of Gieseker semi-stable torsion-free coherent sheaves with a fix Hilbert polynomial exists as a projective variety.

We will explain the equivalent relation later in this chapter.

Before the works of Gieseker, there has been a lot of works by many people on trying to construct the moduli space for vector bundles over a curve or complex projective spaces. Among them are Weil, Mumford, Narasimhan, Ramanan, Seshadri, Barth, Horrocks, Hartshorne and many others. For the surface case, Maruyama [10] also proved the above theorem.

In case the base manifold is a curve, Mumford defined stability and generalized it to higher dimensional case as follow (before Gieseker) :

Definition 2 (Mumford Stability) . Let E be a rank r holomorphic vector bundle over a projective variety X with ample line bundle L . E is called Mumford stable if for any nontrivial coherent subsheaf S of E , we have

$$\mu_S < \mu_E.$$

E is called Mumford semi-stable if

$$\mu_S \leq \mu_E,$$

where μ_E is called the slope of E and defined as

$$\mu_E = \frac{(c_1(E) \cdot c_1(L)^{n-1}) \cdot [X]}{\text{rank} E}.$$

To define μ_S for a torsion free coherent sheaf which is not necessary locally free, we need to make sense of $c_1(S)$. One way to do this is to define $c_1(S)$ as the first Chern class of the double dual of the determinant of S .

Remark. Using Mumford's Geometric Invariant Theory, Seshadri proved that over a curve, Mumford stable bundles form a quasi-projective moduli space with a canonical compactification by equivalent classes of Mumford semi-stable bundles. It can be checked that Mumford stability and Gieseker stability are the same over a curve. For higher dimensional base manifold, Mumford stable implies Gieseker stable and Gieseker semi-stable implies Mumford semi-stable.

The following standard theorem tells us that each Mumford semi-stable bundle is built from Mumford stable bundle in an essentially unique way. In particular, a Gieseker stable bundle can also be decomposed into Mumford stable bundles. This theorem is important in our proof for the existence of almost Hermitian Einstein metric on a Gieseker stable bundle. It is also useful in defining the equivalent relation for semi-stable bundles.

Theorem 3 (Jordan-Hölder Theorem). *If E is a Mumford semi-stable torsion-free coherent sheaf over X , there exists a filtration of E by torsion free subsheaves E_i 's*

$$E = E_0 \supset E_1 \supset E_2 \supset \dots \supset E_{k+1} = 0$$

such that $Q_j = E_j/E_{j+1}$ is Mumford stable and $\mu(Q_j) = \mu(E)$ for each j . Moreover,

$$\text{Gr}(E) = Q_1 \oplus Q_2 \oplus \dots \oplus Q_k$$

is uniquely determined by E up to isomorphism.

A semi-stable torsion-free coherent sheaf is called sufficiently smooth if $\text{Gr}(E)$ is locally free. In the case where E is Mumford stable, then sufficiently smoothness is the same as E being a smooth vector bundle

since in that case the Jordan-Hölder filtration consists of only one term, namely E itself. But in the semi-stable category, sufficiently smoothness is the natural analog for smoothness for stable object.

Now, we can explain the equivalent relation among semi-stable bundles. Two semi-stable bundles are said to be equivalent if their corresponding graded sheaves $Gr(E)$ in their Jordan-Hölder theorem are isomorphic. Notice that this equivalent relationship on the set of Mumford stable bundles is simply the equivalent relation of isomorphism classes. The second part of Gieseker's theorem states that the set of equivalent classes of Gieseker semi-stable torsion-free coherent sheaves carries a natural projective-algebraic structure.

Now, we would like to define these stabilities over a compact Kähler manifold X which is not necessarily projective. For Mumford stability, we shall replace $c_1(L)$ by the cohomology class of ω in the defining inequality. For Gieseker stability, we have used Hilbert polynomial in its definition. By Riemann-Roch theorem and Chern-Weil theory, it can be expressed in terms of characteristic classes of E and L which, in turn, can be computed by curvature of E and L . Therefore, only the curvature of L is used instead of L itself. Up to a constant, the curvature of L is our Kähler class, therefore, we can define the notion of stability even though the Kähler metric ω may not be an integral form. For simplicity, we shall continue to write ω as the first Chern form of a positive line bundle. However, when E is singular, we would have problems in defining its curvature and we should continue to use the Chern classes expression for χ .

When E is a torsion-free coherent sheaf on X , its Chern character is defined to be the alternating sum of Chern characters of a locally free resolution of it. Notice that the Chern character defines in this way is independent of the choice of the resolution of E .

SYMPLECTIC STRUCTURES ON SPACE OF CONNECTIONS

Now, we want to study this moduli space of vector bundles problem in the realm of symplectic geometry. From the Gieseker's considerations, we shall introduce a symplectic structure on the space of connections \mathfrak{A} for each k . They are all invariant with respect to the Gauge group action.

When k gets large, the finite dimensional GIT picture considered by Gieseker will approximate an infinite dimensional theory described by the space of connections and the gauge group action. Since we study the moduli problem by looking at large k behavior, we shall call such a the-

ory asymptotics GIT. (More careful treatment about the approximation will be treated in a future paper.) Instead, we shall study the infinite dimensional GIT/symplectic quotient. In the following sections, we will demonstrate that the asymptotics GIT and the infinite dimensional symplectic quotient corresponds to each other in the case of moduli space of vector bundles.

First we shall use twisted transgression ([8]) to introduce a symplectic form on the space of connections \mathfrak{A} for each k .

Let us recall that after we tensor E with high enough powers of L (the ample line bundle over X), E can be described using its global holomorphic sections, $H^0(X, E \otimes L^k)$. This is similar to getting a holomorphic map from X to an Grassmannian $Gr(r, N(k))$. Here r is the rank of E , and $N(k)$ goes to infinity as k goes to infinity. Over the Grassmannian, there is a canonical rank r vector bundle, the Universal bundle \mathbb{E} which gives E by pulling back \mathbb{E} to X via the above holomorphic map.

There is canonical map from $Map(X, Gr(r, N))$ to the space of connections over X by pulling back the Universal connection \mathbb{D} on \mathbb{E} to E :

$$\Phi : Map(X, Gr(r, N)) \longrightarrow \mathfrak{A}.$$

Moreover, this map induces homotopic equivalence between these two spaces as N goes to infinity (which happens as k goes to infinity). By letting N equal infinity, this map is in fact surjective. We will first introduce a symplectic structure on $Map(X, Gr(r, N))$ by twisted transgression and induces symplectic structures on the space of connections over X .

First, we look at evaluation map :

$$X \times Map(X, Gr(r, N)) \longrightarrow Gr(r, N).$$

Over the Grassmannian, we have a canonical Chern character form $ch(\mathbb{E}, \mathbb{D})$ defined using the Universal connection over the Universal bundle \mathbb{E} . Then we define the twisted transgression of $ch(\mathbb{E}, \mathbb{D})$ ([8]) with respect to the harmonic Todd form $Td(X)$ on X by

$$\Omega^{[*]} = \Pi_{2*}(ev^* ch(\mathbb{E}, \mathbb{D}) \wedge Td(X)),$$

where Π_2 is the projection to the second factor $Map(X, Gr(r, N))$. $\Omega^{[*]}$ is an even degree differential form on $Map(X, Gr(r, N))$.

The degree two part $\Omega^{[2]}$ of this differential form corresponds to the symplectic approach to the Gieseker GIT theory for moduli space of

bundles. $\Omega^{[2]}$ induces a symplectic form (which can be degenerated for a finite k) Ω on the space of connections so that Φ preserves these two symplectic structures.

We shall now describe Ω on \mathfrak{A} with a fix k (we call such a two-form by Ω_k):

$$\Omega_k(D_A)(B, C) = \int_X Tr_E[B \wedge e^{(k\omega I_E + \frac{i}{2\pi} R_A)} \wedge C]_{sym} Td(X),$$

where D_A is a connection on E , and B, C are tangent vectors of \mathfrak{A} at D_A which can be identified as $End(E)$ valued one-forms on X , and R_A is the curvature tensor of the connection D_A . In addition, we have to take anti-symmetrized product of those terms inside the integral sign (see [8] for more details). It is a close differential form and non-degenerate at any D_A for large enough k . It is because when k is large enough, the term $k\omega I_E$ will dominate the curvature term. Moreover, the Gauge group acts symplectically on \mathfrak{A} with respect to any one of these symplectic structures.

In [8], the author also extends it to higher degree forms (to study family situations) and relates these forms to family index theorems and their equivariant extensions with respect to the gauge group action. In this paper, we are only interested in holomorphic maps and connections defining holomorphic structures on E .

When we let the parameter k goes to infinity, we will obtain the standard symplectic form Ω on $\mathfrak{A}(E)$:

$$\Omega(D_A)(B, C) = \int_X Tr BC \wedge \frac{\omega^{n-1}}{(n-1)!}.$$

Notice that this symplectic form is a constant form in the sense that it does not depend on the connection D_A . In the next section, we shall study the symplectic quotient of \mathfrak{A} by \mathfrak{G} with respect to these symplectic forms.

3. Moment map equations

In this section, we shall study moment maps that correspond to the symplectic action of the Gauge group on the space of connections via the whole family of symplectic structures that we introduced in the last section. In order to construct a symplectic quotient, we will need to choose a suitable coadjoint orbit and solve the moment map equation

by requiring that the image of the moment map lies inside the chosen coadjoint orbit. The corresponding moment map equations will also be called the almost Hermitian Einstein equations for reasons which will be explained. For more details of the explanations of these equations and their generalizations in symplectic realm, reader can refer to [8].

By setting k to equal infinity, this has the effect of linearizing the moment map equations, and we recover the Hermitian Einstein equations. We then discuss its relationships with the notion of Mumford stability of vector bundles, and also show the (Gieseker) stability property for almost Hermitian Einstein bundles. In the next section, we will complete the picture by solving the moment map equation (almost Hermitian Einstein equations) for Gieseker stable bundles.

Let Ω_k be the symplectic form that we defined in last section on the space of connections \mathfrak{A} . As we saw, it is preserved by the action of Gauge transformations. Then the moment map is given as follows [8] :

$$\mu_k : \mathfrak{A} \longrightarrow Lie \mathfrak{G}^*,$$

$$\mu_k(D_A) = [e^{(k\omega I + \frac{i}{2\pi}R)}Td_X]^{2n}.$$

Here we have identified the dual of the Lie algebra of the Gauge group, $Lie \mathfrak{G}^*$, with the space of endomorphism valued top forms $\Omega^{2n}(X, End(E))$ on X via integration.

A moment map can be considered as an equivariant extension of the symplectic form with respect to the group actions. In our situation here, for each k , we get a \mathfrak{G} equivariant closed form on \mathfrak{A} , namely $\Omega_k + \mu_k$, depending only on the choice of a connection D_L on L and the Todd form of X . Nevertheless, we proved in [8] that the equivariant cohomology class it represents, $[\Omega_k + \mu_k] \in H_{\mathfrak{G}}^2(\mathfrak{A})$, is independent of such choices.

In order to construct corresponding symplectic quotients, we first need to choose a coadjoint orbit in $Lie \mathfrak{G}^*$. The simplest such choice would be 0, however, $\mu^{-1}(0)$ may not be non-empty in general. We shall choose those coadjoint orbits which consist of a single point. The set of all these coadjoint orbits can be identified with the space of $2n$ -forms on X (in the sense of distributions). We shall choose a harmonic $2n$ -form on X as our coadjoint orbit. A harmonic $2n$ -form is always proportional to the volume form $\frac{\omega^n}{n!}$. In fact, the constant is determined by the topology of the bundle and given by the normalized Euler characteristic:

$\chi(X, E \otimes L^k)/\text{rank}(E)$. Hence, our moment map equation is given by:

$$[e^{\frac{i}{2\pi}R_A+k\omega I_E}Td(X)]^{(2n)} = \frac{1}{rk(E)}\chi(X, E \otimes L^k)\frac{\omega^n}{n!}I_E.$$

Notice that this equation is a fully nonlinear elliptic system for large k . The nonlinearity of the equation comes from those terms involving products of the curvature of E with itself. Moreover, when E is a line bundle, this equation behaves like a complex Monge Ampère equation.

LETTING k GO TO INFINITY

To study the limiting case for k going to infinity, we expand Ω_k and μ_k in powers of k as follows:

$$\Omega_k(D_A)(B_1, B_2) = k^{n-1} \cdot \int_X Tr B_1 \wedge B_2 \wedge \frac{\omega^{n-1}}{(n-1)!} + O(k^{n-2}).$$

and

$$\mu_k(D_A) = k^n \frac{\omega^n}{n!}I_E + k^{n-1}R_A \wedge \frac{\omega^{n-1}}{(n-1)!} + O(k^{n-2}).$$

The leading order term of Ω_k defines a (everywhere non-degenerate) symplectic form Ω on \mathfrak{A} , which moreover is a constant form on \mathfrak{A} in the sense that there is no dependence of D_A in its expression. The moment map defined by this constant symplectic form Ω is

$$\mu(D_A) = R_A \wedge \frac{\omega^{n-1}}{(n-1)!}.$$

which is the same as the first nontrivial term in the expansion of μ_k , (since moment map is uniquely determined only up to addition of any constant central element and therefore we can throw away the first term in the expansion of μ_k without any harm).

The limiting moment map therefore becomes

$$R_A \wedge \frac{\omega^{n-1}}{(n-1)!} = \mu_E \frac{\omega^n}{n!}I_E,$$

where $\mu_E = \frac{1}{rk(E)} \langle c_1(E) \cup \frac{c_1(L)^{n-1}}{(n-1)!}, [X] \rangle$ is the slope of E with respect to the polarization L . For simplicity, we have normalized the (symplectic) volume of X to be one.

Using $\frac{\omega^n}{n!}$ and a metric on X , we can convert this equation into an equation of zero forms, it reads as :

$$\wedge R_A = \mu_E I_E,$$

where \wedge is the adjoint to the multiplication operator $L = \omega \wedge (\dots)$. In local coordinates, this equation becomes

$$g^{\alpha\bar{\beta}} R_{j\alpha\bar{\beta}}^i = \mu_E \delta_j^i,$$

where $\sqrt{-1}g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$ is the Kähler form ω on X , and $(g^{\alpha\bar{\beta}})$ is the inverse matrix to $(g_{\alpha\bar{\beta}})$. This equation is called the Hermitian Einstein equations or the Hermitian Yang Mills equations. It is called Einstein because $\wedge R_A$ is a Ricci curvature of E . These equations are more linear in nature compared to previous equations

$$[e^{\frac{i}{2\pi} R_A + k\omega} I_E Td(X)]^{(2n)} = \frac{1}{rk(E)} \chi(X, E \otimes L^k) \frac{\omega^n}{n!} I_E.$$

It is because it depends on the curvature term only linearly, and all terms involving powers of the curvature are of lower order in k which disappear when we let k go to infinity. We shall call the previous moment map equations as almost Hermitian Einstein equations.

In order to understand the infinite dimensional GIT for moduli of holomorphic vector bundles, we want to form symplectic quotients for each large k . Therefore, we need to solve these fully nonlinear almost Hermitian Einstein equations and relate them to Geometric Invariant Theory used by Gieseker. To do that, we first should have a good understanding of the linear theory which shall serve as an approximation. The original nonlinear theory will then be analysed using singular perturbation in the last section of this paper.

LINEARIZATION

Narasihan and Seshadri [12] in dimension one, Donaldson [3] in dimension two, and Uhlenbeck and Yau [14] in general proved that in the limiting situation where we let k equal infinity, the symplectic theory does correspond to the 'limit' of the GIT, which is the Mumford stability.

To be more precise, let E be an irreducible holomorphic vector bundle over X . M. Lubke [9] observed that if E admits a Hermitian Einstein metric, then it must be Mumford stable (which should be regarded as putting k to be infinity in Gieseker's notion of stability). For the converse, we need to solve the Hermitian Einstein equation on any Mumford stable vector bundle.

Theorem 4 (Donaldson, Uhlenbeck & Yau). *Let E be a Mumford stable bundle over a compact Kähler manifold X . Then there exists*

a unique Hermitian metric h on E which solves the Hermitian Einstein equation:

$$\wedge R_A = \mu_E I_E,$$

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Now, we are going to prove that if E is irreducible and sufficiently smooth and admits almost Hermitian Einstein connections whose curvatures are bounded independent of k , then it is (Gieseker) stable. The main difficult part of solving the moment map equations (almost Hermitian Einstein equations) will be treated in the next section. We shall call such bundle E as almost Hermitian Einstein bundle. We shall see that an almost Hermitian-Einstein bundle is always Mumford semi-stable and hence has a Jordan-Hölder filtration. E is called sufficiently smooth if the associated graded sheaf $Gr(E)$ is locally free. This is the natural smoothness assumption on holomorphic vector bundle analog to the usual smoothness in Hermitian-Einstein metric situation. If we want to extend to the torsion-free sheaf case, then we expect to only assume boundedness of curvature tensors in the L^2 sense.

Proposition 3.1. *Let E be an irreducible sufficiently smooth holomorphic vector bundle of rank r over a compact Kähler manifold of dimension n . Suppose that E is almost Hermitian Einstein. Then E is Gieseker stable.*

Let us first recall the moment map equations or the almost Hermitian Einstein equations:

$$[e^{(k\omega I + \frac{i}{2\pi}R)} Td_X]^{2n} = \frac{1}{rk(E)} \chi(X, E \otimes L^k) \frac{\omega^n}{n!} I_E.$$

We shall let $\eta = 1/k$ and look at the small η behavior. Now, the moment map equations become

$$[e^{(\omega I + \eta \frac{i}{2\pi}R)} Td_X^\eta]^{TOP} = \chi_E I_E \frac{\omega^n}{n!},$$

where Td_X^η is the harmonic Todd polynomial for X with variable η , that is,

$$Td_X^\eta = 1 + \eta \frac{c_1}{2} + \eta^2 \frac{c_1^2 + c_2}{12} + \dots$$

If we expand the equation in power of η , we get

$$\begin{aligned} \frac{\omega^n}{n!} I_E + \eta \left[\frac{i}{2\pi} R \wedge \frac{\omega^{n-1}}{(n-1)!} + \frac{n}{2} \mu_X \frac{\omega^n}{n!} \right] + O(\eta^2) \\ = \frac{\omega^n}{n!} I_E + \eta \left[\mu_E I_E \frac{\omega^n}{n!} + \frac{n}{2} \mu_X \frac{\omega^n}{n!} \right] + O(\eta^2). \end{aligned}$$

For an almost Hermitain Einstein bundle, we mean the above can be solved by a connection for all sufficiently small positive real number η such that the (C^k) norm of the curvature is bounded independent of η .

Proof of Proposition. For simplicity, we assume that the volume of X is one, that is $\int_X \frac{\omega^n}{n!} = 1$. We shall first see that E is Mumford semi-stable. From the equation and the boundedness of the curvature, we have a Hermitian metric h on E such that for any positive constant δ its curvature R satisfies the following estimate :

$$|\Lambda R - \mu_E I_E| < \delta.$$

According to [6], such a bundle is said to have an approximate Hermitian Einstein Structure. It is not hard to show that an approximate Hermitian Einstein bundle is Mumford semi-stable; the proof goes essentially the same as the one for Mumford stability for a Hermitian Einstein bundle.

Recall that a Mumford semi-stable bundle has a Jordan-Hölder filtration

$$E = E_0 \supset E_1 \supset E_2 \supset \dots \supset E_{k+1} = 0.$$

By assumption, each E_i is a holomorphic vector bundle. In the following, we are going to show that for each i ,

$$\frac{\chi(X, E \otimes L^k)}{\text{rank} E} > \frac{\chi(X, E_i \otimes L^k)}{\text{rank} E_i}$$

for large enough k .

In general, suppose that we are given a subbundle S of E with $\text{rank} S = s$. Let $h = h(\eta)$ be the Hermitian metric that solves equation $(aHE)_\eta$. Then we decompose E orthogonally with respect to h :

$$E = S \oplus Q,$$

where Q is the quotient bundle for $S \rightarrow E$. In general we have a holomorphic exact sequence :

$$0 \rightarrow S \rightarrow E \rightarrow Q \rightarrow 0.$$

With respect to this orthogonal decomposition, we can also decompose the Hermitian connection D on E and its curvature R as follow:

$$D = \begin{pmatrix} D_S & A^* \\ A & D_Q \end{pmatrix}$$

and

$$R = \begin{pmatrix} R_S - A^*A & \partial A^* \\ -\bar{\partial}A & R_Q - AA^* \end{pmatrix}.$$

Here, we have $D = \partial + \bar{\partial}$ as the decomposition of D into sum of its (1,0)-part and (0,1)-part. Since S is holomorphic, A is a (1,0)-form with valued in $S^* \otimes Q$ and $\partial A = 0$ (which is equivalent to $\bar{\partial}A^* = 0$). From $\bar{\partial}A^* = 0$, A^* represents certain cohomology class. In fact, A^* is the extension class for the above holomorphic exact sequence: $[A^*] \in Ext_{\mathcal{O}_X}^1(Q, S) = H^1(X, Q^* \otimes S)$.

Use $R|_S = R_S - A^*A$. We take the trace of it over S and integrate it over X to obtain

$$\int_X |A|^2 < C \cdot B + s \cdot \mu_S.$$

Since $\lim_{\eta \rightarrow 0} \eta |R|_{C^1} = 0$ for some family of Hermitian metric depending on η , one can prove by elliptic estimate that $\lim_{\eta \rightarrow 0} \eta |A|^2 = 0$.

Now, from the equation and the decomposition of the curvature, we have

$$\begin{aligned} \frac{\chi_E - \chi_S}{\eta} &= \frac{i}{2\pi} \int_X Tr A (\omega + \eta (\frac{i}{2\pi} R + Td_1)) A^* \wedge \frac{\omega^{n-2}}{(n-2)!} \\ &\quad - \frac{i}{2\pi} \int_X Tr A^* (\omega + \eta (\frac{i}{2\pi} R + Td_1)) A \wedge \frac{\omega^{n-2}}{(n-2)!} \\ &\quad - \eta \mathcal{T}_1 - \eta^2 \mathcal{T}_2 - \dots \end{aligned}$$

Choose a small η such that $|\eta (\frac{i}{2\pi} R + Td_1)| \leq 1/2$. Then the two integrals on the right side are greater than or equal to $\int |A|^2$. To deal with those \mathcal{T}_l terms. Notice that each \mathcal{T}_l is an integral of product of terms like $A^*A, \bar{\partial}A, \partial A^*, R_S, R_Q$ and some closed even forms of X . The number of these terms in the product of \mathcal{T}_l is less than or equal to $l + 1$. We claim that each $\eta^l \mathcal{T}_l$ is not greater than a small fraction of $\int |A|^2$. If this is the case, then we have proved the right-hand side of the above equation is positive since A cannot be trivial (otherwise it will violate the irreducibility assumption of E).

Therefore, we have to prove the above estimates of $\eta^l \mathcal{T}_l$. First, we observe that η times any one of A^*A , $\bar{\partial}A$, ∂A^* , R_S , R_Q is very small by our assumption $\lim_{\eta \rightarrow 0} \eta |R|_{C^1} = 0$ and the previous lemmas. Then we see that if there is a term of A^*A in the expression of \mathcal{T}_l , we are done already. It is because there are now at most l terms left in the integral, and when they are multiplied by η^l , they become very small and we can control them by a small fraction of the L^2 norm of A , $(small) \cdot \int |A|^2$.

Now, if there is no A^*A term in \mathcal{T}_l but there is a $\bar{\partial}A$ (or ∂A^*), then we integrate that derivative by parts. If the derivative hits a R_S or R_Q term, it will be zero by Bianchi identity. If it hits the closed form of X , it is also zero. So, the only possibility is when it hits ∂A^* (or $\bar{\partial}A$), then we interchange ∂ and $\bar{\partial}$. Using $\bar{\partial}A = 0$, it will only produce terms which are products of curvature and A^*A or AA^* . Thus using previous argument, we are done in this case.

The final case is there is no A term and only curvature terms appear. Then we can change them to have only R appear but not R_S or R_Q since their differences are terms involving A^*A or AA^* . Those terms involving integral of R 's will cancel out automatically. Therefore, we have proved that if S is a subbundle of E , then $\chi_S < \chi_E$.

Suppose S is any coherent subsheaf of E (of $s = \text{rank } S < \text{rank } E$) and S is a Gieseker destabilizing subsheaf for E . Since E is Mumford semi-stable, we have μ_S is not larger than μ_E . In order for S to be a candidate as a Gieseker destabilizing subsheaf of E , μ_S must equal to μ_E . Without loss of generality, we can assume that S is Gieseker stable and S is a Gieseker destabilizing subsheaf for E . Consider the following exact sequence coming from the first step in the Jordan-Hölder filtration,

$$0 \rightarrow E_1 \rightarrow E \rightarrow Q_0 \rightarrow 0.$$

Consider the composition map from S to E to Q_0 . Since S is Gieseker stable, Q_0 is Mumford stable and they have the same slope (which is the slope of E), this map must be either zero or an isomorphism. However, by the irreducibility assumption of E , this cannot be an isomorphism. Therefore, we can lift the subsheaf S of E to a coherent subsheaf of E_1 .

We are going to show that the rank of S is strictly less than the rank of E_1 . Suppose not. Then the quotient sheaf T_1 of E_1 by S is a torsion sheaf:

$$0 \rightarrow S \rightarrow E_1 \rightarrow T_1 \rightarrow 0.$$

For large enough positive integer k , we have

$$\begin{aligned}\chi(X, E_1 \otimes L^k) &= \dim H^0(E_1 \otimes L^k) \\ &\geq \dim H^0(S \otimes L^k) \\ &= \chi(X, S \otimes L^k).\end{aligned}$$

Since $\text{rank } E = \text{rank } S$ by assumption, we have

$$\frac{\chi(X, S \otimes L^k)}{\text{rank } E} \leq \frac{\chi(X, E_1 \otimes L^k)}{\text{rank } E_1}.$$

By our previous argument, this is smaller than $\frac{\chi(X, E \otimes L^k)}{\text{rank } E_1}$ for k sufficiently large contradicting to our assumption that S is a Gieseker destabilizing subsheaf for E . Therefore, rank of S is strictly smaller than rank of E_1 .

Now we look at the next step in the Jordan-Hölder filtration

$$0 \rightarrow E_2 \rightarrow E_1 \rightarrow Q_1 \rightarrow 0,$$

and S is a sheaf of E_1 of smaller rank and with the same slope. Repeating the previous argument, we see that S is in fact a subsheaf of E_2 of strictly smaller rank. Inductively, we can conclude that $S = E_{k+1}$ which is the zero sheaf. As a result, S does not exist and hence E is a Gieseker stable bundle.

4. Singular perturbation

In this section, we shall prove the existence of almost Hermitian Einstein metric on a Gieseker stable sufficiently smooth holomorphic vector bundle E over a compact Kähler manifold X . The equations involved are in a fully nonlinear elliptic system of partial differential equations.

The method that we are going to use is the singular perturbation argument. By the Jordan-Hölder filtration of E , we can decompose E as Mumford stable bundles and their extension classes. On a Mumford stable bundle, there is a unique Hermitian Einstein metric by the theorem of Uhlenbeck and Yau. If we try to use their method to construct an almost Hermitian Einstein metric on E , then we expect the metric to blow up as η goes to zero. In fact, after suitable rescaling, these metric will 'converge' to the direct sum of Hermitian Einstein metric on

each component of Mumford stable bundles and ‘forget’ the extension classes.

Our proof will confirm the above observation and in fact reverse the blowing up. We will perturb from the direct sum of Hermitian Einstein metric (which is at the infinity of the complex gauge orbit for E). However, the linearized operator will have a large kernel. The situation is similar to Taubes’ ([13]) proof of the existence of anti-self-dual connection on $SU(2)$ bundles over compact four dimensional manifolds if the second Chern class of E is big enough. In his proof, he glued in concentrated instantons and try to perturb them. First he needs to use the conformal symmetry of Yang Mills theory to rescale the neighborhood of those points where concentrated instanton is added. Then to perturb the equation, there is still a finite dimensional obstruction which can be killed by adding enough instantons and therefore the second Chern class of the bundle has to be big enough.

Instead of the conformal symmetry, we have a complexified Gauge group as our symmetry group. We need to rescale the equation order by order using this complexified Gauge group. Then we can identify the obstruction for perturbation is exactly the coefficients of the Hilbert polynomial of E . As a result, if E is Gieseker stable, we will prove that there exists an almost Hermitian Einstein metric on it.

Let us first recall the Jordan-Hölder filtration for a Mumford semi-stable bundle, in particular, a Gieseker stable bundle.

Theorem 5 (Jordan-Hölder Theorem). *If E is a Mumford semi-stable bundle over X , there exists a filtration of E by torsion free subsheaves E_i ’s*

$$E = E_0 \supset E_1 \supset E_2 \supset \dots \supset E_{k+1} = 0$$

such that $Q_j = E_j/E_{j+1}$ is Mumford stable and $\mu(Q_j) = \mu(E)$ for each j . Moreover,

$$Gr(E) = Q_0 \oplus Q_1 \oplus \dots \oplus Q_k$$

is uniquely determined by E up to isomorphism.

Let us first prove the existence result in a very particular case. We assume that only two components appear in the filtration and they are both locally free.

Proposition 4.2. *Let E be a Gieseker stable bundle over a compact Kähler manifold X . Suppose that*

$$0 \rightarrow S \rightarrow E \rightarrow Q \rightarrow 0$$

is a Jordan-Hölder filtration of E , and both S and Q are Mumford stable bundles over X . Then E is an almost Hermitian Einstein bundle.

Proof. Let us denote the i^{th} coefficient of χ_E by χ_E^i , that is,

$$\chi_E = 1 + \eta\chi_E^1 + \eta^2\chi_E^2 + \eta^3\chi_E^3 + \dots .$$

As a warm up, we will first assume that $\chi_S^2 < \chi_E^2$. Notice that since $\mu(S) = \mu(E) = \mu(Q)$, we have $\chi_S^1 = \chi_E^1 = \chi_Q^1$.

Now, the almost Hermitian Einstein equation on E is:

$$\frac{i}{2\pi}R_{A_E} \wedge \frac{\omega^{n-1}}{(n-1)!} = \mu_E \frac{\omega^n}{n!} I_E + \sum_{j=1}^{n-1} \eta^j T_{j+1}$$

for sufficiently small positive η where

$$T_j = \chi_E^j \frac{\omega^n}{n!} - \sum_{k=0}^j \left(\frac{i}{2\pi}R_{A_E}\right)^k td_X^{j-k} \cdot \frac{\omega^{n-j}}{(n-j)!}.$$

Since both S and Q are Mumford stable bundles, there exist unique Hermitian Einstein metrics h_{0S} and h_{0Q} on them, with connections D_{0S}, D_{0Q} and curvatures R_{0S}, R_{0Q} , by the theorem of Uhlenbeck and Yau. Therefore, by using $\mu(S) = \mu(E) = \mu(Q)$, we have

$$\wedge R_{0S} = \mu_E I_S$$

and

$$\wedge R_{0Q} = \mu_E I_Q.$$

Let B be a holomorphic $(0,1)$ -form on X with values in $Hom(Q, S)$ such that its cohomology class represents the extension class of E being an extension of Q by S :

$$0 \rightarrow S \rightarrow E \rightarrow Q \rightarrow 0.$$

Notice that the extension class of E in $Ext^1(Q, S)$ is uniquely determined by the above exact sequence up to scalar multiple. Therefore, there exists a unique harmonic B which satisfies

$$\int |B|^2 = (\chi_E^2 - \chi_S^2) \cdot (rkS).$$

The existence of such a B is equivalent to $\chi_S^2 < \chi_E^2$, which is from the assumption of E being Gieseker stable. We shall denote $-B^*$ by

A. Now, we want to construct a Hermitian connection D_E on E of the following form:

$$D_E = \begin{pmatrix} D_{0S} + h_S^{-1} \partial h_S & -A^* - \bar{\partial} \phi^* \\ A + \partial \phi & D_{0Q} + h_Q^{-1} \partial h_Q \end{pmatrix},$$

where h_S (resp. h_Q) is a positive self-adjoint endomorphism of S (resp. Q) with respect to the background metrics, and ϕ is a homomorphism from S to Q . Therefore, the curvature of D_E is of the form:

$$R_E = \begin{pmatrix} R_{0S} + \bar{\partial}(h_S^{-1} \partial h_S) & \partial^{new}(A^* + \bar{\partial} \phi^*) \\ -(A^* + \bar{\partial} \phi^*)(A + \partial \phi) & \\ \bar{\partial}(A + \partial \phi) & R_{0Q} + \bar{\partial}(h_Q^{-1} \partial h_Q) \\ -(A + \partial \phi)(A^* + \bar{\partial} \phi^*) & \end{pmatrix}.$$

Let us denote (see [14])

$$u_S = \log h_S$$

and

$$u_Q = \log h_Q.$$

Now, we will regard u_S, u_Q and ϕ as the variables for the almost Hermitian Einstein equation for a fix small η . It is clear that the equation is solvable for η equal to zero. However, the linearized operator at $\eta = 0$ is not invertible. Therefore, we make the following rescaling :

$$u_S \rightarrow \eta u_S,$$

$$u_Q \rightarrow \eta u_Q,$$

$$A \rightarrow \eta^{1/2} A,$$

$$\phi \rightarrow \eta \phi.$$

We just rescale each variable by multiplying by η in this case. However, in the general case, the rescaling is more complicated as we will soon see. Now, we rewrite the equation in terms of these new variables, that is,

$$D_E = \begin{pmatrix} D_{0S} + \eta h_S^{-1} \partial h_S & -\eta^{1/2} A^* - \eta \bar{\partial} \phi^* \\ \eta^{1/2} A + \eta \partial \phi & D_{0Q} + \eta h_Q^{-1} \partial h_Q \end{pmatrix}.$$

Because both background metrics on S and Q are Hermitian Einstein, the constant term in η of both sides of the equation are equal. After subtracting $\mu_E \frac{\omega^n}{n!} I_E$ from both sides of the equation and dividing both sides by η , the almost Hermitian Einstein equation will become the following form (we divide the equation into three equations according to different blocks in the decomposition):

$$\begin{aligned}
 (1)_\eta : \\
 & \frac{i}{2\pi} (\bar{\partial}(h_S^{-1} \partial h_S) - A^* A) \wedge \frac{\omega^{n-1}}{(n-1)!} \\
 & = \chi_E^2 \frac{\omega^n}{n!} I_S - \left[\left(\frac{i}{2\pi} R_{A_{0S}} \right)^2 \right. \\
 & \quad \left. + \frac{i}{2\pi} R_{A_{0S}} \cdot td_X^1 + td_X^2 \right] \wedge \frac{\omega^{n-2}}{(n-2)!} + O(\eta^{1/2}).
 \end{aligned}$$

$$\begin{aligned}
 (2)_\eta : \\
 & \frac{i}{2\pi} (\bar{\partial}(h_Q^{-1} \partial h_Q) - AA^*) \wedge \frac{\omega^{n-1}}{(n-1)!} \\
 & = \chi_E^2 \frac{\omega^n}{n!} I_Q - \left[\left(\frac{i}{2\pi} R_{A_{0Q}} \right)^2 \right. \\
 & \quad \left. + \frac{i}{2\pi} R_{A_{0Q}} \cdot td_X^1 + td_X^2 \right] \wedge \frac{\omega^{n-2}}{(n-2)!} + O(\eta^{1/2}).
 \end{aligned}$$

$$(3)_\eta : \quad \frac{i}{2\pi} \bar{\partial} \partial \phi \wedge \frac{\omega^{n-1}}{(n-1)!} = O(\eta^{1/2}).$$

Notice that in $(3)_\eta$, we should have a term $\bar{\partial} A \wedge \frac{\omega^{n-1}}{(n-1)!} / \eta^{\frac{1}{2}}$. However, using the fact that B is harmonic, this term is equal to zero identically.

Next, we claim that these three equations have a unique solution at $\eta = 0$ and the corresponding linearized operator is indeed invertible. For this purpose, we need to use a refined statement from Uhlenbeck and Yau. Although they did not write it down as a theorem, the proof is already inside their works.

Theorem 6 (Uhlenbeck & Yau). *Let E be a Mumford stable bundle over a compact Kähler manifold X , and h_0 be any Hermitian*

metric on E . Suppose H_0 is an endomorphism of E such that $\int \text{Tr}_E H_0 = 0$. Then there exists a unique positive self-adjoint endomorphism h of E with $\text{deth} = 1$, which solves

$$\wedge \bar{\partial}(h^{-1}\partial h) = H_0.$$

Moreover, the linearized operator from $\mathfrak{B}^{k+2,\alpha}$ to $\mathfrak{B}^{k,\alpha}$ at $\log h$ is invertible.

Here $\mathfrak{B}^{k,\alpha}$ denotes the Sobolev space of $C^{k,\alpha}$ self-adjoint endomorphism u of E with $\int \text{Tr} u = 0$.

Now, in equation (1), we shall take $*H_0$ to be

$$\begin{aligned} \chi_E^2 \frac{\omega^n}{n!} I_S - [(\frac{i}{2\pi} R_{A_{0S}})^2 + \frac{i}{2\pi} R_{A_{0S}} \cdot td_X^1 + td_X^2] \wedge \frac{\omega^{n-2}}{(n-2)!} \\ + \frac{i}{2\pi} A^* A \wedge \frac{\omega^{n-1}}{(n-1)!}. \end{aligned}$$

By the choice of B , we have $\int \text{Tr} H_0 = 0$. Therefore, by the above theorem and the fact that S is Mumford stable, $(1)_{\eta=0}$ always has a solution, and the corresponding linearized equation at the solution is invertible provided that we restrict to the space of u_S such that $\int \text{Tr}_S u_S = 0$. By the same reason, equation (2) can be solved uniquely in the same way. For $(3)_{\eta=0}$, it is

$$\frac{i}{2\pi} \bar{\partial}\partial\phi \wedge \frac{\omega^{n-1}}{(n-1)!} = 0.$$

If ϕ is a solution of it, then we have $\bar{\partial}\phi^* = 0$. That is ϕ^* defines a holomorphic morphism from S to Q . However, both S and Q are Mumford stable bundles of the same slope. By the general properties of Mumford stable bundles (see chapter two), ϕ is either zero or an isomorphism. But ϕ cannot be an isomorphism, otherwise, we have $\chi_S = \chi_Q = \chi_E$ which violates our assumption that E is Gieseker stable. Therefore, the last equation has zero as its only solution and it is also a linear equation. That is, it is invertible too.

Now, we are almost ready to perturb these solutions to obtain the almost Hermitian Einstein metric on E . However, there is a one-dimensional kernel for both u_S and u_Q that we still need to take care of. Let us first introduce the suitable Banach spaces and bounded operator that we are going to do perturbation. Let $\mathfrak{B}^{k,\alpha}$ be the space of triples (u_S, u_Q, ϕ) , where u_S is a self-adjoint endomorphism of S with

$\int Tr_S u_S = 0$, and u_Q is similarly defined. ϕ is a homomorphism from S to Q , and all these homomorphisms are assumed to be of the class $C^{k,\alpha}$. The above three equations at $\eta = 0$ are therefore uniquely solved in these space. However, the almost Hermitian Einstein equation does not define an operator on these space for η is not zero. For the equation

$$[e^{(\omega I + \eta \frac{i}{2\pi} R)} T d_X^\eta]^{TOP} - \chi_E I_E \frac{\omega^n}{n!} = 0,$$

if we take the trace of the left side and integrate it over X , then we will get zero. However, if we only do it on the upper left block, the part corresponding to the bundle S , then it is only zero up to the second order in η . This is true because of our choice of B and the rescaling. The same is true for the lower right block, that corresponding to the bundle Q part. So, if we try to define a nonlinear differential operator \mathfrak{L} from $\mathfrak{B}^{k+2,\alpha}$ to $\mathfrak{B}^{k,\alpha}$, then we need to carefully rescale A again. Instead of looking at $\eta^{1/2}A$, we should look at $\eta^{1/2} \cdot t \cdot A$, where t is a function depending on (u_S, u_Q, ϕ) and η such that it goes to one as η goes to zero. Now, the Hermitian connection looks like :

$$D_E = \begin{pmatrix} D_{0S} + h_S^{-1} \partial h_S & -\eta^{1/2} t A^* - \bar{\partial} \phi^* \\ \eta^{1/2} t A + \partial \phi & D_{0Q} + h_Q^{-1} \partial h_Q \end{pmatrix}.$$

The insertion of t is so that the integral of the trace of the equation OVER S is zero. The existence of t can be proved by the implicit function theorem, and t can be written down in terms of the variables explicitly. Now, we automatically get the same result for the Q part because integrating the trace of the equation over the whole bundle E is always zero.

As a result, we can define the nonlinear differential operator \mathfrak{L} as follow:

$$\mathfrak{L} : \mathfrak{B}^{k+2,\alpha} \longrightarrow \mathfrak{B}^{k,\alpha},$$

$$\mathfrak{L} \cdot \begin{pmatrix} u_S & \phi \\ \phi^* & u_Q \end{pmatrix} = \frac{1}{\eta^2} \left([e^{(\omega I + \eta \frac{i}{2\pi} R)} T d_X^\eta]^{TOP} - \chi_E I_E \frac{\omega^n}{n!} \right),$$

where R is the curvature for the above rescaled connection. Notice that the operator \mathfrak{L} is well-defined even for $\eta = 0$ because D_{0S} and D_{0Q} are both Hermitian Einstein of the same slope.

From the above discussion, we know that $\mathfrak{L} = 0$ can be solved when η equals zero, and the linearized operator is invertible there. By the

Implicit Function Theorem in Banach space, the equation $\mathfrak{L} = 0$ would therefore have solution for all small enough η which depends smoothly in η positive. Moreover, the $(C^{k,\alpha})$ norm of their curvatures are bounded. Therefore, E is almost Hermitian Einstein in this case.

Next, we will only assume $\chi_S < \chi_E$ for small enough η . There exists an integer $m > 1$ such that

$$\chi_S^j = \chi_E^j, \text{ for } j = 1, 2, \dots, m$$

and

$$\chi_S^{m+1} < \chi_E^{m+1}.$$

Now, the representative B for the extension class of E will be chosen to be the unique harmonic form such that

$$\int |B|^2 = (\chi_E^{m+1} - \chi_S^{m+1}) \cdot (rkS),$$

which is possible by the above assumption. As before, $A = -B^*$. Before the rescaling, the connection on E looks like :

$$D_E = \begin{pmatrix} D_{0S} + h_S^{-1} \partial h_S & -\eta^{m/2} A^* - \bar{\partial} \phi^* \\ \eta^{m/2} A + \partial \phi & D_{0Q} + h_Q^{-1} \partial h_Q \end{pmatrix}.$$

We need to rescale the variables order by order in η . First, we note that the connection $D_{0E} = \begin{pmatrix} D_{0S} & -\eta^{m/2} A^* \\ \eta^{m/2} A & D_{0Q} \end{pmatrix}$ solves the equation in zeroth order in η . Consider

$$D_{1E} = \begin{pmatrix} D_{0S} + \eta h_{1S}^{-1} \partial h_{1S} & -\eta^{m/2} A^* \\ \eta^{m/2} A & D_{0Q} + \eta h_{1Q}^{-1} \partial h_{1Q} \end{pmatrix}.$$

We claim that there exist unique pair (h_{1S}, h_{1Q}) for each small η which solves the almost Hermitian Einstein equation up to order one (where the uniqueness is under the normalisation that $\int Tr_S \log h_{1S} = 0$ and $\int Tr_Q \log h_{1Q} = 0$). Since these variables have no contribution to the zeroth order (in η) part of the equation, we only need to look at the

first order terms, which are :

$$\begin{aligned}
 & \frac{i}{2\pi} (\bar{\partial}(h_{1S}^{-1} \partial h_{1S})) \wedge \frac{\omega^{n-1}}{(n-1)!} \\
 (1) \quad & = \chi_E^2 \frac{\omega^n}{n!} I_S - \sum_{k=0}^2 \left(\frac{i}{2\pi} R_{A_{0S}} \right)^k \cdot td_X^{2-k} \frac{\omega^{n-2}}{(n-2)!},
 \end{aligned}$$

$$\begin{aligned}
 & \frac{i}{2\pi} (\bar{\partial}(h_{1Q}^{-1} \partial h_{1Q})) \wedge \frac{\omega^{n-1}}{(n-1)!} \\
 (2) \quad & = \chi_E^2 \frac{\omega^n}{n!} I_Q - \sum_{k=0}^2 \left(\frac{i}{2\pi} R_{A_{0Q}} \right)^k \cdot td_X^{2-k} \frac{\omega^{n-2}}{(n-2)!}.
 \end{aligned}$$

By our assumption that $\chi_S^2 = \chi_E^2 = \chi_Q^2$, S, Q , Mumford stable and the theorem of Uhlenbeck and Yau, there exist a unique (normalized) solution pair (h_{1S}, h_{1Q}) .

In order to solve the equation up to order two, we consider the following connection :

$$D_{2E} = \begin{pmatrix} D_{1S} + \eta^2 (h_{2S}^{-1} \partial h_{2S}) & -\eta^{m/2} A^* \\ \eta^{m/2} A & D_{1Q} + \eta^2 (h_{2Q}^{-1} \partial h_{2Q}) \end{pmatrix}.$$

We should notice that the background metric has been changed to a new one, which is $h_0 \begin{pmatrix} h_{1S}^\eta & \\ & h_{1Q}^\eta \end{pmatrix}$. For example, the operator ∂ is with respect to this new background metric, the A is also rescaled such that its L^2 -norm satisfies the previous equality under this new metric. However, we are only perturbing everything in the first order of η , there is no change for the lower order parts of the equation.

As before, by adding terms involving (h_{2S}, h_{2Q}) will not affect the equation up to order one. Therefore, the almost Hermitian Einstein equation up to second order in η will be :

$$\begin{aligned}
 & \frac{i}{2\pi} \bar{\partial}(h_{2S}^{-1} \partial h_{2S}) \wedge \frac{\omega^{n-1}}{(n-1)!} \\
 (1) \quad & = \chi_E^3 \frac{\omega^n}{n!} I_S - \sum_{k=0}^3 \left(\frac{i}{2\pi} R_{A_{1S}} \right)^k td_X^{3-k} \frac{\omega^{n-3}}{(n-3)!},
 \end{aligned}$$

$$\begin{aligned}
 & \frac{i}{2\pi} \bar{\partial}(h_{2Q}^{-1} \partial h_{2Q}) \wedge \frac{\omega^{n-1}}{(n-1)!} \\
 (2) \quad & = \chi_E^3 \frac{\omega^n}{n!} I_Q - \sum_{k=0}^3 \left(\frac{i}{2\pi} R_{A_{1Q}} \right)^k t d_X^{3-k} \frac{\omega^{n-3}}{(n-3)!}.
 \end{aligned}$$

By the same reasoning as before, there exists a unique (normalized) pair (h_{2S}, h_{2Q}) such that the connection D_{2E} solves the almost Hermitian Einstein equation up to the second order in η .

Repeat this process to get $(h_{1S}, h_{1Q}), (h_{2S}, h_{2Q}), \dots$. When we arrive at the $(\frac{m}{2})^{th}$ step, we will have a off diagonal term $\eta^{m/2} A$ that we need to deal with (for simplicity, we have assumed m is an even integer). However, its contribution to the equation would be zero because of the harmonicity of A . But if we are in $(j = m/1 + 1)^{th}$ order, we will need to add a term $\eta^{m/2+1} \phi_{m/2+1}$ to the variables. Then in the $(\frac{m}{2} + 1)^{th}$ order in η , there will be a third equation

$$(3) \quad \frac{i}{2\pi} \bar{\partial} \partial \phi_{m/2+1} \wedge \frac{\omega^{n-1}}{(n-1)!} = T_{m/2+1},$$

where $T_{m/2+1}$ is some expression in A and the background curvature. Since S and Q are Mumford stable of the same slope, we have $H^0(X, Hom(Q, S)) = 0$. By the standard Hodge theory, equation (3) has a unique solution $\phi_{m/2+1}$ and is invertible there. But we also have to solve equations (1) and (2). In general, if we try to solve the equation for all order $j > m/2$, then there will be contribution from the A and ϕ_l $l < j$. To be precise, we look at

$$D_{jE} = \begin{pmatrix} D_{(j-1)S} + \eta^j h_S^{-1} \partial h_{jS} & -\eta^{m/2} A^* - \eta^{m/2+1} \bar{\partial} \phi_{m/2+1}^* - \dots \eta^j \bar{\partial} \phi_j^* \\ \eta^{m/2} A + \eta^{m/2+1} \partial \phi_{m/2+1} + \dots + \eta^j \partial \phi_j & D_{(j-1)Q} + \eta^j h_Q^{-1} \partial h_{jQ} \end{pmatrix}.$$

We can see that in order $j > m/2$, there will be terms in the almost Hermitian Einstein equation coming from the products of A or ϕ_l 's with the background. However, these extra terms are all in the off-diagonal part as long as $j < m$. Therefore, in the diagonal parts (the S part and

Q part) of the equation, we still face the same equation as before :

$$\begin{aligned}
 & \frac{i}{2\pi}(\bar{\partial}(h_{jS}^{-1}\partial h_{jS})) \wedge \frac{\omega^{n-1}}{(n-1)!} \\
 (1) \quad & = \chi_E^{j+1} \frac{\omega^n}{n!} I_S - \sum_{k=0}^{j+1} \left(\frac{i}{2\pi} R_{A_{jS}}\right)^k \cdot td_X^{2-k} \frac{\omega^{n-j-1(2)}}{(n-j-1)!},
 \end{aligned}$$

$$\begin{aligned}
 & \frac{i}{2\pi}(\bar{\partial}(h_{jQ}^{-1}\partial h_{jQ})) \wedge \frac{\omega^{n-1}}{(n-1)!} \\
 (2) \quad & = \chi_E^{j+1} \frac{\omega^n}{n!} I_Q - \sum_{k=0}^{j+1} \left(\frac{i}{2\pi} R_{A_{jQ}}\right)^k \cdot td_X^{2-k} \frac{\omega^{n-j-1}}{(n-j-1)!}.
 \end{aligned}$$

(Notice, if $j > m$, then there will be terms like A^*A in the diagonal part which would make the trace to be nonzero.) Therefore, there always exists a unique normalized solution pair (h_{jS}, h_{jQ}) for equations (1) and (2). Now, for the off-diagonal term, although there might be contribution from the A and ϕ , but equation (3) can always be solved without any traceless assumption by the Hodge theory as in the previous case. As a result, the almost Hermitian Einstein equation can be solved up to order j for all $j < m$.

When $j = m$, we do have an extra term A^*A in the diagonal whose integral of its trace over S is not zero. Nevertheless, this just compensates with those coming from the curvature of S , namely, $(\chi_E^{m+1} - \chi_S^{m+1})(rkS) > 0$, by the choice of A (or the same as for B). Hence, we can still solve the almost Hermitian Einstein equation uniquely (after normalization) up to order m .

For higher order, we shall introduce t and do a perturbation argument. By replacing $\eta^{m/2}A$ by $\eta^{m/2} t A$ for some suitable function t (with the property that t goes to one as η goes to zero), we can manage to make the two diagonals corresponding to the S part and Q part of the equation have the property that taking the trace over S (or Q) and integrating it over X will get zero. First of all, we only need to do it for the S part, and the Q part will follow. For the S part, it is because, we already have this property for the equation up to order m , and for the even higher order terms, we can perturb the L^2 norm of A a little bit to adjust them. To be more precise, the existence of t can be proved by implicit function theorem easily.

After all these careful arrangements, we can finally define the non-linear differential operator \mathfrak{L} as follow:

$$\mathfrak{L} : \mathfrak{B}^{k+2,\alpha} \longrightarrow \mathfrak{B}^{k,\alpha},$$

$$\mathfrak{L} \cdot \begin{pmatrix} u_S & \phi \\ \phi^* & u_Q \end{pmatrix} = \frac{1}{\eta^{m+1}} \left([e^{(\omega I + \eta \frac{i}{2\pi} R)} T d_X^\eta]^{TOP} - \chi_E I_E \frac{\omega^n}{n!} \right),$$

where R is the curvature for the connection :

$$D = \begin{pmatrix} D_{mS} + \eta^m h_S^{-1}(\partial h_S) & -\eta^{m/2} tA^* - \bar{\partial}\Phi^* \\ \eta^{m/2} tA + \partial\Phi & D_{mQ} + \eta^m h_Q^{-1}(\partial h_Q) \end{pmatrix},$$

where $u_S = \log h_S$, $u_Q = \log h_Q$ and

$$\Phi = \eta^{m/2+1} \phi_{m/2+1} + \dots + \eta^m \phi_m + \eta^m \phi.$$

From the above discussion, we know that $\mathfrak{L} = 0$ can be solved by the triple $(u_S, u_Q, \phi) = (0, 0, 0)$ when η equals zero and the linearized operator is invertible there. By the Implicit Function Theorem in Banach space, the equation $\mathfrak{L} = 0$ would therefore have solution for all small enough η which depends smoothly in η positive. Moreover, the $(C^{k,\alpha})$ norm of their curvatures are bounded. Therefore, E is almost Hermitian Einstein in this case.

Hence we have proved the proposition. q.e.d.

Next, we are going to move forward to study the case where there are more than two components in the Jordan-Hölder filtration for the Gieseker stable bundle E .

Proposition 4.3. *Let E be a Gieseker stable bundle over a compact Kähler manifold X . Suppose that*

$$E = E_0 \supset E_1 \supset E_2 \supset \dots \supset E_{k+1} = 0$$

denote its Jordan-Hölder filtration as a Mumford semi-stable bundle. If each E_j is a vector bundle, then E is an almost Hermitian Einstein bundle.

Proof. For simplicity, we will assume that $\chi_{E_j}^2 < \chi_E^2$ for all $j > 0$. The more general case can be treated using the same method as in the

Proof. of the previous proposition. Notice that since these E_j 's are the components of the Jordan-Hölder filtration of E , they are have the

same slope as E does, it implies that $\chi_{E_j}^1 = \chi_E^1$. Denote $Q_j = E_j/E_{j+1}$, then they are all Mumford stable bundles and also have the same slope as E .

By the Uhlenbeck and Yau’s theorem, there is a unique Hermitian Einstein metric on each of the Q_j s. Denote the Hermitian Einstein connection on Q_j by D_j . If we write the connection on $Gr(E) = Q_0 \oplus Q_1 \oplus \dots \oplus Q_k$ as follow:

$$D = \begin{pmatrix} D_k & 0 & \dots & 0 \\ 0 & D_{k-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_0 \end{pmatrix},$$

then it solve the almost Hermitian Einstein equation up to zeroth order in η with the same reason as in the proof of the previous proposition. Next, we are going to choose the second fundamental forms for the successive extension one by one. We will start from the bottom of the filtration. Consider the last extension sequence :

$$0 \rightarrow E_k \rightarrow E_{k-1} \rightarrow Q_{k-1} \rightarrow 0.$$

But we have $E_k = Q_k$ and therefore this exact sequence becomes

$$0 \rightarrow Q_k \rightarrow E_{k-1} \rightarrow Q_{k-1} \rightarrow 0.$$

On those Q_j ’s, we already have a background (Hermitian Einstein) metric, now we shall choose a harmonic second fundamental form $A_{k-1,k}^*$ (with respect to these metrics) to represent this extension class such that its L^2 -norm is normalized so that

$$\int |A_{k-1,k}|^2 = (\chi_E^2 - \chi_{Q_k}^2)(rkQ_k).$$

This is possible because of our assumption that $\chi_{Q_k}^2 = \chi_{E_k}^2 < \chi_E^2$. Now, we have a Hermitian connection on E_{k-1} , namely, $\begin{pmatrix} D_k & -A_{k-1,k}^* \\ A_{k-1,k} & D_{k-1} \end{pmatrix}$.

Suppose we have chosen the second fundamental form and get a Hermitian metric on E_j . Then as the next step, we will be looking at the following exact sequence :

$$0 \rightarrow E_j \rightarrow E_{j-1} \rightarrow Q_{j-1} \rightarrow 0.$$

Thus with respect to the newly formed Hermitian metric on E_j and the Hermitian Einstein metric on Q_{k-1} , we pick the harmonic second fundamental form $A_{j-1,k}^* + A_{j-1,k-1}^* + \dots + A_{j-1,j}^*$, where $A_{j-1,l}^*$ is its E_l component (that is $A_{j-1,l}^*$ is a $(0,1)$ form with valued in $Hom(Q_{j-1}, Q_k)$). The suitable normalization turns out to be

$$\int |A_{j-1,j}|^2 = (\chi_E^2 - \chi_{E_j}^2)(rkE_j);$$

the positivity of the right-hand side is guaranteed by the stability assumption of E . Notice that, this equality is equivalent to

$$\int |A_{j-1,j}|^2 - \int |A_{j,j+1}|^2 = (\chi_E^2 - \chi_{Q_j}^2)(rkQ_j),$$

which we will be using in the followings.

At this stage, we can introduce the rescaling :

$$A_{j-1,j} \rightarrow \eta^{1/2} A_{j-1,j},$$

$$A_{j,l} \rightarrow \eta A_{j,l} \quad , \text{ if } |j - l| > 1.$$

Now, the domain Banach space would be the $\mathfrak{B}^{k+2,\alpha}$ such that an element of it would be like $(u_j, \phi_{i,j}, i, j = 0, 1, 2, \dots, k; i < j)$, where u_j is a trace-free endomorphism of Q_j , and $\phi_{i,j}$ is a homomorphism from Q_i to Q_j . Let us write $h_j = e^{u_j}$. The connection on E would be :

$$\begin{pmatrix} D_k + \eta h_k^{-1} \partial h_k & -\eta^{1/2} A_{k-1,k}^* - \eta \bar{\partial} \phi_{k-1,k}^* & -\eta A_{k-2,k}^* - \eta \bar{\partial} \phi_{k-2,k}^* & \dots \\ \eta^{1/2} A_{k-1,k} + \eta \partial \phi_{k-1,k} & D_{k-1} + \eta h_{k-1}^{-1} \partial h_{k-1} & -\eta^{1/2} A_{k-2,k-1}^* - \eta \bar{\partial} \phi_{k-2,k-1}^* & \dots \\ \eta A_{k-2,k} + \eta \partial \phi_{k-2,k} & \eta^{1/2} A_{k-2,k-1} + \eta \partial \phi_{k-2,k-1} & D_k + \eta h_{k-2}^{-1} \partial h_{k-2} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Notice that the power $\eta^{1/2}$ only attaches to $A_{j-1,j}$, and all the others have a η attaches to them. By the harmonicity of the successive second fundamental forms and the carefully chosen normalization of them, it can be proved in a similar fashion as before that up to the first order in η , the almost Hermitian Einstein equation can be solved by a unique element in $\mathfrak{B}^{k+2,\alpha}$.

In order to apply the implicit function theorem, we have to introduce the functions t_k, t_{k-1}, \dots, t_1 , each of them is a function of η and the variables such that t_j goes to one as η goes to zero for each j . By replacing $\eta^{1/2} A_{j-1,j}$ by $\eta^{1/2} t_j A_{j-1,j}$, the function t_j is so chosen

(uniquely) such that on each diagonal block (corresponding to the Q_{j-1} part), taking the trace of the equation over that block and integrate it over X will give zero. After this procedure, we can then define our nonlinear differential operator \mathfrak{L} as in the previous proposition which has the property that when $\eta > 0$, the equation $\mathfrak{L} = 0$ is the almost Hermitian Einstein equation, and the equation at $\eta = 0$ can be solved such that the corresponding linearized operator is invertible there. Hence, by the implicit function theorem, $\mathfrak{L} = 0$ can be solved for small η . Moreover, the resulting curvature is bounded in $\mathfrak{B}^{k,\alpha}$ independent of η . That is, we have obtained an almost Hermitian Einstein bundle E , and the proposition is proved. q.e.d.

Acknowledgment

There are many people who have taught me mathematics over the years. I wish to thank all of them very much, my special thanks are given to Professors R. Bott, I. M. Singer and S. T. Yau. The author would also like to thank G. Daskalopoulos.

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