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A DECOMPOSITION OF SMOOTH SIMPLY-CONNECTED h-COBORDANT 4-MANIFOLDS

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Introduction and the Statement

In [1] S. Akbulut obtained an example of the exotic manifold by cutting off the contractible submanifold from the standard manifold and regluing it via nontrivial involution of the boundary.

In these notes we give a proof of a decomposition theorem stated below, which generalizes the example of Akbulut.

Another proof of the theorem was independently obtained by C.L. Curtis, M.H. Freedman, W.C. Hsiang, and R. Stong in [4].

The author would like to thank S. Akbulut for many useful discussions, constant strong support and for bringing [2], [3] to his attention.

Throughout these notes all maps and manifolds are smooth, and immersions are in general position (or do their best if they have to obey some extra conditions). We also make the convention that if a star appears in place of a subindex, we consider a union of all objects in the family, where the index substituted by the star runs over its range. For example, $D_* \stackrel{\text{def}}{=} \bigcup_i D_i$.

Theorem. *Let U be a smooth, 5-dimensional, simply-connected h-cobordism with $\partial U = M_1 \sqcup (-M_2)$. Let $f : M_1 \to M_2$ be the homotopy equivalence induced by U. Then the following hold:*

1. *There are decompositions*

$$M_1 = M \natural_\Sigma W_1, \qquad M_2 = M \natural_\Sigma W_2$$

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such that $in_{2*} \circ in_{1*}^{-1} = f_* : H_2(M_1) \to H_2(M_2)$. *Here* in_{2*}, in_{1*} *are the maps induced in the second homology by embeddings of* M *into* M_1 *and* M_2 *respectively, and* W_1, W_2 *are smooth, compact, contractible 4-manifolds, and* $\Sigma = \partial W_1 = \partial W_2 = \partial M$.

2. *These decompositions may be chosen so that* W_1 *is diffeomorphic to* W_2.

In fact, it can be seen from the proof that the whole cobordism can be decomposed into two subcobordisms, one is a product cobordism and one is diffeomorphic to D^5 (as a smooth manifold, without any additional structure).

We will also need the following

Definition. We say that two collections $\{S_i\}_{i=1}^n$, $\{P_i\}_{i=1}^n$ of oriented 2-spheres immersed in an oriented 4-manifold are algebraically dual if $\langle [S_i], [P_j] \rangle = \delta_{ij}$. Here, $[S]$ is a homology class of immersed sphere S and $\langle \cdot, \cdot \rangle$ is the intersection form in the second homology of 4-manifold.

They are geometrically dual if they are algebraically dual and, moreover, $\mathrm{card}(S_i \cap P_j) = \delta_{ij}$.

Proof of Theorem

First, observe that U has a handlebody with no 1- and 4-handles. Let N be the middle level of U between 2- and 3-handles. Then we have two diffeomorphisms

$$\phi_1 : M_1 \sharp S_{11}^2 \times S_{12}^2 \sharp \dots \sharp S_{n1}^2 \times S_{n2}^2 \to N,$$
$$\phi_2 : M_2 \sharp S_{11}^2 \times S_{12}^2 \sharp \dots \sharp S_{n1}^2 \times S_{n2}^2 \to N.$$

By enriching the handlebody of U by 2-3 canceling pairs of handles and choosing ϕ_1 and ϕ_2, we can assume that $(\phi_2^{-1} \circ \phi_1)_*|_{H_2(M_1)} = f_*$ and $(\phi_2^{-1} \circ \phi_1)_*[S_{ij}^2] = [S_{ij}^2]$, $i = 1, \dots, n$, $j = 1, 2$. Then the two embeddings $c_1 : \bigsqcup S_{i1}^2 \vee S_{i2}^2 \to N$, $c_2 : \bigsqcup S_{i1}^2 \vee S_{i2}^2 \to N$, representing cores of products of spheres in decompositions ϕ_1 and ϕ_2, are homotopic. We have two algebraically dual collections of embedded 2-spheres $\{S_i = c_1(S_{i1}^2) \subset N\}_{i=1}^n$, $\{P_i = c_2(S_{i2}^2) \subset N\}_{i=1}^n$, and surgery along $\{S_i\}$ gives M_1; along $\{P_i\}$ gives M_2. Put $V_0 = \mathrm{Nd}_N(S_* \cup P_*)$, the closed regular neighborhood of $S_* \cup P_*$ in N.

Here we give a sketch of the rest of the construction and then work out the details.

1. Manifold V_0 has a free fundamental group and its second homology are generated by classes of spheres S_i and P_i, $i = 1, \dots, n$. We enlarge V_0 to obtain the bigger manifold V_1 so that collections of spheres $\{S_i\}$ and $\{P_i\}$ have geometrically dual collections of immersed spheres inside of V_1 and the properties mentioned above are satisfied.

2. We add 1-handles and essential 2-handles to V_1 so that the fundamental group vanishes and no new two-dimensional homology classes appear.

3. Surgery of the manifold obtained in step 2 along collections $\{S_i\}$ and $\{P_i\}$ gives two contractible submanifolds W_1 and W_2 of M_1 and M_2, respectively, and $\overline{M_1 \setminus W_1} \cong \overline{M_2 \setminus W_2}$.

The intersection points of embedded spheres S_i and P_j can be grouped in pairs of points of opposite sign with one extra point of positive sign when $i = j$, which we refer to as wedge points. Consider a disjoint collection of Whitney circles $l_1, l_2, \dots, l_m$ in $S_* \cup P_*$, one for each pair of points considered above. Push these circles to the boundary of the neighborhood of $S_* \cup P_*$ in N and call this new circles $l'_1, l'_2, \dots, l'_m$. Each of l'_i is contractible in $N \setminus S_*$, as well as in $N \setminus P_*$, since the latter two are simply-connected manifolds. Thus, for each l_i we can find two immersed disks D_i and E_i with $\partial D_i = \partial E_i = l_i$, coinciding along the collar of the boundary, and satisfying the following condition:

$$\mathring{D}_* \cap S_* = \emptyset, \qquad \mathring{E}_* \cap P_* = \emptyset. \tag{1}$$

Now we shall show that it is possible to find disks D'_i and E'_i so that they are homotopic rel(collar of the boundary) and the condition similar to (1) is satisfied.

The union of D_i and E_i with appropriate orientations gives us a class $[D_i \cup E_i]$ in the homology of N, which splits into the direct sum $H_2(N) = H_2(M_1) \oplus < [S_i], [P_i] >$. So, we can write $[D_i \cup E_i] = a + \sum \beta_k [S_k] + \sum \gamma_k [P_k]$, where $a \in H_2(M_1)$. Using the fact that $\pi_1(M_1) = \{1\}$ and the decomposition ϕ_1 we can realize class a, considered as a class in $H_2(N)$, by an immersed sphere A in N disjoint from S_*. Classes $\sum \beta_k [S_k]$ and $\sum \gamma_k [P_k]$ can be realized by the immersed spheres B and C in N disjoint from S_* and P_*, respectively. Now taking the connected sums $D_i' = D_i \sharp(-A)\sharp(-B)$, $E_i' = E_i\sharp(-C)$ ('$-$' here denotes reversal of orientation) ambiently along carefully chosen paths, we obtain discs D_i' and E_i' satisfying property similar to (1) and homotopic rel(collar of the boundary).

Consider homotopy $F_t : \bigsqcup D_i^2 \to N$ rel(collar of the boundary), where $F_0(\bigsqcup D_i^2) = D_*'$ and $F_1(\bigsqcup D_i^2) = E_*'$. It can be also viewed as a homotopy of the union of the disks and spheres S_*, P_*, and the spheres stay fixed during the homotopy.

Lemma. *Homotopy F can be perturbed, fixing ends, $S_* \cup P_*$ and the collar of the boundary of disks in the disks, to homotopy F' having the following property: $F|_{[0,\frac{1}{2}]}$ can be decomposed in a sequence of simple homotopies each of which is either a cusp or finger move; analogously, $F|_{[\frac{1}{2},1]}$ can be decomposed in a sequence of inverse cusp moves and Whitney tricks.*

For definitions of cusp, finger move, inverse cusp move and Whitney trick see [5], [7], [6].

Proof of Lemma is given in the next section.

Assume, now, that F has the property provided by Lemma above. Consider $K_* = \bigcup_i G_i = F_{\frac{1}{2}}(\bigsqcup D_i^2)$. Each of K_i may intersect both S_* and P_*, but it is homotopic rel(collar of the boundary) to the disk with interior disjoint from S_*, via a homotopy increasing number of intersection points. This means that the intersection points of $\mathring{K}_*$ and S_* can be grouped in pairs so that for each pair there is an embedded Whitney disk X_k with interior disjoint from the rest of the picture. The same argument shows that there are disks Y_k for intersection points of K_* and P_*: see figure 1.

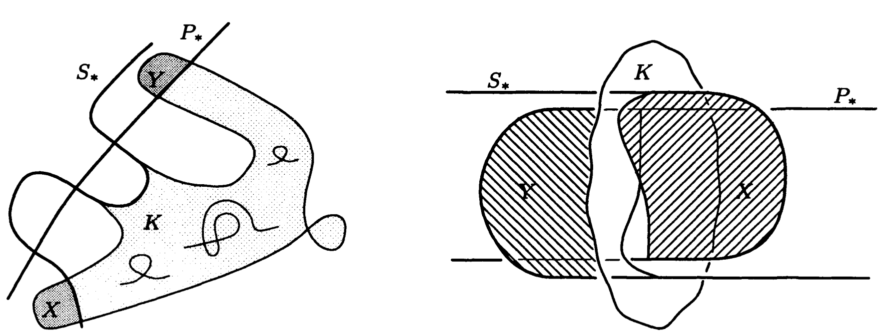

Figure 1.

Using embedded disks from collection $\{X_k\}$ we can push all disks K_i off S_*, producing the new disks K_i'. Each K_i' has an interior disjoint from S_* and its boundary is a Whitney circle for a pair of points of opposite sign in $S_* \cap P_*$. Applying an immersed Whitney trick to P_* we eliminate all the intersections with S_*, except those required by algebraic conditions and produce the collection $\{S_i^\perp\}$ geometrically dual to $\{S_i\}$. Note that, since disks K_i' are, in general, immersed and may have non-trivial relative normal bundle in N, we may have to introduce (self-)intersection points to spheres $\{S_i^\perp\}$ during this process. In the same way, using disks from collection $\{Y_i\}$ we can obtain the immersed Whitney disks K_i'' disjoint from P_*. They allow us to create a collection $\{P_i^\perp\}$ of immersed spheres dual to $\{P_i\}$. Note, that all described homotopies can be performed in a regular neighborhood of $K_* \cup X_* \cup Y_*$.

Thus, one can see that the manifold $V_1 = V_0 \cup \overline{\mathrm{Nd}_N(K_* \cup X_* \cup Y_*)}$ has the following properties:

1. $H_2(V_1)$ is generated by homology classes of embedded spheres S_i, P_i.
2. The collection of spheres $\{S_i\}$ has the geometrically dual collection $\{S_i^\perp\}$ of immersed spheres in V_1. Similarly, collection of spheres $\{P_i\}$ has geometrically dual collection $\{P_i^\perp\}$.
3. $\pi_1(V_1)$ is a free group.

Consider a handlebody of N starting from V_1. Put $V_2 = V_1 \cup$ (union of 1-handles). V_2 still has properties similar to (1), (2), (3) for V_1.

Let $g_1, \dots, g_l$ be free generators of $\pi_1(V_2)$. If we fix paths from a basepoint to attaching spheres of 2-handles, then they represent elements of $\pi_1(\partial V_2)$, say $h_1, \dots, h_L$. Since $V_2 \cup$ (all 2-handles) is a simply connected manifold, each g_i has a lift $\tilde{g}_i$ in $\pi_1(\partial V_2)$, such that it belongs to the normal subgroup of $\pi_1(\partial V_2)$ generated by $h_1, \dots, h_L$. In other words $\tilde{g}_i = (\alpha_1 h_{i_1}^{\pm 1} \alpha_1^{-1})(\alpha_2 h_{i_2}^{\pm 1} \alpha_2^{-1}) \dots (\alpha_k h_{i_k}^{\pm 1} \alpha_k^{-1})$. Duplicating 2-handles, equipping them with appropriate orientations and choosing paths joining attaching circles of 2-handles to the basepoint we may write $\tilde{g}_i = h_1' h_2' \dots h_k'$, where $h_1', h_2', \dots, h_k'$ are elements of $\pi_1(\partial V_2)$ obtained from attaching spheres of new handles with new paths to the basepoint. Now, handles in the decomposition of $\tilde{g}_i$ are all distinct, and we can slide the first handle over all others along paths joining feet of the handles to the basepoint. The attaching circle of the resulting handle, call it G_i, is freely homotopic to g_i in V_2. Adding such G_i's to V_2

for each generator of $\pi_1(V_2)$ we obtain the simply-connected manifold V_3. Since $g_1, \dots, g_l$ are free generators of the fundamental group, we do not create any additional two-dimensional homology classes.

Surgery of V_3 along collections of embedded spheres $\{S_i\}$ and $\{P_i\}$ gives two contractible sub-manifolds W_1 and W_2 of M_1 and M_2, respectively. Put $M \overset{\text{def}}{=} \overline{N \setminus V_3} \cong \overline{M_1 \setminus W_1} \cong \overline{M_2 \setminus W_2}$. So we have the decompositions:

$$M_1 = M \sharp_\Sigma W_1, \qquad M_2 = M \sharp_\Sigma W_2.$$

The property of induced maps in the homology stated in Theorem is obvious. The first part of Theorem is proved.

Proof of the second part is based on the

Fact. *If W_1, W_2 are homotopy balls built in the proof of the first part of Theorem and S^4 is a 4-dimensional sphere with standard smooth structure, then*

$$W_1 \sharp_\Sigma W_1 \cong S^4, \qquad W_1 \sharp_\Sigma W_2 \cong S^4.$$

Denote the boundary connected sum by '$\natural$'. Then we can write

$$M_1 = M \sharp_\Sigma W_1 \cong (M \sharp_\Sigma W_1) \sharp (W_1 \sharp_\Sigma W_2) \cong (M \natural W_1) \sharp_{\Sigma \sharp \Sigma} (W_1 \natural W_2)$$

and

$$M_2 = M \sharp_\Sigma W_2 \cong (M \sharp_\Sigma W_2) \sharp (W_1 \sharp_\Sigma W_1) \cong (M \natural W_1) \sharp_{\Sigma \sharp \Sigma} (W_2 \natural W_1).$$

See figure 2.

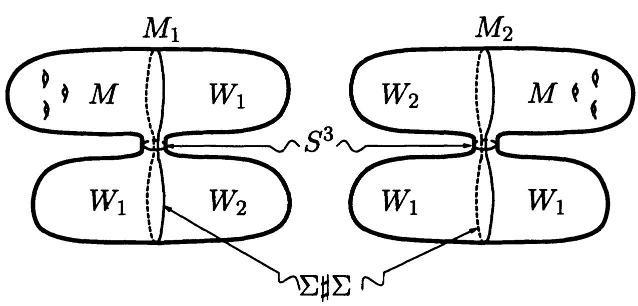

Figure 2.

In order to prove Fact we will use the art of Kirby calculus.

First, we build a handlebody of V_3.

Consider V_0', a closed neighborhood of $S_* \cup P_* \cup$(arcs joining wedge points in $S_i \cap P_i$ and $S_{i+1} \cap P_{i+1}$). If there were no intersections between S_* and P_*, except a wedge point for each pair S_i, P_i, then the handlebody would look like as shown on figure 3.

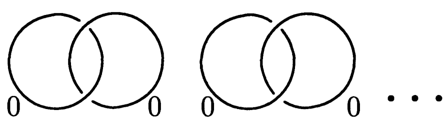

Figure 3.

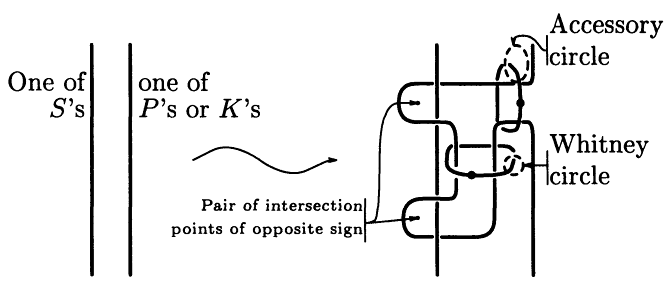

Figure 4.

Introducing a pair of intersection points of opposite signs corresponds to the move in Kirby calculus shown in figure 4.

Here, we introduce two 1-handles, one corresponding to a Whitney circle, another to an accessory circle of a newly introduced pair of intersections. The handlebody of V_0' is obtained by applying several moves shown in figure 4 to the picture in figure 3. Again, if disks K_i were embedded and disjoint from $S_* \cup P_*$, then to obtain the handlebody of $V_0 \cup \overline{\mathrm{Nd}(K_*)}$ one has to attach 2-handles to Whitney circles for each pair of intersection points of S_* and P_*. Then we have to introduce intersections between K_* and $S_* \cup P_*$ and self-intersections of K_*. Corresponding moves are shown on figures 4 and 5.

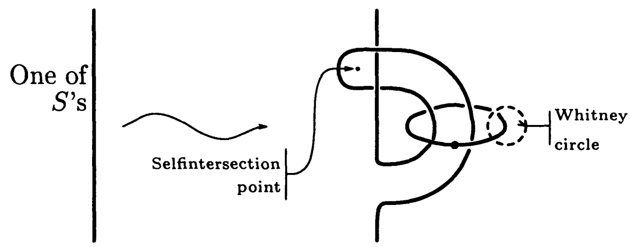

Figure 5.

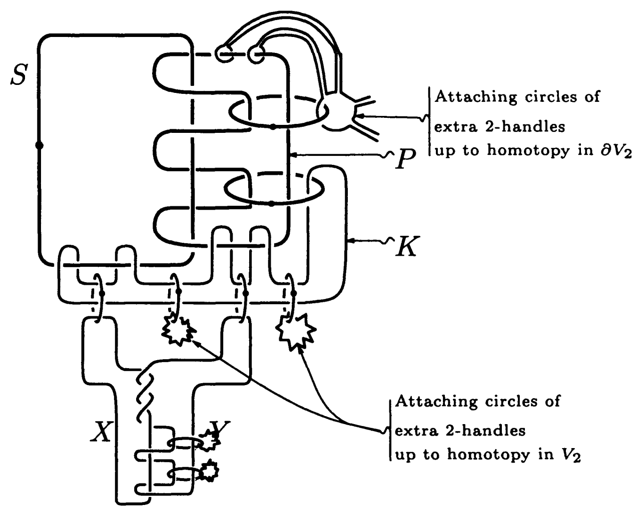

Figure 6.

Introducing a self-intersection corresponds to adding one 1-handle to the picture. Addition of disks from collections $\{X_i\}$, $\{Y_i\}$ corresponds to attaching 2-handles to the circles linked once to the 1-handles corresponding to the Whitney circles, and unlinked from other 1-handles. They may link each other according to the fact that boundaries of X's and Y's are not necessarily disjoint. This phenomenon is illustrated on figure 2. Then, as in previous steps, add 1-handles to the picture reflecting intersections of X_* and Y_*, as in figure 6.

To obtain the handlebody of V_2 we have to attach several 1-handles. The link on figure 6 shows all possible phenomena which can occur. Thus, the handlebody of V_2 has 1-handles of seven types coming from the following:

1. Whitney circles of intersections of S_* and P_*;
2. accessory circles of intersections of S_* and P_*;
3. Whitney circles of intersections of K_* and $S_* \cup P_*$;
4. accessory circles of intersections of K_* and $S_* \cup P_*$;
5. Whitney and accessory circles of self-intersections of K_*;
6. Accessory circles of intersections of X_* and Y_*;
7. Extra 1-handles.

For each 1-handle of types 1 and 3 there is a 2-handle attached to the circle linked to this 1-handle geometrically once and algebraically zero times to other 1-handles.

The attaching circles $h_1, \dots, h_l$ of other 2-handles are homotopic to free generators of $\pi_1(V_2)$. As generators we may choose the cores $g_1, \dots, g_l$ of 1- handles of types 2, 4, 5 and 6.

Consider the homotopy $F : S^1 \times I \to V_2$, $F(\cdot, 0) = g_i$, $F(\cdot, 1) = h_i$. We can make it disjoint from $S_* \cup K_* \cup X_* \cup Y_*$. First, intersections of the image of the homotopy with X_* and Y_* can be turned into intersections with K_* by pushing them toward the boundary of $X_* \cup Y_*$. Then, in the same way, we can avoid intersections with K_* by cost of new intersections with P_*. Finally, intersections with S_* can be removed using the geometrically dual collection of immersed spheres $\{S_i^{\perp}\}$. Call this new homotopy F' and let $\{x_1, x_2, \dots, x_k\} = {F'}^{-1}(P_*)$. If we remove a small neighborhood of a union of disjoint arcs joining each x_i to the point on $S^1 \times \{0\}$ from $S^1 \times I$, then restriction of F' on this set is a homotopy of h_i to the curve ${g'}_i$ which is a band connected sum of g_i and meridians of P_*.

This homotopy is disjoint from $S_* \cup P_* \cup K_* \cup X_* \cup Y_*$ and can be pushed to the boundary of V_2.

We obtain the handlebody of W_1 by attaching 2-handles to $h_1, \dots, h_l$ and surgering S_*, which corresponds to putting dots on the circles representing S_* in Kirby calculus.

Here we summarize all the information about the handlebody of W_1. For 2-handles we use same notation as for the corresponding object in the above construction of W_1:

1-handles	2-handles
1. Surgery of S_*.	P_*; attaching circles go geometrically once through corresponding 1-handles S_*.
2. Whitney circles of intersections of S_* and P_*.	K_*; attaching circles go geometrically once through corresponding 1-handles from the left entry of the table.
3. Whitney circles of intersections of K_* and $S_* \cup P_*$.	X_*, Y_*; attaching circles go geometrically once through corresponding 1-handles from the left entry of the table.
4. Accessory circles of all intersections, Whitney circles of (self-)intersections of K_*, Whitney circles of intersections of X_* and Y_*, extra 1-handles.	Extra 2-handles H_*; attaching circles go geometrically once through corresponding 1-handles from the left entry of the table and homotopic to band connected sum of 1-handles with meridians of P_*.

Taking a double of W_1 corresponds to attaching zero-framed 2-handles to the meridians of existent 2-handles, and as many 3-handles as there are 1-handles in the handlebody of W_1.

Remember that attaching circles h_i of 2-handles H_i are homotopic to g'_i. Sliding H_* over dual handles H^*_* we can obtain handles H'_* attached to g'_*. Now ${g'}_1, \dots, {g'}_l$ may be linked to the meridians of handles P_*, K_*, X_* and Y_*, but situation can be improved by sliding handles dual to P_*, K_*, X_* and Y_* over handles dual to H'_*. Further we may slide H'_* over handles dual to P_* to obtain handles attached to meridians of 1-handles from the fourth row of the table, so they can be cancelled.

The framings of H'_* result in a twist of attaching circles of remaining 2-handles.

We undo this twist using dual handles.

With the next step we unlink handles X_* and Y_* from each other using dual 2-handles and cancel them with 1-handles of type 3. Now K_i are attached to meridians of 1-handles of type 1 and also can be cancelled. After cancelling P_* with 1-handles from surgery of S_* we end up with unknotted, unlinked, zero framed handles and the same number of 3-handles, which is clearly S^4.

We need only small changes in our construction to prove the second diffeomorphism in our Fact. The handlebody of W_2 differs from W_1 by putting dots on the attaching circles of P_* rather than S_*. Thus, the handlebody of $W_1 \natural_\Sigma W_2$ is obtained from W_1 by attaching 2-handles to meridians of 1-handles, coming from surgered S_i's, and to meridians of all 2-handles, except P_i's. Using the same trick we can make the homotopy of h_* to g_* disjoint from $P_* \cup K_* \cup X_* \cup Y_*$, rather than $S_* \cup K_* \cup X_* \cup Y_*$. Applying the same procedure to H_* gives us 2-handles G''_* attached to the band connected sum of meridians of 1-handles corresponding to generators of $\pi_1(V_2)$ and meridians of S_*. But now 1-handles corresponding S_* have dual 2-handles, so sliding H_* over them gives handles dual to 1-handles generating $\pi_1(V_2)$. So we may apply the same procedure to simplify the handlebody and end up with S^4. This finishes the proof of Fact and the second part of Theorem.

Proof of Lemma

It is a simple consequence of singularity theory that a homotopy of a surface in 4-manifold can be decomposed (after small perturbation) in a sequence of finger moves, cusps, Whitney tricks and inverse cusp moves. In our case we have to consider, in addition, a finger move along a path in one of the disks joining (self-)intersection of the disks with a point on its boundary. The inverse of this move is a Whitney trick with a Whitney circle intersecting the boundary of the disk. We have to show that it is possible to reorder these simple homotopies so that those increasing number of intersections come first. This is obvious after the following consideration: cusp birth happens in a small neighborhood of the point in the surface and we can assume that part of the surface in this neighborhood stays fixed during the part of homotopy preceding this cusp birth, so we can push this cusp up to the beginning of the homotopy. A finger move can be localized in the neighborhood of an embedded arc joining two points in the surface, thus we may apply the same argument and this finishes the proof of Lemma.

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