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POLYCYCLIC GROUPS AND TRANSVERSELY AFFINE FOLIATIONS

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Introduction

A foliation is called transversely affine if coordinates can be chosen so that the holonomy maps are all affine [3]. It is known that there is a close relation between complete affinely flat manifolds and polycyclic groups. Specifically, all polycyclic groups occur as the fundamental group of such a manifold [10] and it is conjectured that the converse is virtually true. Here we consider foliations of codimension one of manifolds with polycyclic fundamental group and show that in certain natural situations the foliation must have a transverse affine structure. For example, when the manifold is compact with polycyclic fundamental group, any real analytic foliation with exponential growth (of all leaves) is transversely affine (Theorem 4.1). For foliations with less differentiability analogous results are obtained by making additional topological hypotheses on the foliation. These results may be thought of as generalizations of results from [4] and [17] for manifolds of dimension three. In contrast, examples from [8] are described which show that these results do not hold when the fundamental group is merely assumed solvable. The results for foliations are based on a study of smooth actions by polycyclic groups on the real line which, using the main result in [23], yields sharper conclusions than similar results obtained in [18] for actions by more general solvable groups.

1. Polycyclic groups of diffeomorphisms of $\mathbb{R}$

Denote by $\mathrm{Diff}^k(\mathbb{R})$ the group of $\mathscr{C}^k$ diffeomorphisms of the real line, where k is a positive integer, ∞, or ω (real analytic). We will denote by $\mathrm{Aff}(\mathbb{R})$ the subgroup of $\mathrm{Diff}^\omega(\mathbb{R})$ consisting of affine maps ($x \mapsto ax+b$ for some $a \neq 0$, $b \in \mathbb{R}$). Of particular interest to us will be diffeomorphism groups which are polycyclic and have exponential growth. For abstract groups of this type the basis reference is Wolf [22].

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A finitely generated group Γ is said to be *polycyclic* if there is a finite sequence of subgroups

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_n = \{e\}$$

such that for each $i = 1, \cdots, n$, Γ_i is normal in Γ_{i-1} and Γ_{i-1}/Γ_i is finite or infinite cyclic. The following result from [22] gives several equivalent conditions which will be useful.

(1.1) Proposition. *For a solvable group* Γ *the following conditions are equivalent*:

(1) Γ *is polycyclic.*

(2) *Every subgroup of* Γ *is finitely generated.*

(3) Γ *has a subgroup* Γ^* *of finite index whose commutator subgroup* $[\Gamma^*, \Gamma^*]$ *is finitely generated and nilpotent, and* $\Gamma^*/[\Gamma^*, \Gamma^*]$ *is finitely generated free abelian.*

It will be convenient to extend this notion somewhat by saying that a group Γ is *virtually polycyclic* if it has a polycyclic subgroup of finite index.

A finitely generated group Γ is said to have *exponential growth* if, for some (or any) finite generating set, the number $g(n)$ of elements in Γ which can be expressed as a word of length $\leq n$ in the generators satisfies

$$\liminf_n \frac{\log g(n)}{n} > 0.$$

On the other hand, Γ has *polynomial growth* if $g(n) \leq p(n)$ for all $n \in \mathbf{Z}^+$, where $p(x)$ is a polynomial. The following is also proved in [22].

(1.2) Theorem. *If* Γ *is virtually polycyclic then either*

(i) Γ *has nilpotent subgroup of finite index and has polynomial growth, or*

(ii) Γ *has exponential growth.*

Throughout the rest of this section Γ will denote a virtually polycyclic group of diffeomorphisms of the real line. Note that if $\Gamma \subset \mathrm{Aff}(\mathbf{R})$, then $[\Gamma, \Gamma]$ consists of translations. In this case we define the translation number homomorphism $T \colon [\Gamma, \Gamma] \to \mathbf{R}$ as follows: If $\gamma(x) = x + b$, then $T(\gamma) = b$.

(1.3) Lemma. *If* $\Gamma \subset \mathrm{Aff}(\mathbf{R})$ *is a polycyclic subgroup of exponential growth, then* $[\Gamma, \Gamma]$ *contains two elements whose translation numbers have quotient equal to an irrational algebraic number. In particular,* $[\Gamma, \Gamma]$ *is a finitely generated subgroup of rank* ≥ 2 *which is dense in the translation subgroup of* $\mathrm{Aff}(\mathbf{R})$.

Proof. Since Γ is polycyclic, $[\Gamma, \Gamma]$ is finitely generated as an abstract group. Since Γ has exponential growth it is not abelian, so both $[\Gamma, \Gamma]$ and $\Gamma/[\Gamma, \Gamma]$ must be nontrivial. We claim that $[\Gamma, \Gamma]$ has rank ≥ 2. If not, $[\Gamma, \Gamma]$ would be infinite cyclic generated by some γ_0. For $\gamma \in \Gamma$ we would have $\gamma\gamma_0\gamma^{-1}$ equal to γ_0 or γ_0^{-1}. Passing to a subgroup of index ≤ 2 in Γ we could assume $\gamma\gamma_0\gamma^{-1} = \gamma_0$ for all $\gamma \in \Gamma$ which would imply that γ_0 is a central element and further (since $\Gamma/[\Gamma, \Gamma]$ is abelian) that Γ is nilpotent. This contradicts the assumption that Γ has exponential growth. So $[\Gamma, \Gamma]$ has rank ≥ 2, and by rescaling we may assume that $T(\gamma) = 1$ for some $\gamma \in [\Gamma, \Gamma]$. If $T([\Gamma, \Gamma]) \subset \mathbf{Q}$ it would follow that $\operatorname{rank}[\Gamma, \Gamma] = 1$, so there is a $\gamma \in \Gamma$ such that $T(\gamma)$ is irrational. It remains to be shown that all $T(\gamma)$ are algebraic. Regard T as an element of $\operatorname{Hom}([\Gamma, \Gamma]; \mathbf{R}) \cong \mathbf{R}^d$ (where $d = \operatorname{rank}[\Gamma, \Gamma]$). The action of Γ on $[\Gamma, \Gamma]$ by inner automorphisms induces an action of Γ on the vector space $\operatorname{Hom}([\Gamma, \Gamma]; \mathbf{R})$. For $\gamma \in \Gamma$ we denote the corresponding automorphism by γ_*. For any basis of $[\Gamma, \Gamma]$, γ_* will have an integer matrix with respect to the dual basis of $\operatorname{Hom}([\Gamma, \Gamma]; \mathbf{R})$. Since Γ has exponential growth there is no Γ-invariant measure but Lebesgue measure is the unique (up to rescaling) quasi-invariant measure in the sense of [18, §4]. In particular, for each $\gamma \in \Gamma$, there is a nonzero real number $c(\gamma)$ such that $\gamma_*(T) = c(\gamma)T$ and, furthermore, T spans the one-dimensional eigenspace of γ_* for the eigenvalue $c(\gamma)$. Since γ_* has integer matrix the eigenvalues $c(\gamma)$ are algebraic. When T is scaled so that one of its coordinates is one, it follows that the other coordinates of T must also be in the field of algebraic numbers. This completes the proof of (1.3).

We now consider finitely generated subgroups of $\operatorname{Diff}^k(\mathbf{R})$ for $k \geq 2$. For $\Gamma \subset \operatorname{Diff}^k(\mathbf{R})$ and $x \in \mathbf{R}$ the *orbit of x under* Γ is the set $\{\gamma(x) | \gamma \in \Gamma\}$ which we denote by $\Gamma(x)$. $\Gamma(x)$ is said to have *exponential growth* if, for some (or any) fixed finite generating set for Γ, the cardinality of the set $\{\gamma(x) | \gamma$ has minimum word length $\leq n\}$ grows exponentially in n. According to (1.3) of [18], Γ has a nontrivial invariant Borel measure which is finite on compact sets if, and only if, some orbit does *not* have an exponential growth. Note that if $\Gamma(x)$ has exponential growth for some x, then Γ has exponential growth as an abstract group. The group Γ is said to be *minimal* if every orbit is dense in $\mathbf{R}$.

(1.4) Lemma. *Suppose $\Gamma \subset \operatorname{Diff}^k(\mathbf{R})$ $(k \geq 2)$ is a virtually polycyclic group such that every orbit has exponential growth. For such groups the following are equivalent:*

(1) *There is a subgroup Γ^* of finite index in Γ such that the center of $[\Gamma^*, \Gamma^*]$ contains an element without fixed points.*

(2) *Γ is minimal.*

(3) *Γ is conjugate (by a homeomorphism) to a subgroup of* $\operatorname{Aff}(\mathbf{R})$.

Proof. (1) $\Rightarrow$ (2). Assume that condition (1) is satisfied and let $\gamma \in Z([\Gamma^*, \Gamma^*])$ be a diffeomorphism without fixed points. The quotient space $\mathbf{R}/x \sim \gamma(x)$ is diffeomorphic to the circle and the action of $Z([\Gamma^*, \Gamma^*])$ on $\mathbf{R}$ induces an action by the quotient group $Z([\Gamma^*, \Gamma^*])/\langle\gamma\rangle$ on the circle. It is proved in [18] that Γ has a quasi-invariant measure (that is, each element of Γ either preserves the measure or multiplies it by a constant) which is finite on compact sets. Since orbits of Γ^* have exponential growth the measure is *not* Γ^*-invariant. However, the measure is invariant for the subgroup $[\Gamma^*, \Gamma^*]$, and by suitable rescaling we may assume that the induced invariant measure on the circle has total measure 1. The translation numbers [18] of the elements of $[\Gamma^*, \Gamma^*]$ when reduced mod 1 become the rotation numbers [15] for the induced action on the circle. Since the original measure is not Γ^*-invariant, the action of Γ^* by inner automorphisms on the translation numbers of $[\Gamma^*, \Gamma^*]$ determines a subgroup of $\mathrm{Aff}(\mathbf{R})$ which has exponential growth. By (1.3) it follows that at least one of the rotation numbers for the action induced on the circle is irrational. A classical theorem of Denjoy (since $k \geq 2$) implies that the action on the circle is minimal and, therefore, so is the action by Γ.

That (2) $\Rightarrow$ (3) follows from (4.6) of [18] and (3) $\Rightarrow$ (1) is obvious, so (1.4) is proved.

(1.5) Lemma. *If $\Gamma \subset \mathrm{Diff}^{\omega}(\mathbf{R})$ is virtually polycyclic and such that every orbit has exponential growth, then Γ satisfies the equivalent conditions of* (1.4).

Proof. By (1.1) we may assume, by passing to a subgroup of finite index, that $[\Gamma, \Gamma]$ is nilpotent. Let γ be a nontrivial element in the center of $[\Gamma, \Gamma]$. Either γ has no fixed points, or its translation number with respect to a Γ-quasi-invariant measure is zero [18], in which case the support of the measure is contained in the fixed point set of γ. Since γ is real analytic its fixed point set is discrete, which means that the support of the measure consists of atoms. This would imply that the group of translation numbers of $[\Gamma, \Gamma]$ is infinite cyclic. By an argument similar to that in the proof of (1.3) the action induced by Γ on the support set of the measure would have polynomial growth, contradicting the hypothesis that *all* orbits of Γ have exponential growth. Therefore, γ has no fixed points, and condition (1) of (1.4) is satisfied. This proves (1.5).

(1.6) Lemma. *Suppose that every orbit of $\Gamma \subset \mathrm{Diff}^k(\mathbf{R})$ $(k \geq 2)$ has exponential growth and that Γ has a subgroup Γ^* of finite index such that $[\Gamma^*, \Gamma^*]$ is a finitely generated abelian group. Then Γ satisfies the equivalent conditions of* (1.4).

Proof. Clearly Γ is virtually polycyclic. As before there is a nontrivial Γ^*-quasi-invariant measure which is invariant under $[\Gamma^*, \Gamma^*]$. We claim that some element of $[\Gamma^*, \Gamma^*]$ has nonzero translation number and, therefore, no fixed points. If all translation numbers of $[\Gamma^*, \Gamma^*]$ were zero, then the support of the measure would be pointwise fixed by every element of $[\Gamma^*, \Gamma^*]$. In this case Γ^* would induce an action of $\Gamma^*/[\Gamma^*, \Gamma^*]$ on the support of the measure, but this quotient group (abelian) has polynomial growth contradicting the fact that every orbit of Γ^* has exponential growth. So the claim is proved, and (1) of (1.4) follows since $[\Gamma^*, \Gamma^*]$ is abelian.

2. Differentiable conjugacy to affine subgroups

(2.1) Theorem. *If $\Gamma \subset \mathrm{Diff}^k(\mathbf{R})$ $(k \geq 2)$ is virtually polycyclic, minimal, and has exponential growth (as an abstract group), then Γ is equivalent by a $\mathscr{C}^{k-2}$ change of coordinates to a subgroup of* $\mathrm{Aff}(\mathbf{R})$.

Proof. By (4.6) of [18] there is a continuous change of coordinates taking Γ into $\mathrm{Aff}(\mathbf{R})$. By (1.4) and its proof we see that $[\Gamma, \Gamma]/\langle\gamma\rangle$ acts on the circle where γ is any nontrivial element of $[\Gamma, \Gamma]$. By (1.3) we conclude that one of the rotation numbers of this action is an irrational algebraic number and, consequently, satisfies the Roth condition (cf. [19]). By the main result of Yoccoz [23], the map which conjugates the action on the circle to a group of rotations must be at least of class $\mathscr{C}^{k-2}$. The lift of this map to $\mathbf{R}$ is a change of coordinates taking Γ into $\mathrm{Aff}(\mathbf{R})$.

Remark. When $k = \infty$ or ω there is no loss of differentiability, that is, the conjugacy takes place in the *group* $\mathrm{Diff}^k(\mathbf{R})$.

(2.2) Corollary. *If $\Gamma \subset \mathrm{Diff}^\omega(\mathbf{R})$ is virtually polycyclic and every orbit of Γ has exponential growth, then Γ is conjugate by a real analytic diffeomorphism to a subgroup of* $\mathrm{Aff}(\mathbf{R})$.

Proof. (2.2) follows from (2.1) and (1.5).

(2.3) Corollary. *If every orbit of $\Gamma \subset \mathrm{Diff}^k(\mathbf{R})$ $(k \geq 2)$ has exponential growth and Γ has a subgroup Γ^* of finite index such that $[\Gamma^*, \Gamma^*]$ is a finitely generated abelian group, then Γ is equivalent by a $\mathscr{C}^{k-2}$ change of coordinates to a subgroup of* $\mathrm{Aff}(\mathbf{R})$.

Proof. (2.3) follows from (2.1) and (1.6).

Remark. The conclusions of (2.1), (2.2), and (2.3) imply that Γ is actually polycyclic and that $[\Gamma, \Gamma]$ is abelian.

The essential nature of the hypothesis that Γ is polycyclic is illustrated by examples in Hirsch [8]. If $f\colon S^1 \to S^1$ is a covering map of degree

$n \geq 1$, let $G: \mathbf{R} \to \mathbf{R}$ be a covering transformation which generates the fundamental group of S^1 and let $F: \mathbf{R} \to \mathbf{R}$ be a lift of f. Then F and G satisfy the relation $FGF^{-1} = G^n$ and generate an action by a solvable group Γ on $\mathbf{R}$. When $n = 1$ the group is abelian, but when $n > 1$ the group is *not* polycyclic. To see this, consider the special case $f(t) = nt (\operatorname{mod} 1)$, $F(t) = nt$, $G(t) = t + 1$. The group generated by F and G in this case lies in $\operatorname{Aff}(\mathbf{R})$. It is clear for $n > 1$ that the commutator subgroup of Γ is not finitely generated. For example, when $n = 2$, $[\Gamma, \Gamma]$ corresponds to the group of translations by dyadic rational numbers. Also note, for $n > 1$, that Γ has no invariant measure, so every orbit, as well as Γ itself, has exponential growth. A covering map $f: S^1 \to S^1$ is said to be an *expanding map* if $f'(t) > 1$ for all $t \in S^1$. It is proved in [20] that if f is any expanding map of degree $n > 1$ on the circle, then it is conjugate by a (unique) homeomorphism to the map $t \mapsto nt (\operatorname{mod} 1)$. This conjugacy lifts to $\mathbf{R}$, so we conclude that whenever f is an expanding map the corresponding action of Γ on $\mathbf{R}$ is equivalent by a continuous change of coordinates to a subgroup of $\operatorname{Aff}(\mathbf{R})$. On the other hand, the change of coordinates is not usually differentiable. For example, when the conjugating map is differentiable at the (unique) fixed point s of f it must be the case that $f'(s) = n$. This situation can be avoided by a $\mathscr{C}^1$ small perturbation of f. This shows that (2.1) and (2.2) above do not hold for solvable groups which are not polycyclic and that (2.3) is not valid without the assumption that $[\Gamma^*, \Gamma^*]$ is finitely generated. Actually, as observed in [8], when Γ is not polycyclic it can fail to be equivalent to a subgroup of $\operatorname{Aff}(\mathbf{R})$ even by a continuous change of coordinates. To see this, let f be a covering of degree $n > 1$ obtained by composing the map $t \mapsto nt (\operatorname{mod} 1)$ with a diffeomorphism h (even real analytic) of the circle such that $0 < h'(0) < 1/n$. The resulting covering map is no longer an expanding map and the action of Γ on $\mathbf{R}$ is not minimal. This implies that Γ is not even continuously conjugate to a subgroup of $\operatorname{Aff}(\mathbf{R})$. It is true, however, that such Γ are semiconjugate to a subgroup of $\operatorname{Aff}(\mathbf{R})$ [18]. This amounts to saying that the original map is semiconjugate to the map $t \mapsto nt (\operatorname{mod} 1)$. Finally, the essential nature of the hypothesis of exponential growth may be seen by considering the case $n = 1$. Since Γ is abelian of rank 2 in this case, it has polynomial growth. There are diffeomorphisms of the circle (which are even real analytic) which are continuously but not differentiably conjugate to an irrational rotation [23]. (In such cases the rotation number will not be algebraic.) The corresponding Γ is equivalent by a continuous change of coordinates to a subgroup of $\mathrm{Aff}(\mathbf{R})$, but the change of coordinates cannot be as smooth as the conclusions of (2.1), (2.2), and (2.3) require.

3. Transversely affine foliations of codimension one

Suppose M is a connected manifold and $\mathscr{F}$ is a foliation of M. $\mathscr{F}$ is *transversely affine* if M is covered by a collection of $\mathscr{F}$-distinguished open sets $\mathscr{U}$ (that is, the restriction of $\mathscr{F}$ to each U_α in $\mathscr{U}$ is a standard product foliation) such that the submersions $\pi_\alpha \colon U_\alpha \to \mathbf{R}^q$ ($q =$ codimension of $\mathscr{F}$) which define $\mathscr{F}|U_\alpha$ have the property that the transition functions $\tau_{\alpha\beta} = \pi_\beta \circ \pi_\alpha^{-1}$ are affine maps between open subsets of $\mathbf{R}^q$. For codimension-one foliations ($q = 1$) which are transversely oriented and sufficiently smooth to be determined by a differentiable one-form ω (tangent bundle of $\mathscr{F} =$ kernel ω) the condition becomes $d\omega = \theta \wedge \omega$, where θ is a closed one-form. That the existence of such ω and θ is equivalent to $\mathscr{F}$ being transversely oriented and affine is shown in [3]. On the other hand, if $\overline{\omega}$ is any other one-form which also determines the transversely oriented foliation $\mathscr{F}$, then $\overline{\omega} = f\omega$, where $f \colon M \to \mathbf{R}^+$. A short computation shows that $d\overline{\omega} = \overline{\theta} \wedge \overline{\omega}$, where $\overline{\theta} = \theta + d(\log f)$. Thus, θ determines an element of $H^1(M; \mathbf{R})$ which does not depend on the choice of ω. Furthermore, this cohomology class is trivial if, and only if, $\mathscr{F}$ is determined by a closed one-form. If $d\omega = 0$ we can take $\theta = 0$. Conversely, if $\theta = dg$, then take $f = e^{-g}$ to get $\overline{\theta} = 0$. When ω is closed it determines an $\mathscr{F}$-invariant measure in the sense of [15], [16]. We will be interested in foliations without (nontrivial) invariant measures; when such foliations are transversely affine, the cohomology class determined by θ is nontrivial.

Example. Let G be a simply connected Lie group and $H \subset G$ a closed subgroup of codimension one. If $\Gamma \subset G$ is a discrete subgroup, then H acts on G/Γ by left translations. Let $\mathscr{F}$ be the (codimension-one) orbit foliation determined by this action on $M = G/\Gamma$. The universal covering space of M is $\widehat{M} = G$ and the induced foliation $\widehat{\mathscr{F}}$ on $\widehat{M}$ has as its leaf space the one-dimensional manifold $H\backslash G$, which must be diffeomorphic to $\mathbf{R}$ since G is simply connected. Thus, the action of Γ on $H\backslash G$ determines the subgroup of $\mathrm{Diff}^\omega(\mathbf{R})$ (which is defined up to conjugacy) called the (global) *holonomy group* of $\mathscr{F}$. Since G acts transitively on $\mathbf{R}$ (that is, on $H\backslash G$ by right translations) the action is effectively equivalent, by a classical result of Lie, to one of three types: translations by $\mathbf{R}$, the standard action by $\mathrm{Aff}(\mathbf{R})$, or by an action by the universal cover of $\mathrm{SL}(2, \mathbf{R})$. If the group G is amenable (that is, a compact extension of a solvable group [10]), then the third possibility cannot occur and $\mathscr{F}$ will have a transverse affine structure. When G is amenable, Γ must be virtually polycyclic [10], so when $\mathscr{F}$ does not have an invariant measure the fact that it has a transverse affine structure follows also from (2.2). Suppose that Γ is uniform ($G/\Gamma = M$ is compact). This implies that G is unimodular. It follows from a result in [6, p. 90] that the action of G on $H\backslash G$ has an invariant measure if, and only if H is unimodular. Since Γ is uniform, there is a G-invariant measure if, and only if, there is a Γ-invariant measure. Therefore, when M is compact $\mathscr{F}$ has an invariant measure if, and only if, H is unimodular. When G is semisimple with each factor of noncompact type and $\Gamma \subset G$ is a uniform discrete subgroup we claim that the foliation $\mathscr{F}$ cannot have a transverse affine structure. If H were unimodular there would be an $\mathscr{F}$-invariant measure, so G would act on $H\backslash G$ by translations and, if H were not unimodular, G would act effectively as $\mathrm{Aff}(\mathbf{R})$, neither of which can happen since effectively G must act as the universal cover of $\mathrm{SL}(2, \mathbf{R})$ (cf. Lemma 2.3 of [10]).

Suppose $\mathscr{F}$ is an arbitrary codimension-one foliation of a connected manifold M. We say that $\mathscr{F}$ is *covered by a product foliation* if the foliation $\widehat{\mathscr{F}}$ induced on the universal covering space of M is diffeomorphic to a product foliation $L \times \mathbf{R}$, with leaves $L \times \{t\}$, $t \in \mathbf{R}$. The examples considered above have this property. For such foliations it is clear that the leaf space of $\widehat{\mathscr{F}}$ is diffeomorphic to $\mathbf{R}$. However, the converse is not always true even for transversely affine foliations. For example, it is observed in [3] that the standard Reeb foliation of $S^n \times S^1$ has a transverse affine structure. For $n \geq 2$ the universal covering of this foliation has leaf space diffeomorphic to $\mathbf{R}$, but is not a product foliation (one leaf is $S^{n-1} \times \mathbf{R}$, the rest are $\mathbf{R}^n$). There are some situations where the converse is valid. The foliation $\mathscr{F}$ is said to be *minimal* if every leaf of $\mathscr{F}$ is dense in M.

(3.1) Proposition. *Suppose that $\mathscr{F}$ is a transversely affine foliation of a connected compact manifold M. Assume also that $\mathscr{F}$ satisfies at least one of the following conditions.*

(i) *$\mathscr{F}$ is determined by a closed one-form ($[\theta] = 0$ in $H^1(M; \mathbf{R})$).*

(ii) *$\mathscr{F}$ is minimal and $\widehat{\mathscr{F}}$ has leaf space diffeomorphic to $\mathbf{R}$.*

Then $\mathscr{F}$ is covered by a product foliation.

Proof. Assume that $\mathscr{F}$ satisfies (i), that is, $\mathscr{F}$ is determined by a closed one-form ω. Let X be a vector field on M such that $\omega(X) \equiv 1$. Then since ω is closed and $d i_X \omega = 0$, it follows that $L_X \omega = 0$, so the X-flow preserves $\mathscr{F}$. Lifting the X-flow to $\widehat{M}$ we obtain a flow transverse to $\widehat{\mathscr{F}}$ which takes leaves into leaves. The orbit of a single leaf under the lifted flow is both open and closed in $\widehat{M}$, so $\widehat{\mathscr{F}}$ has a product structure. Assume now that $\mathscr{F}$ satisfies (ii). The action of $\pi_1(M)$ on $\widehat{M}$ induces an action on the leaf space of $\widehat{\mathscr{F}}$. Since $\mathscr{F}$ is minimal and transversely affine, this action must be minimal and affine. Let $\overline{M}$ be the covering space of M corresponding to the subgroup of $\pi_1(M)$ which determines translations. The induced foliation $\overline{\mathscr{F}}$ on $\overline{M}$ clearly has an invariant volume, that is, it is determined by a closed one-form $\overline{\omega}$. Let $\overline{X}$ be a vector field on $\overline{M}$ which is perpendicular to $\overline{\mathscr{F}}$ for some metric induced from M and such that $\overline{\omega}(\overline{X}) \equiv 1$. Now proceed as before. This proves (3.1).

The following result of Duminy gives a useful condition for a foliation of a compact manifold to be minimal. A detailed exposition is given in Cantwell and Conlon [2]. Recall that every foliation of a compact manifold has a (nonempty) minimal set (which coincides with M when $\mathscr{F}$ is minimal).

(3.2) Theorem. *If $\mathscr{F}$ is a foliation of class $\mathscr{C}^k$ $(k \geq 2)$ of a compact manifold M such that every leaf has finitely many ends, then every minimal set of $\mathscr{F}$ is either a compact leaf or all of M.*

The examples described in the previous section do occur as global holonomy groups for the universal covering of a foliation of a compact manifold [8]. As before, let $f: S^1 \to S^1$ be a covering map of degree $n \geq 1$. Start with the product foliation of $D^n \times S^1$ ($D^n = n$-disk) determined by projection to S^1. Assuming $n \geq 2$ there is an embedded circle transverse to this foliation which wraps around the S^1 factor n times. Removing a tubular neighborhood of the embedded circle results in a manifold of dimension $n+1$ having two boundary components, each diffeomorphic to $S^{n-1} \times S^1$. Identify these boundary components via a map which preserves the codimension-one foliation of the boundary and projects to f, thus obtaining a codimension one foliation $\mathscr{F}$ of a manifold M without boundary. The foliation $\mathscr{F}$ is covered by a product foliation and has global holonomy group Γ (generated by a lift F of f and generator G of the deck group of S^1). If $f(t) = nt \pmod 1$, $\mathscr{F}$ has a transverse affine structure of class $\mathscr{C}^\omega$. When f is an arbitrary expanding map of degree n, $\mathscr{F}$ has a continuous transverse affine structure. When f is not conjugate to any expanding map, the foliation $\mathscr{F}$ is not minimal. In every case, each leaf of $\mathscr{F}$ is noncompact and has infinitely many ends. Also, when $n \geq 3$, $\pi_1(M)$ is isomorphic to Γ and the fundamental group of each leaf is trivial or infinite cyclic.

4. From polycyclic to transversely affine

In this section we will consider codimension-one foliations $\mathscr{F}$ of class $\mathscr{C}^k$ $(k \geq 2)$ of a connected manifold M. When $\mathscr{F}$ is not transversely orientable we may pass to a two-fold covering space such that the lifted foliation is transversely orientable. In the transversely oriented case $\mathscr{F}$ is determined by a one-form ω. An immersed circle $c\colon S^1 \to M$ is *transverse* to $\mathscr{F}$ if $\omega(\dot{c}(t)) > 0$ for all t. A subset $S \subset M$ is *$\mathscr{F}$-saturated* if it is a union of leaves of $\mathscr{F}$. Standard arguments show that if L is a leaf of $\mathscr{F}$ which is *not* closed, then there is a circle transverse to $\mathscr{F}$ which intersects L. $\mathscr{F}$ is said to have *compact leaf space* if every open cover of M by open $\mathscr{F}$-saturated sets has a finite subcover. When $\mathscr{F}$ has no closed leaves this means that every leaf of $\mathscr{F}$ meets one of finitely many transverse circles, indeed there will exist a single transverse circle which meets every leaf. To see this, suppose $c_1, \cdots, c_n$ are transverse circles with $\mathscr{F}$-saturations $S_1, \cdots, S_n$. Since M is connected, $S_i \cap S_j \neq 0$ for some $i \neq j$. This means that there is a path lying in a single leaf of $\mathscr{F}$ with an endpoint in each of the images of c_i and c_j. Using this path it is easy to construct an immersed circle c^* whose saturation is $S_i \cup S_j$. Replacing c_i and c_j by c^* reduces the number of transverse circles by one. Repeating this process we eventually obtain a transverse circle whose $\mathscr{F}$-saturation is all of M. Novikov [11] would say that such $\mathscr{F}$ have a single component.

(4.1) Theorem. *Suppose $\mathscr{F}$ is a real analytic foliation which has a compact leaf space. If $\pi_1(M)$ is virtually polycyclic and $\mathscr{F}$ does not have any invariant measures, then $\mathscr{F}$ is minimal and transversely affine. If, in addition, M is compact, then $\mathscr{F}$ is covered by a product foliation.*

Proof. Since $\mathscr{F}$ has no invariant measures it does not, in particular, have any closed leaves. Furthermore, when M is compact, nonexistence of invariant measures is equivalent to saying that every leaf of $\mathscr{F}$ has exponential growth [16]. Since $\mathscr{F}$ has a single Novikov component and $\pi_1(M)$ does not contain a free subgroup on two generators, it follows from Solodov [21, Theorem 2.1] that the universal covering space $\widehat{M}$ of M contains an embedded line which is transverse to the lifted foliation $\widehat{\mathscr{F}}$ and meets every leaf. We claim that this transverse line meets every leaf of $\widehat{\mathscr{F}}$ exactly once. If some transverse segment has endpoints in the same leaf of $\widehat{\mathscr{F}}$, then there would be a null-homotopic loop of the form $\alpha * \beta$ with α tangent to $\widehat{\mathscr{F}}$ and β transverse. This could be homotoped to a null-homotopic transverse circle. However this cannot happen for real analytic foliations by a result of Haefliger [5]. The claim follows, that is, the leaf space of $\widehat{\mathscr{F}}$ is $\mathscr{C}^\omega$ diffeomorphic to $\mathbf{R}$. The global holonomy group of $\widehat{\mathscr{F}}$ is virtually polycyclic, and every orbit has exponential growth since $\mathscr{F}$ has no invariant measures. It follows from (1.5) and (2.2) that $\mathscr{F}$ is minimal and transversely affine. The conclusion when M is compact follows from (1.5) and (3.1), and the proof is complete.

For foliations which are not real analytic it is necessary to make additional topological hypotheses on $\mathscr{F}$ and M.

(4.2) Theorem. *Suppose $\mathscr{F}$ is $\mathscr{C}^k$ $(k \geq 2)$ and has compact leaf space but does not have any invariant measures or null-homotopic transverse circles. If $\pi_1(M)$ has a subgroup of finite index whose commutator subgroup is finitely generated and abelian, then $\mathscr{F}$ is minimal and has a transverse affine structure of class $\mathscr{C}^{k-2}$. If M is also compact, $\mathscr{F}$ is covered by a product foliation.*

Proof. The theorem follows from (1.6), (2.3), and (3.1) by the argument used to prove (4.1).

(4.3) Theorem. *Suppose $\mathscr{F}$ is $\mathscr{C}^k$ $(k \geq 2)$ and is minimal but does not have any invariant measures or null-homotopic transverse circles. If $\pi_1(M)$ is virtually polycyclic, then $\mathscr{F}$ has a transverse affine structure of class $\mathscr{C}^{k-2}$. If M is also compact, $\mathscr{F}$ is covered by a product foliation.*

Proof. Since $\mathscr{F}$ is minimal it has compact leaf space, so the result follows from (2.1) and (3.1) by the argument used to prove (4.1).

In the results which follow we will always assume that M is compact. In this case there is a notion of growth type for the leaves of $\mathscr{F}$ [13]–[16]. We say that $\mathscr{F}$ has *exponential growth* if every leaf of $\mathscr{F}$ has exponential growth. This is equivalent to the assertion that $\mathscr{F}$ has no invariant measures for codimension one foliations of compact manifolds.

The following result gives a condition on the leaves of a foliation which is useful in proving minimiality.

(4.4) Lemma. *Suppose $\mathscr{F}$ is a codimension-one foliation of class $\mathscr{C}^k$ $(k \geq 2)$ of a compact manifold of dimension n. Assume that every leaf L of $\mathscr{F}$ satisfies the following:*

(1) *$\pi_1(L)$ is virtually polycyclic.*

(2) *The universal covering space of L has finite-dimensional rational homology.*

Then every minimal set of $\mathscr{F}$ is either a compact leaf or all of M.

Proof. Pass to a finite covering space to assume that the tangent bundle of $\mathscr{F}$ is orientable. If $\mathscr{F}$ has a minimal set which is neither a compact leaf nor all of M, it follows from (3.2) that some leaf L of $\mathscr{F}$ has infinitely many ends. In particular, the space $H_c^1(L; \mathbf{Q}) \cong H_{n-2}(L; \mathbf{Q})$

will be infinite dimensional. It follows from [9, p. 343] and [7, proof of Lemma 6] that the universal covering space of L would have infinite-dimensional rational homology. This proves (4.4).

Remark. If $\mathcal{F}$ is the orbit foliation of a codimension-one locally free action on a connected manifold by an amenable Lie group, then $\mathcal{F}$ satisfies the conditions of (4.4) [10]. On the other hand, the conclusion of (4.4) is not valid for actions by arbitrary Lie groups [14].

If L is a leaf of a foliation $\mathcal{F}$ of M, then any element in the kernel of the induced map $\pi_1(L) \to \pi_1(M)$ is called a *vanishing cycle.* Many types of codimension-one foliations do not have vanishing cyclies, for example, real analytic foliations [5], [11], foliations determined by locally free Lie group actions [13], and Anosov foliations. It is shown in [11] that the existence of a null-homotopic transverse circle implies the existence of a vanishing cycle.

(4.5) Theorem. *Suppose $\mathcal{F}$ is a codimension-one foliation of class $\mathcal{C}^k$ $(k \geq 2)$ of a compact manifold M. Assume that $\mathcal{F}$ has exponential growth, no vanishing cycles, and that the universal cover of every leaf has finite-dimensional rational homology. If $\pi_1(M)$ is virtually polycyclic, then $\mathcal{F}$ is minimal, has a transverse affine structure of class $\mathcal{C}^{k-2}$, and is covered by a product foliation.*

Proof. Since $\mathcal{F}$ has no vanishing cycles, $\pi_1(L)$ is virtually polycyclic for every leaf L. From (4.4) it follows that $\mathcal{F}$ is minimal and the rest of the conclusion follows from (4.3).

A group is said to be *virtually solvable* if it has a solvable subgroup of finite index.

(4.6) Corollary. *Suppose $\mathcal{F}$ is a $\mathcal{C}^\infty$ codimension-one foliation of a compact manifold M of dimension n. Assume that $\mathcal{F}$ has exponential growth, no vanishing cycles, and that the universal cover of every leaf is diffeomorphic to $\mathbf{R}^{n-1}$. Assume also that $\pi_1(M)$ is virtually solvable. Then $\pi_1(M)$ is virtually polycyclic; $\mathcal{F}$ is minimal, has a $\mathcal{C}^\infty$ transverse affine structure, and is covered by a product foliation.*

Proof. Denote by $\widehat{\mathcal{F}}$ the lifting of $\mathcal{F}$ to the universal covering space $\widehat{M}$ of M. $\widehat{\mathcal{F}}$ has no vanishing cycles, so every leaf of $\widehat{\mathcal{F}}$ is diffeomorphic to $\mathbf{R}^{n-1}$. By a result of Palmeira [12, Corollary 3], $\widehat{M}$ is diffeomorphic to $\mathbf{R}^n$. In particular, M is an Eilenberg-Mac Lane space with virtually solvable fundamental group. It follows from (3.3) of [1] that $\pi(M)$ is virtually polycyclic. The rest of the conclusion now follows from (4.5).

Denote by $\mathcal{S}$ the class of virtually solvable groups Γ for which there is a series

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_n = \{e\}$$

such that for each $i = 1, \cdots, n$, Γ_i is normal in Γ_{i-1} and there is a finitely generated subgroup $H_i \subset \Gamma_{i-1}/\Gamma_i$ which has nonexponential growth, and for every $g \in \Gamma_{i-1}/\Gamma_i$ there is a natural number m such that $g^m \in H_i$. Clearly $\mathcal{S}$ includes all virtually polycyclic groups as well as the example Γ considered in §§2 and 3. In [18] an example is given of a finitely generated solvable group which is not in $\mathcal{S}$. (The author does not know of such an example which is finitely presented.) It is shown in [18] that a minimal action on $\mathbf{R}$ by any group $\Gamma \in \mathcal{S}$ must be $\mathscr{C}^0$ conjugate to a subgroup of $\mathrm{Aff}(\mathbf{R})$. The example of §2 shows that the smoothness cannot be improved without additional hypotheses.

(4.7) Corollary. *Suppose M is compact with $\pi_1(M) \in \mathcal{S}$ and that $\mathcal{F}$ is a $\mathscr{C}^k$ $(k \geq 2)$ codimension-one foliation of M having exponential growth and no null-homotopic transverse circles. Assume that every leaf L of $\mathcal{F}$ is a $K(\pi, 1)$ with $\pi_1(L)$ virtually polycyclic. Then $\mathcal{F}$ is minimal, has a $\mathscr{C}^{k-2}$ transverse affine structure, and is covered by a product foliation. Furthermore, $\pi_1(M)$ is virtually polycyclic.*

Proof. From (4.4) it follows that $\mathcal{F}$ is minimal. By the argument used to prove (4.3) it follows that $\mathcal{F}$ has a $\mathscr{C}^0$ transverse affine structure and is covered by a product foliation. In particular, $\widehat{M}$ has the same homotopy type as the universal cover of a leaf (contractible), i.e., M is a $K(\pi, 1)$. By (3.3) of [1], $\pi_1(M)$ must be virtually polycyclic. The result now actually follows from (4.3).

Remarks. (1) The topological hypotheses on $\mathcal{F}$ in (4.5) (respectively, (4.6) and (4.7)) are satisfied if $\mathcal{F}$ is the orbit foliation of a locally free action by an amenable (respectively, solvable) Lie group.

(2) The result from [12] cited in the proof of (4.6) is valid for $\mathscr{C}^k$ $(k \geq 2)$ foliations if the leaves are $\mathscr{C}^\infty$. One concludes in that case that the foliation has a $\mathscr{C}^{k-2}$ transverse affine structure.

(3) In (4.7), the hypothesis on L may be weakened to (2) of (4.4) with the weakened conclusion that the foliation has a $\mathscr{C}^0$ transverse affine structure.

(4) It may be true that the hypothesis on $\pi_1(M)$ in (4.2) may be weakened to virtually polycyclic, in which case (4.5) would be redundant. Virtually polycyclic groups have the property that any coset space has finitely many ends; this follows, for example, from Lemma 6 of [7]. On the other hand, this is not true of solvable groups such as the example Γ from §2. A technical problem is that the holonomy of $\mathcal{F}$ is determined by a pseudogroup but *not* by a group action.

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